Appendix D

Floquet Analysis for an Infinite Array of Regular Electrodes

The equations for analysis of multi-strip couplers were derived in Chapter 5 using the Green's function method, making use of the quasi-static approximation. The equations therefore exclude electrode interactions, and in consequence do not predict the presence of any stop bands. In addition, for frequencies outside the stop bands the interactions cause a small change to the surface wave velocity, and this is not correctly predicted by the quasi-static approximation. To rectify these omissions, this appendix gives an account based on the work of Bløtekjaer *et al.* [475, 476], in which the solution is obtained by applying Floquet's theorem and using the effective permittivity discussed in Chapter 3. The permittivity is taken to be adequately represented by Ingebrigtsen's approximation, assuming that the only acoustic wave present is a piezoelectric Rayleigh wave. A similar approach was used by Emtage [477], and an alternative method using perturbation theory [118] gives equivalent results. In the analysis here the surface electric field and charge density are written in terms of space harmonics with coefficients given by Legendre functions, in a manner similar to an earlier analysis of helical waveguides given by Chu [478].

The use of the effective permittivity implies that mass loading due to the electrodes is neglected, and it is also assumed that the electrodes have negligible thickness and resistivity. The electrodes are taken to have a length, in the y-direction, much greater than their width, and the surface waves are assumed to be free of diffraction and to have wavefronts parallel to the y-axis. Thus all quantities, including the surface potential and charge density, can be taken to be invariant with y.

It is a feature of the solution that the equations become very complex when high frequencies are considered. In view of this, the analysis is given here for frequencies $\omega < 2\pi v_0/p$, where v_0 is the free-surface velocity and p the pitch of the electrodes. In most practical cases this range includes the frequencies of interest, and it includes the first stop-band, which occurs for frequencies near $\pi v_0/p$. The analysis for higher frequencies is discussed briefly in Section D.4.

D.1. General Solution for Low Frequencies

For a piezoelectric half-space, the surface potential $\phi(x)$ and charge density $\sigma(x)$ are related by the effective permittivity $\varepsilon_s(\beta)$, which gives the ratio of the Fourier transforms $\bar{\phi}(\beta)$ and $\bar{\sigma}(\beta)$. For the analysis here it is convenient to use the parallel electric field $E_x(x)$ at the surface, rather than the potential $\phi(x)$. Taking $\bar{E}_x(\beta)$ as the transform of $E_x(x)$, we have $\bar{E}_x(\beta) = -j\beta\bar{\phi}(\beta)$. Thus, with $\varepsilon_s(\beta)$ defined by equation (3.24) of Section 3.2, we have

$$\bar{\sigma}(\beta) = j\varepsilon_s(\beta) \operatorname{sgn}(\beta)\, \bar{E}_x(\beta) \tag{D.1}$$

with sgn $(\beta) = 1$ for $\beta > 0$ and sgn $(\beta) = -1$ for $\beta < 0$. The solution must satisfy this relation and the boundary conditions that $E_x(x) = 0$ on the electrodes and $\sigma(x) = 0$ in the gaps between the electrodes.

The configuration of the electrodes is shown in Figure 5.1. The electrodes are centred at $x = mp$, with $-\infty \leqslant m \leqslant \infty$, and have width a. For the electrode centred at $x = mp$, the voltage is V_m and the current entering the electrode is I_m. For the analysis here we consider solutions in which these take the forms

$$\begin{aligned} V_m &= V_0 \exp(-j\kappa mp), \\ I_m &= I_0 \exp(-j\kappa mp), \end{aligned} \tag{D.2}$$

where a term exp $(j\omega t)$ is implicit. The factor κ has the rôle of a propagation constant. These equations are unaffected if a multiple of $2\pi/p$ is added to κ, and it is convenient to restrict κ to the range

$$0 \leqslant \kappa \leqslant 2\pi/p. \tag{D.3}$$

This restriction does not affect the generality of the solution.

With the electrode voltages given by equation (D.2), it is reasonable to assume that the electric field has the property $E_x(x + p) = E_x(x) \exp(-j\kappa p)$, and it then follows that $E_x(x) \exp(j\kappa x)$ must be a periodic function, with period p. Thus $E_x(x)$ can be written in the form

$$E_x(x) = \sum_{n=-\infty}^{\infty} E_n\, e^{-j2\pi nx/p}\, e^{-j\kappa x} \tag{D.4}$$

and similarly the charge density $\sigma(x)$ can be written

$$\sigma(x) = \sum_{n=-\infty}^{\infty} \sigma_n\, e^{-j2\pi nx/p}\, e^{-j\kappa x}, \tag{D.5}$$

where the coefficients E_n and σ_n are to be determined. These equations have the forms given by Floquet's theorem, which is readily proved for the case of a wave in an unbounded medium with periodic properties [114]. However, here we are concerned with a bounded medium having a periodic boundary condition, and for such cases Floquet's theorem is not always valid [114]. Nevertheless, it will be seen that a solution with the form of equations (D.4) and (D.5) can be obtained for the present case.

In the β-domain, each term in equations (D.4) and (D.5) has the form of a delta function located at $\beta = -(\kappa + 2\pi n/p)$. The ratio of E_n to σ_n is thus given by equation (D.1) for this value of β. In view of equation (D.3), we have sgn $(-\kappa - 2\pi n/p) = -S_n$, with S_n defined by

$$S_n = 1, \quad \text{for } n \geqslant 0; \qquad S_n = -1, \quad \text{for } n < 0. \tag{D.6}$$

Noting also that $\varepsilon_s(-\beta) = \varepsilon_s(\beta)$, we have

$$\frac{\sigma_n}{E_n} = -jS_n\varepsilon_s(\kappa + 2\pi n/p). \tag{D.7}$$

Now, $\varepsilon_s(\beta)$ is taken to be given by Ingebrigtsen's approximation, equation (3.38), which is plotted in Figure 3.2. This function is almost constant except for β close to $\pm k_0$, where $k_0 > 0$ is the wavenumber for surface wave propagation on a free surface at frequency ω. This condition is expressed by writing $\varepsilon_s(\beta) = \varepsilon_s(\infty)$ for all β except for values near $\pm k_0$. We also assume that the frequency is restricted such that

$$0 \leqslant k_0 < 2\pi/p, \tag{D.8}$$

which implies that $\omega < 2\pi v_0/p$. In view of this restriction, and of equation (D.3), it is found that $\kappa + 2\pi n/p$ can only be close to $\pm k_0$ if $n = 0$ or -1, and so equation (D.7) becomes

$$\frac{\sigma_n}{E_n} = -jS_n\varepsilon_s(\kappa + 2\pi n/p), \quad \text{for } n = 0, -1, \tag{D.9a}$$

$$= -jS_n\varepsilon_s(\infty), \qquad \text{for } n \neq 0, -1. \tag{D.9b}$$

If k_0 is close to $2\pi/p$, the additional case $n = 1$ must be included in equation (D.9a); however this is excluded here by assuming that k_0 is at least a few percent below $2\pi/p$.

The solution for σ_n and E_n is found using properties of Legendre functions, given in Appendix C. Consider the functions

$$E'(x) = \sum_{n=-\infty}^{\infty} S_{n-r}P_{n-r}(\cos\Delta)\, e^{-j2\pi nx/p}\, e^{-j\kappa x}$$

$$\sigma'(x) = -j\varepsilon_s(\infty) \sum_{n=-\infty}^{\infty} P_{n-r}(\cos\Delta)\, e^{-j2\pi nx/p}\, e^{-j\kappa x}, \tag{D.10}$$

where $\Delta = \pi a/p$ and a and p are respectively the width and pitch of the electrodes. From equation (C.15) it can be seen that $E'(x)$ is zero at the electrode locations, irrespective of the value of the integer r. From equation (C.14), $\sigma'(x)$ is zero in the gaps between the electrodes, for any r. If $\varepsilon_s(\beta)$ were independent of β, equations (D.9) would be satisfied by equations (D.10), with $r = 0$. However, because $\varepsilon_s(\beta)$ has different values for $n = 0, -1$ it is necessary to add terms with several values of r. It is found that terms with $r = -1, 0, 1$ are sufficient, so that the field and charge density are given by

$$E_x(x) = \sum_{n=-\infty}^{\infty} \sum_{r=-1}^{1} \alpha_r S_{n-r}P_{n-r}(\cos\Delta)\, e^{-j2\pi nx/p}\, e^{-j\kappa x}, \tag{D.11}$$

$$\sigma(x) = -j\varepsilon_s(\infty) \sum_{n=-\infty}^{\infty} \sum_{r=-1}^{1} \alpha_r P_{n-r}(\cos\Delta)\, e^{-j2\pi nx/p} e^{-j\kappa x}, \tag{D.12}$$

where the coeffients α_r are to be determined. Since these are linear combinations of equations (D.10), the required boundary conditions are satisfied. We also have, writing $P_n(\cos\Delta)$ as P_n for brevity,

$$\frac{\sigma_n}{E_n} = \frac{-j\varepsilon_s(\infty)\,[\alpha_{-1}P_{n+1} + \alpha_0 P_n + \alpha_1 P_{n-1}]}{\alpha_{-1}S_{n+1}P_{n+1} + \alpha_0 S_n P_n + \alpha_1 S_{n-1}P_{n-1}} \tag{D.13}$$

and this is required to satisfy equations (D.9). Now, from equation (D.6), it can be seen that $S_{n-1} = S_n = S_{n+1}$ for $n \neq 0$ or -1, and hence equation (D.9b) is already satisfied. The ratios of the coefficients α_r are therefore determined by equation (D.9a). It is convenient to define A_0 and A_{-1} by

$$A_r = \frac{\varepsilon_s(\kappa + 2\pi r/p) + \varepsilon_s(\infty)}{\varepsilon_s(\kappa + 2\pi r/p) - \varepsilon_s(\infty)}, \quad \text{for } r = 0, -1. \tag{D.14}$$

Substituting equation (D.13) into equation (D.9a), it is then found that

$$\frac{\alpha_1}{\alpha_0} = \frac{A_{-1} + \cos\Delta}{A_0 A_{-1} - \cos^2\Delta},$$

$$\frac{\alpha_{-1}}{\alpha_0} = \frac{A_0 + \cos\Delta}{A_0 A_{-1} - \cos^2\Delta}, \tag{D.15}$$

where use has been made of the relations $P_{-1} = P_0 = 1$ and $P_{-2} = P_1 = \cos\Delta$.

The effective permittivity $\varepsilon_s(\beta)$ is taken to be given by Ingebrigtsen's approximation, equation (3.38), so that

$$\varepsilon_s(\beta) = (\varepsilon_0 + \varepsilon_p^T)\,\frac{\beta^2 - k_0^2}{\beta^2 - k_m^2}, \tag{D.16}$$

where k_0 and k_m are respectively the wavenumbers for surface waves on a free surface and on a metallised surface. We thus have $\varepsilon_s(\infty) = \varepsilon_0 + \varepsilon_p^T$ and, using equation (D.14),

$$A_0 = (2\kappa^2 - k_0^2 - k_m^2)/(k_m^2 - k_0^2) \tag{D.17}$$

and

$$A_{-1} = [2(\kappa - 2\pi/p)^2 - k_0^2 - k_m^2]/(k_m^2 - k_0^2). \tag{D.18}$$

Electrode Voltages and Currents. The electrode voltages V_m are found by integrating the field to give the potential, evaluating this at the electrode centres $x = mp$. Using equation (D.4) for the field, the electrode voltages are found to be $V_m = V_0 \exp(-j\kappa mp)$, with V_0 given by

$$V_0 = -\frac{jp}{2\pi}\sum_{n=-\infty}^{\infty}\frac{E_n}{n + s}, \tag{D.19}$$

where we have defined

$$s = \kappa p/(2\pi), \tag{D.20}$$

and, in view of equation (D.3), we have $0 \leqslant s \leqslant 1$. The coefficients E_n in equation (D.19) are identified by comparing equation (D.4) with equation (D.11), and the summation over n can be done by using Dougall's expansion, equation (C.12). We thus have

$$V_0 = -\frac{jp}{2\sin(\pi s)}\sum_{r=-1}^{1}(-1)^r\alpha_r P_{r+s-1}(-\cos\Delta). \tag{D.21}$$

The current I_m entering the electrode centred at $x = mp$ is found by integrating the charge density over the area of the electrode and differentiating with respect to time. The integral over x has limits $x = mp \pm a/2$, and the electrode length, in the y-direction, is W. Using equation (D.5) for the charge density, it is found that $I_m = I_0 \exp(-j\kappa mp)$, with

$$I_0 = \frac{j\omega Wp}{2\pi}\int_{-\Delta}^{\Delta}\sum_{n=-\infty}^{\infty}\sigma_n\,\mathrm{e}^{-j(n+s)\theta}\,\mathrm{d}\theta, \tag{D.22}$$

where $\theta = 2\pi(x - mp)/p$ and $\Delta = \pi a/p$. The coefficients σ_n can be evaluated by comparing equations (D.5) and (D.12). Equation (D.22) may then be evaluated using equation (C.16) of Appendix C, giving the result

$$I_0 = \omega Wp(\varepsilon_0 + \varepsilon_p^T)\sum_{r=-1}^{1}\alpha_r P_{r+s-1}(\cos\Delta), \tag{D.23}$$

where the relation $\varepsilon_s(\infty) = (\varepsilon_0 + \varepsilon_p^T)$ has been used.

Equations (D.21) and (D.23) give the electrode voltages and currents in terms of the coefficients α_r, which are related to the propagation constant κ by equations (D.15), (D.17) and (D.18). If the ratio I_0/V_0 is specified these equations determine the relative values of the α_r and the value of κ, and hence the field and charge density, equations (D.11) and (D.12), are determined. The following sections describe some particular cases.

D.2. Propagation Outside the Stop Band

For propagation on a free surface, the surface-wave wavenumber is $\pm k_0$. Strong coupling to a surface wave is therefore expected only if one or more of the wavenumbers $\kappa + 2\pi n/p$ in the Floquet expansion is near to $\pm k_0$. Usually, at most one of the wavenumbers is close to $\pm k_0$. However, in the special case $k_0 \simeq \pi/p$, it is possible to have $\kappa \simeq k_0$ and $\kappa - 2\pi/p \simeq -k_0$, so that coupling occurs for $n = 0$ and for $n = -1$. In this case two surface waves propagating in opposite directions are coupled by the structure, and this leads to the presence of a stop band. This occurs for frequencies close to $\pi v_0/p$. In this section it will be assumed that the frequency is not close to this value. The stop band will be considered in Section D.3.

It can be assumed that, if coupling to surface waves occurs, it occurs for $\kappa \approx k_0$. The alternative case, when $\kappa - 2\pi/p \approx -k_0$, is essentially the same solution, and can therefore be neglected. In view of this, we can take $\varepsilon_s(\kappa - 2\pi/p) = \varepsilon_s(\infty)$ and thus from equation (D.14) we have $A_{-1} = \infty$. Equations (D.15) then give $\alpha_{-1} = 0$ and

$$\alpha_0/\alpha_1 = A_0 = (2\kappa^2 - k_0^2 - k_m^2)/(k_m^2 - k_0^2), \tag{D.24}$$

where A_0 has been obtained from equation (D.17). Thus the α_{-1} term in the electric field and charge density, equations (D.11) and (D.12), is not required here. This can also be deduced more directly by noting that equation (D.9b) must be valid for all $n \neq 0$. We now consider separately three cases, in which the electrodes are short-circuited or open-circuited, or have some more general termination.

(a) Shorted Electrodes. If the electrodes are all connected together the voltages V_m must all be zero, and hence $V_0 = 0$. Thus, noting that $\alpha_{-1} = 0$, equation (D.21) gives

$$\alpha_0/\alpha_1 = P_s(-\cos\Delta)/P_{s-1}(-\cos\Delta). \tag{D.25}$$

Using equation (D.24), this gives a relation determining the propagation constant κ, which for this case is denoted k_{sc}. Thus,

$$\kappa^2 = k_{sc}^2 = k_0^2 + \tfrac{1}{2}(k_m^2 - k_0^2)\left[1 + \frac{P_s(-\cos\Delta)}{P_{s-1}(-\cos\Delta)}\right]. \tag{D.26}$$

Since k_m is close to k_0 it can be concluded that k_{sc} is also close to k_0. In the above equation $s = k_{sc}p/(2\pi)$, so that the equation is transcendental. However, since the Legendre functions vary slowly with s, a good approximation is obtained by using $s = k_0 p/(2\pi)$, and the right side is then independent of k_{sc}. The surface-wave velocity in the structure is $v_{sc} = \omega/k_{sc}$. Noting that $k_m \approx k_0$, the velocity is approximately given by

$$v_{sc} \approx v_0 + \tfrac{1}{2}(v_m - v_0)\left[1 + \frac{P_s(-\cos\Delta)}{P_{s-1}(-\cos\Delta)}\right], \tag{D.27}$$

where $v_0 = \omega/k_0$ and $v_m = \omega/k_m$.

(b) Open-Circuit Electrodes. With the electrodes disconnected electrically the currents I_m are zero, and hence $I_0 = 0$. From equation (D.23) we have

$$\alpha_0/\alpha_1 = -P_s(\cos\Delta)/P_{s-1}(\cos\Delta), \tag{D.28}$$

and κ may then be obtained using equation (D.24). For this case κ is denoted by k_{oc}, and we thus have

$$\kappa^2 = k_{oc}^2 = k_0^2 + \tfrac{1}{2}(k_m^2 - k_0^2)\left[1 - \frac{P_s(\cos\Delta)}{P_{s-1}(\cos\Delta)}\right]. \tag{D.29}$$

Here again, k_{oc} must be close to k_0, and a good approximation is obtained by setting $s = k_0 p/(2\pi)$. The surface-wave velocity is $v_{oc} = \omega/k_{oc}$ and is approximately given by

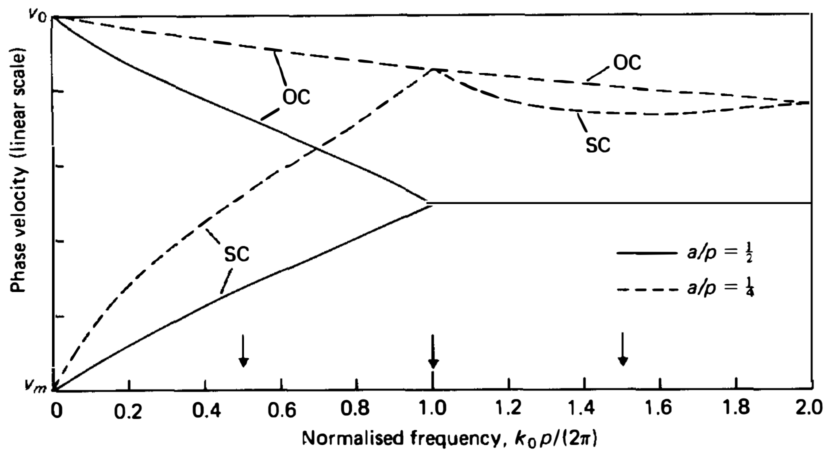

FIGURE D.1. Surface wave velocity in an array of open-circuited (OC) or shorted (SC) electrodes. Arrows indicate locations of stop bands, where the analysis is invalid.

$$v_{oc} \approx v_0 + \tfrac{1}{2}(v_m - v_0)\left[1 - \frac{P_s(\cos\Delta)}{P_{s-1}(\cos\Delta)}\right] \tag{D.30}$$

The velocities v_{sc} and v_{oc} are shown as functions of frequency in Figure D.1, for several metallisation ratios. For $\omega < 2\pi v_0/p$ the velocities are given by equations (D.27) and (D.30), but for higher frequencies the analysis of Section D.4 below must be used. Some experimental results confirming the forms of the curves, for $a/p = \frac{1}{2}$ and $0 < \omega < 2\pi v_0/p$, were obtained by Williamson [134] using a Y, Z lithium niobate substrate and an electrostatic probe.

With $s = k_0 p/(2\pi)$, the difference $k_{sc}^2 - k_{oc}^2$ can be obtained by subtracting equation (D.29) from equation (D.26). Using equation (C.6) of Appendix C, this gives

$$k_{sc}^2 - k_{oc}^2 = \frac{k_m^2 - k_0^2}{k_0 p} \frac{2\sin(\pi s)}{P_{-s}(-\cos\Delta)P_{-s}(\cos\Delta)}. \tag{D.31}$$

Noting that $k_m \approx k_0$, this is in good agreement with the result from the quasi-static analysis, equation (5.23).

(c) General Terminations. The above equations give the propagation constant κ when the ratio I_0/V_0 is either infinite or zero. More generally, I_0/V_0 can take intermediate values. The electrode voltages and currents must of course have the form given by equations (D.2).

The ratio I_0/V_0 is given by equations (D.21) and (D.23), noting that $\alpha_{-1} = 0$ here. The ratio can be expressed as

$$\frac{I_0}{V_0} = 2j\omega W(\varepsilon_0 + \varepsilon_p^T)\sin(\pi s)\frac{P_{s-1}(\cos\Delta)}{P_{s-1}(-\cos\Delta)}\frac{X_+}{X_-}, \tag{D.32}$$

where

$$X_\pm = 1 \pm \frac{\alpha_1}{\alpha_0}\frac{P_s(\pm\cos\Delta)}{P_{s-1}(\pm\cos\Delta)} \tag{D.33}$$

and $s = \kappa p/(2\pi)$. The ratio α_1/α_0 is related to κ by equation (D.24), and hence equation (D.32) gives the ratio I_0/V_0 if κ is specified. Alternatively, the equation gives κ if I_0/V_0 is specified.

The above result may be shown to be in close agreement with the quasi-static analysis. From equation (D.24) it can be seen that α_1/α_0 is small except when κ is close to k_0. Thus, since the Legendre functions vary slowly with s, the functions $X_\pm$ in equation (D.33) can be evaluated approximately by setting $s = k_0 p/(2\pi)$. The ratio of Legendre functions in equation (D.33) is then related to k_{oc} or k_{sc} by equation (D.29) or equation (D.26). This gives

$$X_+ \approx 2(\kappa^2 - k_{oc}^2)/(2\kappa^2 - k_0^2 - k_m^2).$$

For X_- the same result is obtained, with k_{oc} replaced by k_{sc}. Thus I_0/V_0 is given by equation (D.32), with

$$\frac{X_+}{X_-} \approx \frac{\kappa^2 - k_{oc}^2}{\kappa^2 - k_{sc}^2} = 1 - \frac{k_{sc}^2 - k_{oc}^2}{k_{sc}^2 - \kappa^2} \tag{D.34}$$

and with $s = \kappa p/(2\pi)$. Also, $k_{sc}^2 - k_{oc}^2$ is given by equation (D.31). This result agrees well with the quasi-static analysis, equation (5.18), except that the latter has $k_0^2 - \kappa^2$ in the denominator instead of $k_{sc}^2 - \kappa^2$.

D.3. Stop Bands

When k_0 is close to π/p, it is no longer valid to ignore the α_{-1} term in the equations of Section D.1, and it is then found that the propagation constant κ can be complex. In this section we consider the stop band for the two cases when the electrodes are either shorted or open-circuited. It is sufficient to assume that $k_0 \approx \pi/p$, because the solution for other k_0 has already been given in Section D.2.

(a) Shorted Electrodes. For shorted electrodes V_0, given by equation (D.21), is zero. The propagation constant κ must be close to k_0, and since $k_0 \approx \pi/p$ we have $s = \kappa p/(2\pi) \approx \frac{1}{2}$. The Legendre functions vary slowly with s, and so it is a good approximation to set $s = \frac{1}{2}$ in these functions. Equation (D.21) thus gives

$$(\alpha_1 + \alpha_{-1})/\alpha_0 = P_{-1/2}(-\cos\Delta)/P_{1/2}(-\cos\Delta). \tag{D.35}$$

The left side of this equation is related to κ by equations (D.15), (D.17) and (D.18).

To simplify the equations, approximate forms of the functions A_0 and A_{-1} are used. Noting that κ, k_0 and k_m are all close to π/p, equations (D.17) and (D.18) give

$$A_0 \approx (2\kappa - k_0 - k_m)/(k_m - k_0)$$

and

$$A_{-1} \approx (4\pi/p - 2\kappa - k_0 - k_m)/(k_m - k_0). \tag{D.36}$$

The solution for κ is then found by using equations (D.15). It is convenient to define the functions

$$F_{\pm}(\Delta) = \mp\cos\Delta + P_{1/2}(\pm\cos\Delta)/P_{-1/2}(\pm\cos\Delta) \tag{D.37}$$

and the expression

$$\frac{\Delta v}{v} = (v_0 - v_m)/v_0 \approx p(k_m - k_0)/\pi. \tag{D.38}$$

It is then found that

$$(\kappa - \pi/p)^2 \approx (\omega - \omega_1)(\omega - \omega_{sc})/v_0^2, \tag{D.39}$$

where

$$\omega_1 = \frac{\pi v_0}{p}\left[1 + \frac{1}{2}\frac{\Delta v}{v}(\cos\Delta - 1)\right] \tag{D.40}$$

and ω_{sc} is given by

$$\omega_1 - \omega_{sc} = \frac{\pi v_0}{p}\frac{\Delta v}{v}F_-(\Delta). \tag{D.41}$$

The function $F_-(\Delta)$ is positive, so that $\omega_1 > \omega_{sc}$. Equation (D.39) thus shows that κ is complex for $\omega_{sc} < \omega < \omega_1$, and hence the frequencies ω_{sc} and ω_1 give the two edges of the stop band. These frequencies are shown in Figure D.2, as functions of metallisation ratio. The width of the stop band, $\omega_1 - \omega_{sc}$, is given by equation (D.41) and is shown in Figure D.3.

(b) Open-Circuited Electrodes. For the open-circuit case $I_0 = 0$, and with $s = \frac{1}{2}$ equation (D.23) gives

$$(\alpha_1 + \alpha_{-1})/\alpha_0 = -P_{-1/2}(\cos\Delta)/P_{1/2}(\cos\Delta). \tag{D.42}$$

Using equations (D.36) and (D.15), the solution for κ is

$$(\kappa - \pi/p)^2 \approx (\omega - \omega_1)(\omega - \omega_{oc})/v_0^2, \tag{D.43}$$

where ω_1 is given by equation (D.40) and ω_{oc} is given by

$$\omega_{oc} - \omega_1 = \frac{\pi v_0}{p}\frac{\Delta v}{v}F_+(\Delta) \tag{D.44}$$

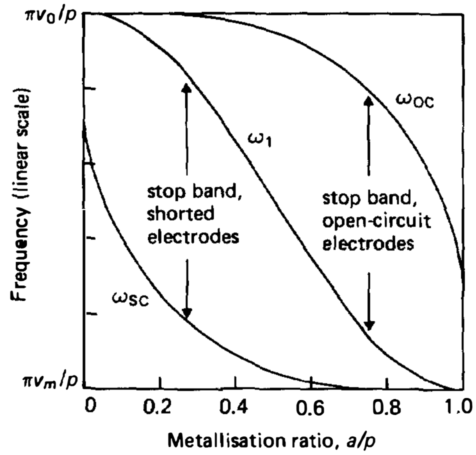

FIGURE D.2. Frequencies of the stop band edges.

with $F_+(\Delta)$ give by equation (D.37). Here $\omega_{oc} > \omega_1$, so κ is complex for frequencies in the range $\omega_1 < \omega < \omega_{oc}$. The frequency ω_{oc} is shown in Figure D.2, and the width of the stop band is shown in Figure D.3.

D.4. Solution at Higher Frequencies

The above sections give the solution for frequencies $\omega < 2\pi v_0/p$, that is, for $0 \leqslant k_0 < 2\pi/p$. For higher frequencies a similar approach can be used, with the field and charge density given by expressions similar to equations (D.11) and (D.12), but the range of values for r must be increased. Suppose, for example, that $E_x(x)$ and $\sigma(x)$ are given by equations (D.4) and (D.5), but the coefficients E_n and σ_n are now given by

$$E_n = \sum_{r=-R}^{R} \alpha_r S_{n-r} P_{n-r}(\cos \Delta) \tag{D.45}$$

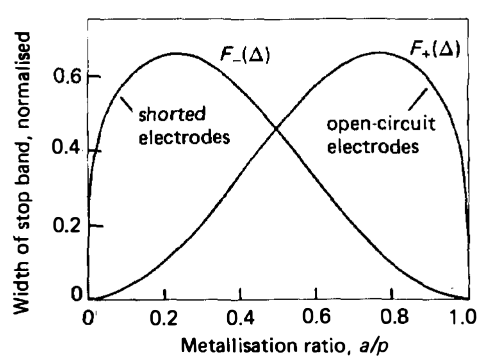

FIGURE D.3. Width of stop band, as a function of metallisation ratio.

and

$$\sigma_n = -j\varepsilon_s(\infty) \sum_{r=-R}^{R} \alpha_r P_{n-r}(\cos \Delta). \tag{D.46}$$

These equations give

$$\frac{\sigma_n}{E_n} = -jS_n\varepsilon_s(\infty), \quad \text{for } n \geqslant R, \quad \text{or} \quad \text{for } n \leqslant -R - 1.$$

We also have, from equation (D.7),

$$\frac{\sigma_n}{E_n} = -jS_n\varepsilon_s(\kappa + 2\pi n/p), \tag{D.47}$$

where $0 \leqslant \kappa \leqslant 2\pi/p$. Equations (D.45) and (D.46) therefore give a valid solution if $\varepsilon_s(\kappa + 2\pi n/p) = \varepsilon_s(\infty)$ for $n \geqslant R$ and for $n \leqslant -R - 1$. In the cases of interest, $\varepsilon_s(\beta)$ always approaches a limit for large β, and hence this condition can be satisfied by choosing a large enough value for R. The relative values of the coefficients α_r can then be found by substituting equations (D.45) and (D.46) into equation (D.47). It should be noted that, for $-R - 1 < n < R$, the method places no restrictions on the values of $\varepsilon_s(\kappa + 2\pi n/p)$, and is therefore very versatile.

The analysis using this method is given by Bløtekjaer *et al.* [475, 476]. In view of the complexity of the results, the discussion here is restricted to a consideration of the surface-wave velocities for either shorted or open-circuit electrodes, for frequencies outside the stop bands. For these cases, the analysis can be presented in a somewhat simpler form, similar to that of Datta and Hunsinger [479]. The effective permittivity $\varepsilon_s(\beta)$ is taken to be given by Ingebrigtsen's approximation, and it is a good approximation to assume that $\varepsilon_s(\kappa + 2\pi n/p)$ is equal to $\varepsilon_s(\infty)$ for all except one value of n. This of course excludes the stop bands. It is found that negative values of r are not required, so the coefficients E_n and σ_n can be written as

$$E_n = \sum_{r=0}^{R} \alpha_r S_{n-r} P_{n-r}(\cos \Delta),$$

$$\sigma_n = -j\varepsilon_s(\infty) \sum_{r=0}^{R} \alpha_r P_{n-r}(\cos \Delta). \tag{D.48}$$

These equations give $\sigma_n/E_n = -jS_n\varepsilon_s(\infty)$ for $n \geqslant R$ and for $n \leqslant -1$. It is assumed that $\varepsilon_s(\kappa + 2\pi n/p) = \varepsilon_s(\infty)$ for $n \neq R - 1$, which implies that

$$2\pi(R - 1)/p \leqslant k_0 \leqslant 2\pi R/p, \tag{D.49}$$

and that, from equation (D.47),

$$\frac{\sigma_n}{E_n} = -jS_n\varepsilon_s(\infty), \quad \text{for } n \neq R - 1. \tag{D.50}$$

Substituting equations (D.48) into equation (D.50), with $n = R - 2$, gives $\alpha_{R-1} = -\alpha_R P_{-2}(\cos \Delta)$. Using $n = R - 3$, $R - 4, \ldots$ gives the relative values of $\alpha_{R-2}, \alpha_{P-3}, \ldots$. These can be expressed in the form

$$\alpha_{R-m+1} = -\sum_{i=2}^{m} \alpha_{R-m+i} P_{-i}(\cos \Delta), \quad \text{for } 2 \leqslant m \leqslant R. \tag{D.51}$$

This determines the relative values of all the α_r, except for α_0. The additional relation required is obtained by setting $n = R - 1$, using equation (D.47) for σ_n/E_n. This gives

$$\sum_{i=0}^{R-1} \alpha_i P_{R-i-1}(\cos \Delta) = \alpha_R A_{R-1}, \tag{D.52}$$

where A_{R-1} is defined by equation (D.14), taking $r = R - 1$. Using Ingebrigtsen's approximation, equation (D.16), we have

$$A_{R-1} = (2k^2 - k_0^2 - k_m^2)/(k_m^2 - k_0^2), \tag{D.53}$$

where k has been defined by

$$k = \kappa + 2\pi(R - 1)/p. \tag{D.54}$$

When the electrodes are either shorted or open-circuited, it will be found that k is close to k_0, and can therefore be identified as the surface-wave wavenumber.

The electrode voltages and currents are calculated as in Section D.1, giving

$$V_0 = -\frac{jp}{2 \sin (\pi s)} \sum_{r=0}^{R} \alpha_r (-1)^r P_{-r-s}(-\cos \Delta) \tag{D.55}$$

and

$$I_0 = \omega W p(\varepsilon_0 + \varepsilon_p^T) \sum_{r=0}^{R} \alpha_r P_{-r-s}(\cos \Delta), \tag{D.56}$$

where, as before, $s = \kappa p/(2\pi)$ and $0 \leqslant s \leqslant 1$. For shorted electrodes V_0 is set to zero, and with equations (D.51)–(D.54) the wavenumber k is then determined. This wavenumber is denoted k_{sc}. For open-circuited electrodes I_0 is set to zero, and the solution for k is denoted k_{oc}.

As an example, we give the solutions for $R = 2$. Defining

$$B_{\pm}(\Delta) = \cos \Delta + \frac{[P_{s+1}(\pm \cos \Delta) \mp \cos (\Delta) P_s(\pm \cos \Delta)] \cos \Delta}{P_{-s}(\pm \cos \Delta)}$$

the solution for shorted electrodes is

$$k_{sc}^2 = k_0^2 + \tfrac{1}{2}(k_m^2 - k_0^2)[1 - B_{-}(\Delta)].$$

and for open-circuited electrodes

$$k_{oc}^2 = k_0^2 + \tfrac{1}{2}(k_m^2 - k_0^2)[1 - B_{+}(\Delta)].$$

The corresponding velocities are shown on Figure D.1.