Appendix C

Elemental Charge Density for Regular Electrodes

In Chapter 4 it was shown that the properties of transducers using regular electrodes can be conveniently expressed in terms of a function $\bar{\varrho}_f(\beta)$, defined as the Fourier transform of an elemental charge density $\varrho_f(x)$. The main purpose of this appendix is to demonstrate the validity of an analytic expression for $\bar{\varrho}_f(\beta)$. However, before doing this Section C.1 summarises some properties of Legendre functions that are needed in this appendix and elsewhere.

C.1. Some Properties of Legendre Functions

Properties of Legendre functions are given in many texts, in particular by Erdelyi [474], who gives all the properties quoted here.

The Legendre function with variable x and degree v is written as $P_v(x)$. Generally, x and v may be complex, but for analysis of surface-wave devices they are real, and x is in the range $-1 \leqslant x \leqslant 1$. The function may be evaluated using the expansion

$$P_v(x) = \sum_{m=0}^{\infty} a_m, \quad \text{for } |x| \leqslant 1, \tag{C.1}$$

where $a_0 = 1$ and

$$a_m = \frac{(m - 1 - v)(m + v)(1 - x)}{2m^2} a_{m-1}.$$

Some plots of $P_v(x)$, regarded as functions of v, are shown in Figure C.1. The recursion relation is

$$vP_v(x) = (2v - 1)xP_{v-1}(x) - (v - 1)P_{v-2}(x) \tag{C.2}$$

and the symmetry relation is

$$P_v(x) = P_{-v-1}(x) \tag{C.3}$$

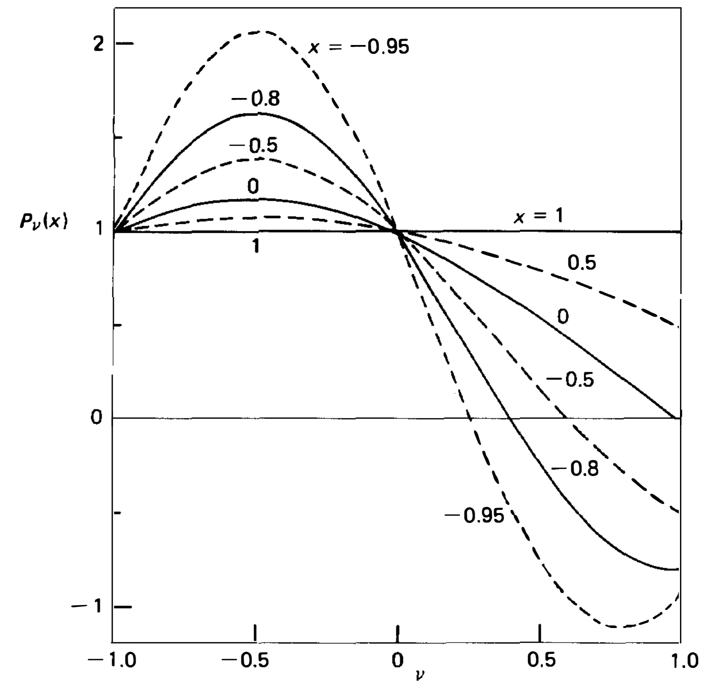

FIGURE C.1. Legendre functions $P_\nu(x)$.

so that, as a function of ν, $P_\nu(x)$ is symmetrical about $\nu = -\frac{1}{2}$. From the series of equation (C.1) it can be seen that $P_\nu(-1)$ is infinite if ν is not an integer. For $\nu = -\frac{1}{2}$, $P_\nu(x)$ is related to the elliptic integral by

$$P_{-\frac{1}{2}}(x) = 2K(m)/\pi, \qquad m = [(1 - x)/2]^{1/2}, \tag{C.4}$$

where $K(m)$ is the complete elliptic integral of the first kind. The Mehler–Dirichlet formula is

$$P_\nu(\cos \Delta) = \frac{1}{\pi\sqrt{2}} \int_{-\Delta}^{\Delta} \frac{\exp[j(\nu + \frac{1}{2})\phi]}{\sqrt{\cos \phi - \cos \Delta}} \, d\phi, \quad \text{for } 0 < \Delta < \pi. \tag{C.5}$$

For positive and negative values of x and ν the Legendre functions are related by

$$P_\nu(x)P_{-\nu}(-x) + P_\nu(-x)P_{-\nu}(x) = \frac{2 \sin \pi\nu}{\pi\nu}, \tag{C.6}$$

as noted by Bløtekjaer *et al.* [475]. This relation may be proved in two stages. First, using the differentiation formulae [474], it can be shown that the left side of the equation is independent of the variable x. Then the left side is evaluated for $x = 0$ by expressing $P_\nu(0)$ in terms of gamma functions [474] and making use of the properties of gamma functions.

For the particular case when ν is an integer, the series of equation (C.1) truncates, so that $P_\nu(x)$ becomes a polynomial. This is referred to as a Legendre Polynomial, and is often written $P_n(x)$. The relations given above are valid for $P_n(x)$, with ν replaced by the integer n. Thus

$$P_0(x) = 1, \tag{C.7}$$

$$P_1(x) = x, \tag{C.8}$$

and, for larger n, $P_n(x)$ may be obtained from the recursion relation, equation (C.2), giving

$$P_2(x) = (3x^2 - 1)/2, \tag{C.9}$$

$$P_3(x) = (5x^3 - 3x)/2, \tag{C.10}$$

and so on. Some Legendre polynomials are shown in Figure C.2. The polynomials are orthogonal over the interval $-1 < x < 1$, and have the symmetry

$$P_n(-x) = (-1)^n P_n(x). \tag{C.11}$$

The Legendre function $P_{-\nu}(\cos \Delta)$ is expressed as a sum of the polynomials by Dougall's expansion:

$$P_{-\nu}(\cos \Delta) = \frac{\sin(\pi\nu)}{\pi} \sum_{n=-\infty}^{\infty} \frac{S_n P_n(-\cos \Delta)}{\nu + n}, \tag{C.12}$$

where

$$S_n = 1, \quad \text{for } n \geqslant 0; \qquad S_n = -1, \quad \text{for } n < 0. \tag{C.13}$$

Also, for $0 < \Delta < \pi$,

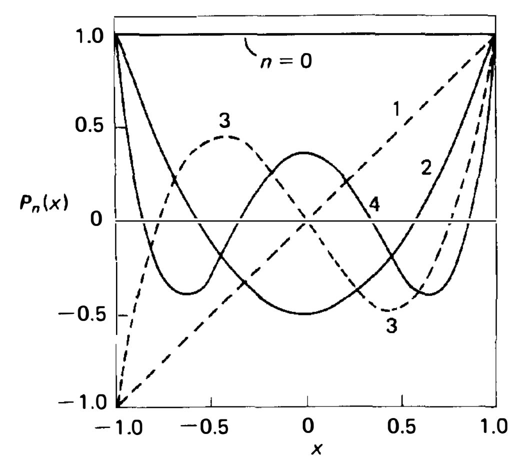

FIGURE C.2. Legendre polynomials $P_n(x)$.

$$\sum_{n=-\infty}^{\infty} P_n(\cos\Delta)\,e^{jn\theta} = \frac{(-1)^m\sqrt{2}\,e^{-j\theta/2}}{\sqrt{\cos\theta - \cos\Delta}}, \quad \text{for } |\theta - 2m\pi| < \Delta,$$

$$= 0, \quad \text{for } \Delta < |\theta - 2m\pi| \leqslant \pi. \quad \text{(C.14)}$$

This is valid for any value of θ; the integer m is chosen such that $|\theta - 2m\pi| \leqslant \pi$. Another series valid for $0 < \Delta < \pi$ is

$$\sum_{n=-\infty}^{\infty} S_n P_n(\cos\Delta)\,e^{jn\theta} = \frac{j(-1)^m\sqrt{2}\,\operatorname{sgn}(\theta - 2m\pi)\,e^{-j\theta/2}}{\sqrt{\cos\Delta - \cos\theta}},$$

$$\text{for } \Delta < |\theta - 2m\pi| < \pi,$$

$$= 0, \quad \text{for } |\theta - 2m\pi| < \Delta, \quad \text{(C.15)}$$

where S_n is given by equation (C.13) and the function $\operatorname{sgn}(x) = 1$ for $x > 0$ and $\operatorname{sgn}(x) = -1$ for $x < 0$. Relations similar to equations (C.14) and (C.15) are given by Bløtekjaer *et al.* [476].

Combining equation (C.14) with the Mehler–Dirichlet formula, equation (C.5), we have, for $0 < \Delta < \pi$,

$$\int_{-\Delta}^{\Delta} \sum_{n=-\infty}^{\infty} P_{n+m}(\cos\Delta)\,e^{-j(n+\nu)\theta}\,d\theta = 2\pi P_{m-\nu}(\cos\Delta). \quad \text{(C.16)}$$

In surface-wave device analysis, this equation is useful when calculating the current flowing into one electrode.

C.2. Elemental Charge Density

The elemental charge density $\varrho_f(x)$ is defined by considering an infinite array of regular electrodes, with width a and pitch p, as in Figure 4.10. The electrodes are deposited on the surface of an anisotropic non-piezoelectric half-space. One of the electrodes is centred at the origin $x = 0$ and has unit voltage applied, while all the other electrodes are grounded. In this situation the charge density on the electrodes is, by definition, the elemental charge density $\varrho_f(x)$. This section shows that the Fourier transform of $\varrho_f(x)$ is given by

$$\bar{\varrho}_f(\beta) = (\varepsilon_0 + \varepsilon_p)\frac{2\sin(\pi s)}{P_{-s}(-\cos\Delta)} P_n(\cos\Delta), \quad \text{for } n \leqslant \frac{\beta p}{2\pi} \leqslant n+1, \quad \text{(C.17)}$$

where $\Delta = \pi a/p$, $s = (\beta p)/(2\pi) - n$, and ε_p is a function of the permittivity components ε_{ij} of the half-space material, given by equation (3.4). The parameter s is in the range $0 \leqslant s \leqslant 1$.

To verify equation (C.17) it must be shown that the charge density and the associated potential satisfy Laplace's equation, in a form given by the electrostatic analysis of Section 3.1, and that the boundary conditions are satisfied. Thus the charge density must be zero in the gaps between the electrodes, and the potential on each electrode must be uniform and equal to the applied voltage.

In the x-domain, the charge density $\varrho_f(x)$ is given by the inverse Fourier transform of $\bar{\varrho}_f(\beta)$, so that

$$\varrho_f(x) = \int_{-\infty}^{\infty} \bar{\varrho}_f(\beta) \exp(j\beta x)\, \mathrm{d}\beta/(2\pi), \tag{C.18}$$

where $\bar{\varrho}_f(\beta)$ is given by equation (C.17). In view of the form of equation (C.17), this can be integrated from $\beta = 2\pi n/p$ to $\beta = 2\pi(n+1)/p$, and then summed with respect to the integer n. The summation has the form of equation (C.14). We define a normalised x-coordinate by

$$\theta = 2\pi x/p. \tag{C.19}$$

The result is

$$\begin{aligned} \varrho_f(x) &= \frac{\varepsilon_0 + \varepsilon_p}{p} \frac{2\sqrt{2}(-1)^m}{\sqrt{\cos\theta - \cos\Delta}} \Gamma(\theta, \Delta), \quad \text{for } |x - mp| < a/2, \\ &= 0, \qquad \text{for } a/2 < |x - mp| \leqslant p/2, \end{aligned} \tag{C.20}$$

where the function $\Gamma(\theta, \Delta)$ is given by

$$\Gamma(\theta, \Delta) = \int_0^1 \frac{\sin(\pi s)\cos[(s - \tfrac{1}{2})\theta]}{P_{-s}(-\cos\Delta)} \mathrm{d}s. \tag{C.21}$$

Thus $\varrho_f(x)$ satisfies the boundary condition in being zero in the gaps between the electrodes. The form of $\varrho_f(x)$ is shown, for $a/p = \frac{1}{2}$, in Figure 4.10.

The associated electric field at the surface has an x-component $E_x(x)$, with Fourier transform $\bar{E}_x(\beta)$. If the potential at the surface is $\phi(x)$, with transform $\bar{\phi}(\beta)$, we have $\bar{E}_x(\beta) = -j\beta\bar{\phi}(\beta)$. The electrostatic analysis in Section 3.1 shows that the charge density and the potential are related, in the β-domain, by equation (3.12), and it follows that

$$\bar{E}_x(\beta) = -j\,\mathrm{sgn}(\beta)\bar{\varrho}_f(\beta)/(\varepsilon_0 + \varepsilon_p). \tag{C.22}$$

Using equation (C.17) for $\bar{\varrho}_f(\beta)$ we have

$$\bar{E}_x(\beta) = -\frac{2j\sin(\pi s)}{P_{-s}(-\cos\Delta)} S_n P_n(\cos\Delta), \quad \text{for } n \leqslant \frac{\beta p}{2\pi} \leqslant n + 1, \tag{C.23}$$

where S_n is defined in equation (C.13). Transforming this to the x-domain involves a summation with the form of equation (C.15), and gives the result

$$\begin{aligned} E_x(x) &= \frac{2\sqrt{2}(-1)^m\,\mathrm{sgn}(\theta - 2m\pi)}{p\sqrt{\cos\Delta - \cos\theta}} \Gamma(\theta, \Delta), \quad \text{for } a/2 < |x - mp| \leqslant p/2, \\ &= 0, \qquad \text{for } |x - mp| < a/2, \end{aligned} \tag{C.24}$$

where $\Gamma(\theta, \Delta)$ is defined in equation (C.21). Thus $E_x(x)$ is zero on the electrodes, which are therefore equipotentials as required.

To show that the potential $\phi(x)$ is correct at the electrode locations, it is sufficient to evaluate it at the electrode centres $x = mp$. In the β-domain we have $\bar{\phi}(\beta) = j\bar{E}_x(\beta)/\beta$, with $\bar{E}_x(\beta)$ given by equation (C.23). Transforming to the x-domain, the integral involves a summation over n, and for $x = mp$ the summation has the form of Dougall's expansion, equation (C.12). We thus find

$$\begin{aligned} \phi(mp) &= 1, \quad \text{for } m = 0, \\ &= 0, \quad \text{for } m \neq 0, \end{aligned} \tag{C.25}$$

and hence the electrode potentials are correct. This completes the proof, since all the boundary conditions are satisfied, and Laplace's equation is satisfied in the form given by equation (C.22).

C.3. Net Charges on Electrodes

The total charge per unit length on the electrode centred at $x = mp$ is denoted Q_m, so that

$$Q_m = \int_{mp-a/2}^{mp+a/2} \varrho_f(x)\,\mathrm{d}x. \tag{C.26}$$

These quantities are useful when calculating the capacitance of a transducer. On substituting for $\varrho_f(x)$ from equation (C.20), it is found that the x-integral has the form of the Mehler–Dirichlet formula, equation (C.5), and consequently Q_m can be expressed as

$$Q_m = 2(\varepsilon_0 + \varepsilon_p) \int_0^1 \frac{\sin(\pi s)\cos(2\pi m s)}{P_{-s}(-\cos\Delta)} P_{-s}(\cos\Delta)\,\mathrm{d}s. \tag{C.27}$$

If the metallisation ratio $a/p = \frac{1}{2}$ we have $\cos\Delta = 0$, so that the Legendre functions cancel. The integral is then straightforward, and gives

$$Q_m = \frac{4(\varepsilon_0 + \varepsilon_p)}{\pi(1 - 4m^2)}, \quad \text{for } \cos\Delta = 0. \tag{C.28}$$

When evaluating the capacitance of uniform transducers, a summation of the Q_m is required. This summation can be obtained from the formula

$$N \sum_{m=-\infty}^{\infty} \cos[2\pi x(mN + n)] = \cos(2\pi n x) \sum_{i=-\infty}^{\infty} \delta(x - i/N), \tag{C.29}$$

where n and N are integers, and $N \neq 0$. This follows from equation (A.43) of Appendix A, or by Fourier transformation of both sides. Using equation (C.27) for Q_m it is found that, for $N \geqslant 1$,

$$\sum_{m=-\infty}^{\infty} Q_{mN+n} = \frac{2(\varepsilon_0 + \varepsilon_p)}{N} \sum_{i=0}^{N} \frac{\sin(\pi\nu)\cos(2\pi n\nu)}{P_{-\nu}(-\cos\Delta)} P_{-\nu}(\cos\Delta) \tag{C.30}$$

with $\nu = i/N$.

charge density must be zero in the gaps between the electrodes, and the potential on each electrode must be uniform and equal to the applied voltage.

Another summation formula is needed for analysis of multi-strip couplers, and is again obtained by using equations (A.43) and (C.27). The formula is

$$\sum_{m=-\infty}^{\infty} Q_m \, e^{-jkmp} = 2(\varepsilon_0 + \varepsilon_p) \frac{\sin(\pi\mu)}{P_{-\mu}(-\cos\Delta)} P_{-\mu}(\cos\Delta), \tag{C.31}$$

where μ is defined such that

$$kp = 2\pi(i + \mu)$$

and i is the integer part of $kp/(2\pi)$, so that $0 \leqslant \mu \leqslant 1$.