Appendix B

Reciprocity

The theory of elastic pieozoelectric materials can be used to derive a general reciprocity relation, with a wide range of applications. This appendix discusses the application of the reciprocity relation to transducers consisting of electrodes on the surface of a piezoelectric solid. The main consequence is that the processes of launching and receiving acoustic waves are mathematically related. Thus, if the launching process can be analysed for a particular transducer, reciprocity enables the receiving process to be deduced directly, with little further analysis. We first consider a generalised geometry, and then later discuss surface wave transducers on a half-space.

Throughout this appendix it is assumed that the surface is force free, so that the electrodes cause no mechanical loading. All appropriate quantities are taken to be proportional to exp $(j\omega t)$, with the frequency ω considered constant.

B.1. General Relation for a Mechanically Free Surface

We consider a homogeneous insulating piezoelectric material, whose behaviour is governed by the piezoelectric equations and Newton's laws. Any disturbance in the material gives rise to a displacement $\mathbf{u}$, stress $\mathbf{T}$, potential Φ and electric displacement $\mathbf{D}$, all of which are functions of the coordinates and of time. It is assumed that these functions describe a solution satisfying the constitutive relations of the material, and they will all be assumed to be proportional to exp $(j\omega t)$. A second solution is also assumed, described by the functions $\mathbf{u}'$, $\mathbf{T}'$, Φ' and $\mathbf{D}'$, and these are also proportional to exp $(j\omega t)$. These two solutions are related at each point by an equation called the *real reciprocity relation*, which is written

$$\operatorname{div}\,[\{\mathbf{u}\,.\,\mathbf{T}'\} - \{\mathbf{u}'\,.\,\mathbf{T}\} + \Phi\mathbf{D}' - \Phi'\mathbf{D}] = 0, \tag{B.1}$$

where the vector $\{\mathbf{u}\,.\,\mathbf{T}'\}$ is defined such that its x_j component is given by

$$\{\mathbf{u}\,.\,\mathbf{T}'\}_j = \sum_{i=1}^{3} u_i T'_{ij}, \quad j = 1, 2, 3.$$

Equation (B.1) is valid provided there are no mechanical or electrical sources within the material, and in particular this means that there must be no free charges. The derivation, given by Auld [32, p. 153], uses the piezoelectric constitutive equations [equations (2.9) and (2.10) of Chapter 2] with Maxwell's equations and Newton's laws. The form given above assumes the electric field to be quasi-static, so that $\mathbf{E} = -\operatorname{grad}\Phi$. The equation is valid even if the material is lossy. Other related reciprocity relations are the complex reciprocity relation given by Auld [32, 472] and the relation give by Lewis [473].

The solid is assumed to be enclosed by a mechanically free surface S, with a vacuum in the space outside this surface. The integral of equation (B.1) over the volume of the solid is related to an integral over the surface S by the divergence theorem

$$\int_V \operatorname{div} \mathbf{A}\, \mathrm{d}V = \int_S \mathbf{A}\,.\,\mathbf{n}\,\mathrm{d}S,$$

where $\mathbf{n}$ is the outward-directed vector normal to the surface, with magnitude unity. For a mechanically free surface, with no forces, it may be shown [32] that $\{\mathbf{u}\,.\,\mathbf{T}'\}\,.\,\mathbf{n} = \{\mathbf{u}'\,.\,\mathbf{T}\}\,.\,\mathbf{n} = 0$. Thus, integration of equation (B.1) over the volume of the solid yields

$$\int_S (\Phi\mathbf{D}' - \Phi'\mathbf{D})\,.\,\mathbf{n}\,\mathrm{d}S = 0. \tag{B.2}$$

Note that no electrical boundary condition has been applied at this stage.

The normal component of displacement at the surface ($\mathbf{D}\,.\,\mathbf{n}$ or $\mathbf{D}'\,.\,\mathbf{n}$) can be related to the charge density on a set of electrodes on the surface. The free charges must however be outside the surface S, because the reciprocity relation, equation (B.1), is valid only if there are no sources. To comply with this, the electrodes are assumed to be separated from the surface by an infinitesimal gap. Thus $\mathbf{D}\,.\,\mathbf{n}$, the normal component of $\mathbf{D}$, is equal to $-\sigma$, where σ is the charge density on the adjacent electrode. If σ' is the charge density corresponding to the displacement $\mathbf{D}'$, equation (B.2) becomes

$$\int_S (\Phi\sigma' - \Phi'\sigma)\,\mathrm{d}S = 0. \tag{B.3}$$

In this equation σ and σ' are the charge densities on the sides of the electrodes adjacent to the piezoelectric. There will also be charges on the vacuum side. The reciprocity argument can be applied to the vacuum as well as the piezoelectric, so that equation (B.3) is valid if σ' and σ are the charge densities on the vacuum side of the electrodes. It follows that the equation is also valid if σ and σ' are the total charge densities, including both the piezoelectric and vacuum sides. In the following description σ and σ' are taken to include the charges on both sides.

B.2. Reciprocity For Two-terminal Transducers

For the type of transducer considered here, each electrode is connected to one of two terminals. Transducers with more than two terminals may be analysed by using the results for two-terminal transducers, in conjunction with the superposition principle.

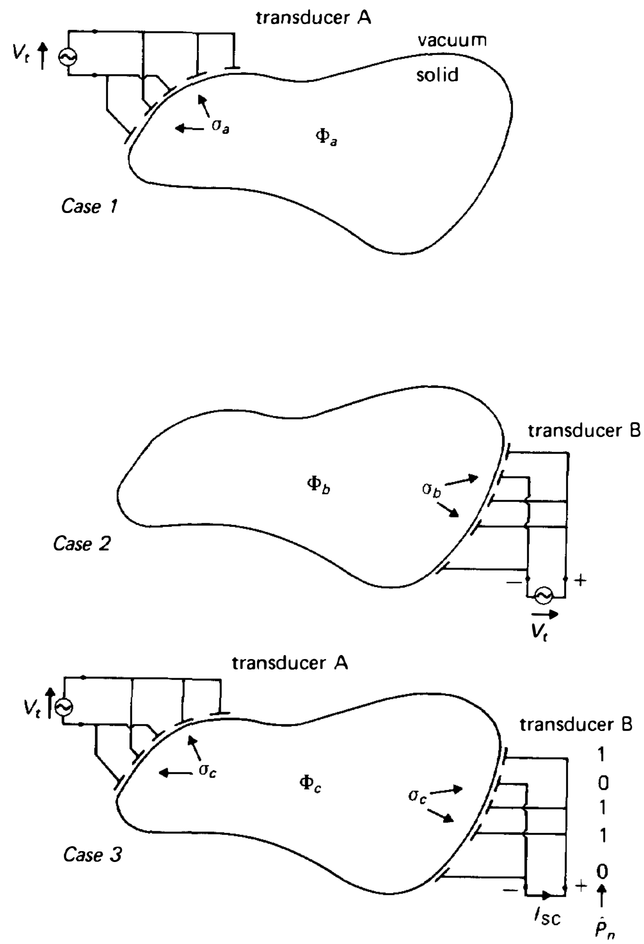

FIGURE B.1. Two-terminal transducers: reciprocity.

It is therefore sufficient to consider two-terminal transducers. The resistivity of the electrodes is assumed to be negligible.

We consider a piezoelectric solid with three different transducer configurations, as shown in Figure B.1. In case 1, transducer A is placed on the surface and a voltage V_t is applied across it. The voltage is proportional to exp $(j\omega t)$, with this factor implicit. A charge density σ_a appears on the electrodes, and the potential, specified everywhere, is Φ_a. The transducer may radiate a variety of types of acoustic waves.

In case 2, transducer B is placed on the surface and a voltage V_t is applied, with transducer A absent. A charge density σ_b appears on the electrodes, and the potential is Φ_b. In case 3 both transducers are present. A voltage V_t is applied to transducer A, while transducer B is shorted and produces a current I_{sc}. The potential is Φ_c and the charge density is σ_c, which includes charges on both transducers.

It is assumed that the launching process can be analysed, so that the functions Φ_a, σ_a, Φ_b and σ_b can be found. Reciprocity enables the current produced by the receiving transducer, I_{sc}, to be expressed in terms of these functions. Using reciprocity in the form of equation (B.3) we have

$$\int_S \Phi_c \sigma_b \, \mathrm{d}S = \int_S \Phi_b \sigma_c \, \mathrm{d}S \tag{B.4}$$

and

$$\int_S \Phi_a \sigma_b \, \mathrm{d}S = \int_S \Phi_b \sigma_a \, \mathrm{d}S, \tag{B.5}$$

where it is assumed that any fields generated by the leads connecting the electrodes can be ignored. Now, in case 3, Φ_c has the same value at all points on the electrodes of transducer B, because this transducer is shorted and its electrodes are assumed to have zero resistivity. In case 2, σ_b is zero everywhere except on the electrodes of transducer B. We therefore have

$$\int_S \Phi_c \sigma_b \mathrm{d}S = \text{const.} \int_S \sigma_b \mathrm{d}S = 0,$$

since the total of all the charges on transducer B must be zero. Equation (B.4) therefore gives

$$\int_S \Phi_b \sigma_c \, \mathrm{d}S = 0. \tag{B.6}$$

It is now assumed that, for case 3, the presence of transducer B does not affect the charge density on transducer A so that, on transducer A, $\sigma_c = \sigma_a$. In most practical cases this assumption is amply justified. The coupling between the two transducers can be regarded as of two types, electrostatic and acoustic. Electrostatic coupling causes a "breakthrough" signal to appear at the terminals of the receiving transducer. This signal is usually small by design; thus any perturbation of the charge density on the *launching* transducer due to electrostatic effects associated with the presence of the receiving transducer can be confidently ignored. Acoustic coupling can occur because transducer B can generate acoustic waves when the waves launched by transducer A are incident on it. However, in most of the cases that we are concerned with, this regeneration is small when the receiving transducer is shorted, as it is here. In any case, the effect of the regeneration is usually to produce multiple-transit signals which are easily identified because their delays differ from that of the main signal.

Thus, assuming that the charge density on transducer A is not affected by the presence of transducer B, we have $\sigma_c = \sigma_a$ on transducer A. Define σ_{cb} as the part of σ_c on the electrodes of transducer B, so that $\sigma_{cb} = 0$ except on transducer B. Then $\sigma_c = \sigma_a + \sigma_{cb}$, and equation (B.6) gives

$$\int_S \Phi_b \sigma_{cb} \, \mathrm{d}S = -\int_S \Phi_b \sigma_a \, \mathrm{d}S. \tag{B.7}$$

To find the current I_{sc} produced by transducer B the electrodes are labelled $n = 1, 2, 3, \ldots$, with areas S_n. The polarity of electrode n is designated $\hat{P}_n$, with a value 1 if it is connected to the positive terminal and 0 if it is connected to the negative terminal, as in Figure B.1. When a voltage V_t is applied to the transducer, as in case 2, the potential of electrode n is $(\hat{P}_n - \frac{1}{2})V_t$. In case 3, the total charge on electrodes connected to the positive terminal is

$$Q_+ = \sum_n \hat{P}_n \int_{S_n} \sigma_{cb} \, \mathrm{d}S,$$

and the total charge on electrodes connected to the negative terminal is

$$Q_- = \sum_n (1 - \hat{P}_n) \int_{S_n} \sigma_{cb} \, \mathrm{d}S.$$

The current $I_{sc} = j\omega Q_+$, and since the total charge must be zero we have $Q_+ = (Q_+ - Q_-)/2$, so that I_{sc} can be written

$$I_{sc} = j\omega \sum_n \int_{S_n} (\hat{P}_n - \tfrac{1}{2})\sigma_{cb} \, \mathrm{d}S. \tag{B.8}$$

Now, for case 2 we have $\Phi_b = (\hat{P}_n - \frac{1}{2})V_t$ on electrode n of transducer B, since the resistivity is assumed to be zero. Also, σ_{cb} is zero except on these electrodes so, in equation (B.8), $(\hat{P}_n - \frac{1}{2})$ may be replaced by Φ_b/V_t. We thus have

$$I_{sc} = \frac{j\omega}{V_t} \int_S \Phi_b \sigma_{cb} \, \mathrm{d}S.$$

Finally, using equations (B.7) and (B.5) we have

$$V_t I_{sc} = -j\omega \int_S \Phi_b \sigma_a \, \mathrm{d}S = -j\omega \int_S \Phi_a \sigma_b \, \mathrm{d}S. \tag{B.9}$$

This equation is the required result, giving the output of the receiving transducer, I_{sc}, in terms of functions obtained by analysis of launching transducers. Analysis of the launching process also gives the impedance of transducer B, so the output voltage and current produced by this transducer can be calculated for any load impedance.

A convenient modification of equation (B.9) is obtained by introducing the functions $\varrho_a = \sigma_a/V_t$ and $\varrho_b = \sigma_b/V_t$. Thus ϱ_a may be defined as the charge density on transducer A when unit voltage is applied, with no other electrodes present on the surface. Equation (B.9) then becomes

$$I_{sc} = -j\omega \int_S \Phi_b \varrho_a \mathrm{d}S = -j\omega \int_S \Phi_a \varrho_b \mathrm{d}S. \tag{B.10}$$

B.3. Symmetry of the Green's Function

If the charge density σ is specified at all points on the surface S enclosing the solid, this determines the potential Φ everywhere, apart from a constant term which is ignored here. Since the relationship is linear it may be expressed in terms of a Green's function $G_1(\mathbf{r}; \mathbf{r}')$, defined as the potential at $\mathbf{r}$ due to the charge density at $\mathbf{r}'$, so that

$$\Phi(\mathbf{r}) = \int_S G_1(\mathbf{r}; \mathbf{r}')\sigma(\mathbf{r}') \, \mathrm{d}S'. \tag{B.11}$$

The subscript is used to distinguish this from the Green's function $G(x, \omega)$ for a half-space (Chapter 3). Using an argument given by Auld [32, p. 366], we show that the Green's function $G_1(\mathbf{r}; \mathbf{r}')$ is symmetrical. The equations involve values of the potential $\Phi(\mathbf{r})$ only at the surface, so the Green's function is specified with both $\mathbf{r}$ and $\mathbf{r}'$ on the surface.

We consider two solutions, one with potential $\Phi_1(\mathbf{r})$ and charge density $\sigma_1(\mathbf{r})$, and the other with potential $\Phi_2(\mathbf{r})$ and charge density $\sigma_2(\mathbf{r})$. Using reciprocity in the form of equation (B.3) we have

$$\int_S \Phi_1(\mathbf{r})\sigma_2(\mathbf{r})\,\mathrm{d}S = \int_S \Phi_2(\mathbf{r})\,\sigma_1(\mathbf{r})\,\mathrm{d}S.$$

Using equation (B.11) for the two potentials gives

$$\iint_S \sigma_2(\mathbf{r})\,G_1(\mathbf{r};\,\mathbf{r}')\,\sigma_1(\mathbf{r}')\,\mathrm{d}S'\,\mathrm{d}S = \iint_S \sigma_1(\mathbf{r})\,G_1(\mathbf{r};\,\mathbf{r}')\,\sigma_2(\mathbf{r}')\,\mathrm{d}S'\,\mathrm{d}S.$$

If we interchange $\mathbf{r}$ and $\mathbf{r}'$ in the integral on the right, it becomes the same as the integral on the left except that $G_1(\mathbf{r};\,\mathbf{r}')$ is replaced by $G_1(\mathbf{r}';\,\mathbf{r})$. Since the integrals must be equal for any choice of the charge densities $\sigma_1(\mathbf{r})$ and $\sigma_2(\mathbf{r})$, we conclude that the two forms of the Green's function are equal, that is,

$$G_1(\mathbf{r}';\,\mathbf{r}) = G_1(\mathbf{r};\,\mathbf{r}'). \tag{B.12}$$

The Green's function is therefore symmetrical.

B.4. Reciprocity for Surface Excitation of a Half-Space

We now consider a half-space with its plane force-free surface normal to the z-axis, and assume that there are no variations in the y-direction, so that the potential and charge density are functions of x only. The potential at the surface is denoted $\phi(x)$ and the charge density, which exists only at the surface, is $\sigma(x)$. Equation (B.11) thus becomes

$$\phi(x) = \int_{-\infty}^{\infty} G_1(x;\,x')\sigma(x')\,\mathrm{d}x'. \tag{B.13}$$

For a half-space the relation between $\phi(x)$ and $\sigma(x)$ must be unchanged if the origin for the x-axis is displaced, and hence $G_1(x;\,x')$ depends only on the distance between x and x'. This is expressed by defining a new Green's function $G(x)$, such that

$$G_1(x;\,x') = G(x - x'), \tag{B.14}$$

and hence

$$\phi(x) = \int_{-\infty}^{\infty} G(x - x')\sigma(x')\,\mathrm{d}x' = G(x) * \sigma(x),$$

so that $G(x)$ is the same as the Green's function introduced in Chapter 3, Section 3.4. Comparing equations (B.12) and (B.14) shows that $G(x)$ is symmetrical:

$$G(-x) = G(x). \tag{B.15}$$

It follows that the Fourier transform of $G(x)$, denoted $\bar{G}(\beta)$, is also symmetrical, so that $\bar{G}(-\beta) = \bar{G}(\beta)$. This gives the symmetry of the effective permittivity $\varepsilon_s(\beta)$. The reciprocal of the permittivity is equal to $|\beta|\bar{G}(\beta)$, from equation (3.41), and hence

$$\varepsilon_s(-\beta) = \varepsilon_s(\beta). \tag{B.16}$$

B.5. Reciprocity for Surface Wave Transducers

In general, a transducer on the surface of a half-space may generate surface waves and bulk waves, and will also generate a potential due to electrostatic effects. For such

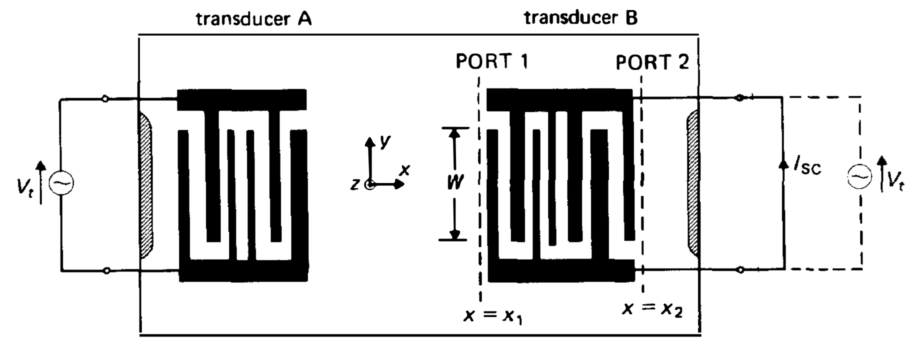

FIGURE B.2. General two-terminal transducers on a plane surface.

cases reciprocity may be used in the form of equation (B.10). However, for devices in which the coupling between transducers is predominantly due to surface waves some useful additional formulae can be derived.

We first consider reception of surface waves by a two-terminal transducer, as shown in Figure B.2. This represents the same situation as case 3 of Figure B.1, but with the transducers on the plane surface of a half-space. The aperture W is assumed to be very large, so that variations in the y-direction can be ignored. From equation (B.10), the short-circuit current produced by transducer B is

$$I_{sc} = -j\omega W \int_{-\infty}^{\infty} \phi_a(x)\varrho_b(x)\,\mathrm{d}x. \tag{B.17}$$

Here $\phi_a(x)$ is the potential produced at the surface by transducer A, when a voltage V_t is applied to it, assuming transducer B to be absent. The function $\varrho_b(x)$ is the charge density on transducer B for unit applied voltage, with transducer A absent. Generally, $\phi_a(x)$ and $\varrho_b(x)$ will also be functions of the frequency, ω.

In general, $\phi_a(x)$ includes bulk wave and electrostatic terms in addition to a surface wave term. However, the bulk wave and electrostatic terms decay with distance. We assume that transducer A is generating surface waves, and assume that the transducer separation is large, so that in the region occupied by transducer B the bulk wave and electrostatic terms are negligible in comparison with the surface wave term. Thus, in the region of transducer B, $\phi_a(x)$ has the form

$$\phi_a(x) = \phi_0 \exp(-jk_0 x), \tag{B.18}$$

where k_0 is the free-surface wavenumber for surface waves at frequency ω, and ϕ_0 is a constant. This is substituted into equation (B.17). If $\bar{\varrho}_b(\beta)$ is the Fourier transform of $\varrho_b(x)$, this gives

$$I_{sc} = -j\omega W \phi_0 \bar{\varrho}_b(k_0). \tag{B.19}$$

It is convenient to express this in terms of the surface wave amplitude at the input port, port 1, of the transducer. The input port is defined by a line at $x = x_1$ near the left edge of the transducer, as on Figure B.2. The exact location of this line is immaterial. The incident surface wave potential is given by equation (B.18), and we define ϕ_{i1} as the value of this potential at $x = x_1$, the input port of transducer B. Thus

$$\phi_{i1} = \phi_0 \exp(-jk_0 x_1)$$

and the output current is given by

$$I_{sc} = -j\omega W \phi_{i1} \bar{\varrho}_b(k_0) \exp(jk_0 x_1). \tag{B.20}$$

Note that this is valid even if the transducer couples to bulk waves; it is only necessary to assume that the incident wave has no bulk wave or electrostatic terms.

The same method may be used to deduce the current produced by transducer B when surface waves are incident from the right instead of from the left. In this case equation (B.17) is still valid, but transducer A must be taken to be on the right of transducer B. The input port of transducer B is now port 2, taken to be at $x = x_2$, and the potential of the incident surface wave at this point is denoted ϕ_{i2}. It is found that the current produced by transducer B is

$$I_{sc} = -j\omega W \phi_{i2} \bar{\varrho}_b(-k_0) \exp(-jk_0 x_2). \tag{B.21}$$

Relation Between Launching and Reception. The amplitudes of the surface waves generated by an isolated transducer are derived in section B.6 below, assuming the surface waves to be of the piezoelectric Rayleigh wave type. If $\sigma(x)$ is the charge density on the transducer, with Fourier transform $\bar{\sigma}(\beta)$, the potential of the surface wave radiated in the $-x$ direction is

$$\phi_s(x) = j\Gamma_s \bar{\sigma}(k_0) \exp(jk_0 x), \tag{B.22}$$

where Γ_s is defined in Section B.6. Now suppose that a voltage V_t is applied to the two-terminal transducer B of Figure B.2, with transducer A absent. For unit applied voltage the charge density is $\varrho_b(x)$, with Fourier transform $\bar{\varrho}_b(\beta)$, so that $\bar{\sigma}(k_0) = V_t \bar{\varrho}_b(k_0)$. The transducer generates a surface wave with potential $\phi_s(x)$, and we define ϕ_{s1} as the potential at port 1, so that $\phi_{s1} = \phi_s(x_1)$. We thus have

$$\phi_{s1} = j\Gamma_s V_t \bar{\varrho}_b(k_0) \exp(jk_0 x_1) \tag{B.23}$$

for the wave launched at ... lent on port 1 the current produced, when the trans... uation (B.20). Comparing with equation (B.23) we h...

$$\left\lfloor \overline{\phi_{i1}} \right\rfloor_{\text{receive}} \quad - \quad \Gamma_s \left\lfloor V_t \right\rfloor_{\text{launch}} \tag{B.24}$$

This equation relates the reception and launching of surface waves at port 1. The reader is reminded that the derivation assumes that there is no mechanical loading, and that the electrodes have zero resistivity. The equation is valid if the transducer couples to bulk waves as well as surface waves; if so, ϕ_{s1} is the surface-wave component of the potential at port 1 for the launching process, while for the receiving process an incident surface wave is assumed, with no bulk wave component.

For port 2 it is readily shown that the same relation, equation (B.24), applies with

ϕ_{i1} and ϕ_{s1} replaced by ϕ_{i2} and ϕ_{s2}, the potentials of incident and launched surface waves at port 2.

B.6. Surface Wave Generation

This section derives the formula for the amplitude of surface waves generated by a transducer, which has already been given in equation (B.22). The formula follows from the nature of the effective permittivity $\varepsilon_s(\beta)$, and therefore assumes that there is no mechanical loading. We also assume that the surface wave is of the piezoelectric Rayleigh wave type, so that $\varepsilon_s(\beta)$ has a zero for $\beta = k_0$, the free-surface wavenumber, and $\varepsilon_s(\beta)$ is real for β close to k_0.

We consider a set of electrodes on the surface, connected to some electrical network that includes at least one current or voltage source, as in Figure B.3. The electrode edges are parallel to the y-axis, and it is assumed that there are no variations in the y-direction. The surface potential and charge density are $\phi(x)$ and $\sigma(x)$, with Fourier transforms $\bar{\phi}(\beta)$ and $\bar{\sigma}(\beta)$, and by definition these are related by

$$\bar{\phi}(\beta) = \frac{\bar{\sigma}(\beta)}{|\beta|\varepsilon_s(\beta)} \tag{B.25}$$

as in Section 3.2, equation (3.24).

In general the potential $\phi(x)$ includes contributions due to surface waves, bulk waves and electrostatic effects. However, the surface wave contribution may be identified by noting that this is the only term which does not decay in amplitude for large positive or negative values of x, remote from the transducer electrodes. Since the surface is unmetallised outside the transducer, the potential for large positive or negative x must have the form $\exp(-jk_0|x|)$. The Fourier transform of such a function must be infinite at $\beta = \pm k_0$, and hence the right side of equation (B.25) must be infinite at these points. Now, the charge density $\sigma(x)$ is localised in a finite region of x, occupied by the transducer electrodes, and it follows that its

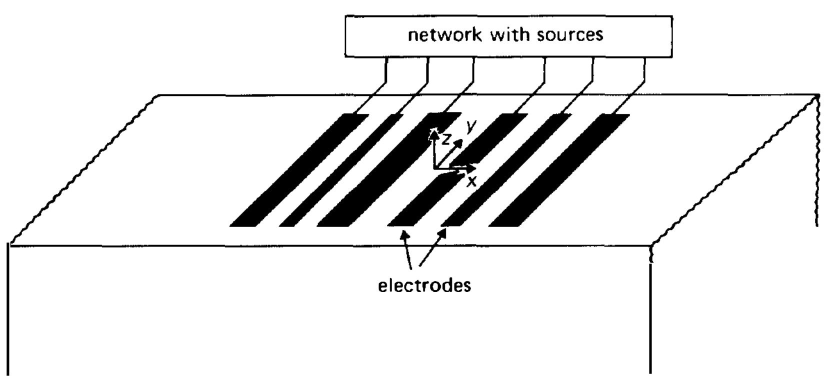

FIGURE B.3. General transducer on a half-space.

transform $\bar{\sigma}(\beta)$ cannot be infinite for any β. The poles of equation (B.25) therefore arise from the zeros of $\varepsilon_s(\beta)$ at $\beta = \pm k_0$.

To evaluate the surface wave potential, it is assumed that the total potential $\phi(x)$ can be written as a sum of two surface wave terms, representing waves of constant amplitude radiated away from the transducer, plus some potential $\phi_l(x)$ which is localised, so that it vanishes for $x \to \pm \infty$. The exact forms assumed for the surface wave terms are not consequential, though they must be defined such that they are harmonic at points remote from the transducer, and they must represent waves propagating away from the transducer. Assuming that the origin of the x-axis is within the transducer, a suitable form for the total potential is

$$\phi(x) = \phi_- U(-x)\, e^{jk_0x} + \phi_+ U(x)\, e^{-jk_0x} + \phi_l(x), \qquad \text{(B.26)}$$

where the constants ϕ_- and ϕ_+ are the amplitudes of the two surface wave potentials. $U(x)$ is the step function, equal to unity for $x > 0$ and zero for $x < 0$. From Appendix A, equation (A.40), the transform of the first term is given by

$$U(-x) \exp(jk_0x) \leftrightarrow \pi\delta(\beta - k_0) + j/(\beta - k_0).$$

This is infinite at $\beta = k_0$. The second term has a similar transform [equation (A.39)], and is infinite at $\beta = -k_0$. Using these and equation (B.25), the charge density has a Fourier transform

$$\begin{aligned}\bar{\sigma}(\beta) &= |\beta|\varepsilon_s(\beta)\bar{\phi}(\beta)\\ &= |\beta|\varepsilon_s(\beta)[\phi_-\pi\delta(\beta - k_0) + j\phi_-/(\beta - k_0)\\ &\qquad + \phi_+\pi\delta(\beta + k_0) - j\phi_+/(\beta + k_0) + \bar{\phi}_l(\beta)],\end{aligned}$$

where $\bar{\phi}_l(\beta)$ is the transform of $\phi_l(x)$.

We now evaluate this function at $\beta = k_0$, noting that at this point $\varepsilon_s(\beta) = 0$. In the square bracket, the two ϕ_+ terms and the term $\bar{\phi}_l(\beta)$ can be omitted because at $\beta = k_0$ they are finite or zero. In addition $\varepsilon_s(\beta)\,\delta(\beta - k_0)$ is zero for all β. Thus for β close to k_0 we only need to consider the remaining term

$$\bar{\sigma}(\beta) = j\phi_-|\beta|\varepsilon_s(\beta)/(\beta - k_0) \qquad \text{(B.27)}$$

and $\varepsilon_s(\beta)$ can be replaced by the first term of its Taylor expansion:

$$\varepsilon_s(\beta) = -(\beta - k_0)/(k_0\Gamma_s), \qquad \text{(B.28)}$$

where the constant Γ_s is defined by

$$\frac{1}{\Gamma_s} = -k_0\left[\frac{d\varepsilon_s(\beta)}{d\beta}\right]_{k_0},$$

as in Section 3.3. Substituting equation (B.28) into equation (B.27) and evaluating at $\beta = k_0$ gives

$$\phi_- = j\Gamma_s\bar{\sigma}(k_0).$$

Similarly, by evaluating $\bar{\sigma}(\beta)$ at $\beta = -k_0$ we find that $\phi_+ = j\Gamma_s\bar{\sigma}(-k_0)$. Thus, for locations remote from the transducer, such that the localised potential $\phi_l(x)$ in equation (B.26) is negligible, the total potential is given by

$$\begin{aligned}\phi(x) &= j\Gamma_s\bar{\sigma}(k_0)\exp(jk_0x), && \text{for } x \ll 0,\\ &= j\Gamma_s\bar{\sigma}(-k_0)\exp(-jk_0x), && \text{for } x \gg 0,\end{aligned} \tag{B.29}$$

which represents surface waves travelling away from the transducer.

An alternative interpretation is to define a surface wave potential $\phi_s(x)$ for all values of x on the free surface outside the transducer. We can thus write

$$\phi_s(x) = j\Gamma_s\bar{\sigma}(\mp k_0)\exp(\mp jk_0x) \tag{B.30}$$

taking the upper signs for $x > 0$ and the lower signs for $x < 0$. At points remote from the transducer this is equal to the total potential $\phi(x)$. For points close to the transducer the surface wave potential can be taken to be given by equation (B.30), though the total potential $\phi(x)$ also includes the term $\phi_l(x)$ which is due to electrostatic effects and, possibly, bulk wave excitation.

An alternative proof is given by Milsom *et al.* [93].