Appendix A

Fourier Transforms and Linear Filters

The theory of surface-wave devices makes extensive use of Fourier transforms and the theory of linear filters. This appendix summarises the results needed in this context. Many of the results are quoted without proof, and for further details the reader is referred to the many texts available, some of which are listed as References [466 – 469].

A.1. Fourier Transforms

The Fourier transform of a function $g(t)$ is denoted $G(\omega)$, while the inverse transform of $G(\omega)$ is $g(t)$. The relation is denoted symbolically by

$$G(\omega) \leftrightarrow g(t). \tag{A.1}$$

The transforms are defined by the integrals

$$G(\omega) = \int_{-\infty}^{\infty} g(t)\, \mathrm{e}^{-j\omega t}\, \mathrm{d}t \quad \text{(forward transform)}, \tag{A.2}$$

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega)\, \mathrm{e}^{j\omega t}\, \mathrm{d}\omega \quad \text{(inverse transform)} \tag{A.3}$$

In this appendix, the variables are written as t and ω throughout. The variable t can be taken to refer to time, in which case ω refers to radian frequency. The variables can of course have other meanings; in particular, we sometimes use the spatial coordinate x instead of t and "wavenumber" β instead of ω. In the equations to follow, the use of t and ω specifies whether a forward or inverse transform is involved. As a standard notation, we write time-domain functions using lower-case symbols and the corresponding frequency-domain functions using the corresponding upper-case symbols. Alternatively, an over-bar is used to indicate the forward transform. Thus, the forward transform of $g(t)$ is written $G(\omega)$ or $\bar{g}(\omega)$. In general, $g(t)$ and $G(\omega)$ may both be complex.

A number of useful theorems follow directly from the definitions of equations (A.2) and (A.3). A basic property is that the Fourier transform is linear, so that the transform of the sum of two functions is the sum of their individual transforms. Some useful symmetry theorems are:

(a) If $g(t)$ is even (so that $g(-t) = g(t)$), then $G(\omega)$ is even, and vice versa.
(b) If $g(t)$ is real and even, then $G(\omega)$ is real and even, and vice versa.
(c) If $g(t)$ and $G(\omega)$ are both real, then they are both even functions.

Transforms of conjugates are given by

$$g^*(t) \leftrightarrow G^*(-\omega), \tag{A.4}$$

$$G^*(\omega) \leftrightarrow g^*(-t). \tag{A.5}$$

Thus if $g(t)$ is real we have

$$G(-\omega) = G^*(\omega), \tag{A.6}$$

and if $G(\omega)$ is real we have

$$g(-t) = g^*(t). \tag{A.7}$$

The scaling theorem states that

$$g(t/a) \leftrightarrow |a| G(a\omega), \tag{A.8}$$

where a is a real constant. In particular, with $a = -1$ we have

$$g(-t) \leftrightarrow G(-\omega). \tag{A.9}$$

The shifting theorems are

$$g(t-a) \leftrightarrow \mathrm{e}^{-ja\omega} G(\omega), \tag{A.10}$$

$$G(\omega - a) \leftrightarrow \mathrm{e}^{jat} g(t), \tag{A.11}$$

and the modulation theorems are

$$g(t) \cos(\omega_0 t) \leftrightarrow \tfrac{1}{2}[G(\omega + \omega_0) + G(\omega - \omega_0)], \tag{A.12}$$

$$g(t) \sin(\omega_0 t) \leftrightarrow \tfrac{1}{2}j[G(\omega + \omega_0) - G(\omega - \omega_0)], \tag{A.13}$$

where ω_0 is a real constant. The differentiation theorems are

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t) \leftrightarrow j\omega G(\omega), \tag{A.14}$$

$$\frac{\mathrm{d}}{\mathrm{d}\omega} G(\omega) \leftrightarrow -jtg(t). \tag{A.15}$$

Parseval's theorem states that

$$\int_{-\infty}^{\infty} g_1(t)\, g_2^*(t)\, \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_1(\omega)\, G_2^*(\omega)\, \mathrm{d}\omega, \tag{A.16}$$

where $G_1(\omega)$ and $G_2(\omega)$ are respectively the transforms of $g_1(t)$ and $g_2(t)$. In particular, if $g_1(t) = g_2(t)$ we have the energy theorem:

$$\int_{-\infty}^{\infty} |g(t)|^2 \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\omega)|^2 \mathrm{d}\omega. \tag{A.17}$$

The convolution operation is denoted by an asterisk separating two functions, and is defined by

$$\begin{aligned} g_1(t) * g_2(t) &= \int_{-\infty}^{\infty} g_1(\tau)\, g_2(t - \tau)\, \mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} g_1(t - \tau)\, g_2(\tau)\, \mathrm{d}\tau \\ &= g_2(t) * g_1(t). \end{aligned} \tag{A.18}$$

Two convolution theorems are

$$g_1(t) * g_2(t) \leftrightarrow G_1(\omega) G_2(\omega), \tag{A.19}$$

$$G_1(\omega) * G_2(\omega) \leftrightarrow 2\pi g_1(t) g_2(t). \tag{A.20}$$

Delta Function. The Dirac delta function $\delta(t)$ is defined such that

$$\int_{-\infty}^{\infty} \delta(t) g(t) \mathrm{d}t = g(0), \tag{A.21}$$

where the function $g(t)$ is taken to be continuous at $t = 0$. The delta function can be regarded as a function whose value is infinite at $t = 0$ and zero elsewhere. However, it is an example of a "generalised function", and is not a function in the usual sense. Strictly, equation (A.21) has a specialised mathematical interpretation but for most practical purposes it can be regarded as a conventional integral. Some resulting properties are, with a real:

$$\int_{-\infty}^{\infty} \delta(t - a) g(t) \mathrm{d}t = g(a), \tag{A.22}$$

$$\delta(t - a) * g(t) = g(t - a), \tag{A.23}$$

$$\delta(t - a) g(t) = \delta(t - a) g(a), \tag{A.24}$$

$$\delta(at) = \frac{1}{|a|} \delta(t). \tag{A.25}$$

If $f(t)$ is a continuous function and is zero at $t = a$, then

$$\delta(t - a) f(t) = 0. \tag{A.26}$$

Fourier Transforms in the Limit. Strictly speaking, two functions $g(t)$ and $G(\omega)$ can be a Fourier transform pair only if the integrals of equations (A.2) and (A.3) converge. Generally, this requires $g(t)$ to approach zero for $t \to \pm \infty$ and $G(\omega)$ to approach zero for $\omega \to \pm \infty$. However, many functions of practical interest do not satisfy these conditions, an example being the sinusoidal waveform $\cos(\omega_0 t)$. For such cases a special approach can often be used. To illustrate this, consider the output of a practical spectrum analyser when the waveform $\cos(\omega_0 t)$ is applied. This gives two peaks at positions corresponding to the frequencies $\pm \omega_0$. The shapes of the peaks correspond to the response of the spectrum analyser, because the latter has finite resolution while the input waveform comprises components (at $\pm \omega_0$) with vanishing bandwidth. Provided the resolution remains finite (however small), and provided the spectrum analyser can be assumed to be linear with respect to inputs of different frequency, the output is well defined and is easily calculated. Mathematically, it can be shown that this can be expressed in terms of the "Fourier transform in the limit" of the function $\cos(\omega_0 t)$, which is given by

$$\cos(\omega_0 t) \leftrightarrow \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0). \tag{A.27}$$

The output is then obtained by convolving this transform (with respect to frequency) with a function representing the response of the spectrum analyser for a single-frequency input. Equation (A.27) thus represents a prescription for calculating the output of any linear spectrum analyser with finite resolution, for an input $\cos(\omega_0 t)$; it can also be regarded as the output in the limit when the resolution falls to zero.

Many of the standard Fourier transform formulae, and all those involving the delta function, are valid only in this limiting sense. The formal justification for the method is based on the theory of generalised functions, and is described by Papoulis [466, p. 269] and Bracewell [467, p. 87], for example.

Some Particular Transforms. In the following, a and ω_0 are real constants.

$$\sin(\omega_0 t) \leftrightarrow j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)], \tag{A.28}$$

$$\exp(-at^2) \leftrightarrow \sqrt{\pi/a}\,\exp(-\omega^2/4a), \qquad \text{for } a > 0, \tag{A.29}$$

$$\exp(\pm jat^2) \leftrightarrow \sqrt{\pi/a}\,\exp\left[\pm j\left(\frac{\pi}{4} - \frac{\omega^2}{4a}\right)\right], \quad \text{for } a > 0, \tag{A.30}$$

$$\delta(t - a) \leftrightarrow \exp(-ja\omega), \tag{A.31}$$

$$\delta(\omega - \omega_0) \leftrightarrow \frac{1}{2\pi}\exp(j\omega_0 t). \tag{A.32}$$

The function $\operatorname{rect}(x) = 1$ for $|x| < \frac{1}{2}$ and is zero for other x, and gives the transforms

$$\operatorname{rect}(t/a) \leftrightarrow a\,\operatorname{sinc}(\tfrac{1}{2}a\omega), \quad \text{for } a > 0, \tag{A.33}$$

and

$$\text{rect}\,(\omega/a) \leftrightarrow \frac{a}{2\pi}\,\text{sinc}\,(\tfrac{1}{2}at), \quad \text{for } a > 0, \tag{A.34}$$

where $\text{sinc}\,(x) = (\sin x)/x$.

The function $\text{sgn}\,(x) = 1$ for $x > 0$ and $\text{sgn}\,(x) = -1$ for $x < 0$, and gives the transforms

$$1/t \leftrightarrow -j\pi\,\text{sgn}\,(\omega), \tag{A.35}$$

$$1/\omega \leftrightarrow \tfrac{1}{2}j\,\text{sgn}\,(t). \tag{A.36}$$

The step function $U(x) = 1$ for $x > 0$ and $U(x) = 0$ for $x < 0$, so that

$$U(x) = \tfrac{1}{2} + \tfrac{1}{2}\,\text{sgn}\,(x). \tag{A.37}$$

Using equations (A.32) and (A.36) we have

$$U(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}. \tag{A.38}$$

Using the modulation theorems, equations (A.12) and (A.13), we have

$$U(t)\exp\,(-j\omega_0 t) \leftrightarrow \pi\delta(\omega + \omega_0) - \frac{j}{\omega + \omega_0} \tag{A.39}$$

and using equation (A.9)

$$U(-t)\exp\,(j\omega_0 t) \leftrightarrow \pi\delta(\omega - \omega_0) + \frac{j}{\omega - \omega_0}. \tag{A.40}$$

Adding the transforms (A.39) and (A.40) gives

$$\exp\,(-j\omega_0|t|) \leftrightarrow \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0) + \frac{2j\omega_0}{\omega^2 - \omega_0^2}. \tag{A.41}$$

An infinite train of delta functions transforms into a function of the same form:

$$\sum_{n=-\infty}^{\infty} \delta(t - na) \leftrightarrow \frac{2\pi}{|a|}\sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m/a). \tag{A.42}$$

From this it follows that

$$\sum_{n=-\infty}^{\infty} e^{jnat} = \frac{2\pi}{|a|}\sum_{m=-\infty}^{\infty} \delta(t - 2\pi m/a), \tag{A.43}$$

which is proved by transforming both sides, making use of equations (A.32) and (A.42).

A.2. Linear Filters

We consider a two-port device with an input waveform $v(t)$ and an output waveform $g(t)$. These waveforms represent physical quantities measured at the two ports, and their precise meaning must be assigned before applying the relations below to a practical device. They usually refer to either voltages or currents. The input $v(t)$ is often defined as the voltage which a specified waveform generator would produce across a matched load.

The term *linear* means that the device obeys superposition with regard to different input waveforms. Suppose that an input waveform $v_1(t)$ gives an output waveform $g_1(t)$, while an input waveform $v_2(t)$ gives an output waveform $g_2(t)$. Then the device is linear if an input waveform $v_1(t) + v_2(t)$ gives an output waveform $g_1(t) + g_2(t)$, irrespective of the forms of $v_1(t)$ and $v_2(t)$. The device is *time-invariant* if an input $v(t - \tau)$ gives an output $g(t - \tau)$ for any input function $v(t)$ and for any value of the delay τ. The term "linear filter" used here refers to a device that is both linear and time-invariant. Most surface-wave devices can be taken to be linear filters provided the power level of the input waveform is not too high; for most practical purposes this is an excellent approximation, though non-linear effects become significant at high power levels (Section 6.3).

It can be assumed here that the input and output waveforms are real. A consequence of linearity and time-invariance is that, if the input $v(t)$ is a real sinusoid, then the output $g(t)$ is also a real sinusoid, with the same frequency but generally with a different amplitude and phase. Thus, if

$$v(t) = \cos \omega_0 t, \quad \text{for } \omega_0 \geqslant 0,$$

then

$$g(t) = A(\omega_0) \cos [\omega_0 t + \phi(\omega_0)], \tag{A.44}$$

where $A(\omega_0)$ and $\phi(\omega_0)$ are functions of frequency but not of time. Since the filter is linear. the input can be written as a sum of two complex exponentials and the output can be regarded as the sum of the responses to these exponentials. We define the *frequency response* $H(\omega)$ such that if

$$v(t) = \exp(j\omega t),$$

then

$$g(t) = H(\omega) \exp(j\omega t). \tag{A.45}$$

This is consistent with equation (A.44) if

$$H(\pm\omega) = A(\omega) \exp[\pm j\phi(\omega)]. \tag{A.46}$$

This defines the frequency response for positive and negative frequencies. Clearly the negative-frequency components are given by

$$H(-\omega) = H^*(\omega). \tag{A.47}$$

Now consider a *general* real input waveform $v(t)$, as in Figure A.1. It is assumed

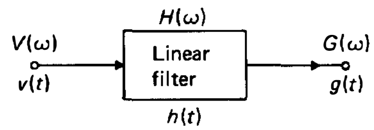

FIGURE A.1. Linear filter, showing notation for input and output waveforms.

that $v(t)$ has a Fourier transform $V(\omega)$ as in equation (A.2), that is, $V(\omega)$ is the *spectrum* of $v(t)$. By equation (A.3), $v(t)$ can be written as an infinite sum of complex exponentials, with coefficients $V(\omega)/2\pi$, and for each component equation (A.45) can be used. The corresponding outputs can be summed to give the total output waveform $g(t)$. This has a spectrum $G(\omega)$ given by equation (A.2), and hence

$$G(\omega) = V(\omega)H(\omega). \tag{A.48}$$

Thus the output spectrum can be obtained for any input waveform once $H(\omega)$ is known. The output waveform $g(t)$ can be obtained by transforming $G(\omega)$. Alternatively, $g(t)$ can be obtained from the *impulse response* $h(t)$, defined as the inverse Fourier transform of the frequency response:

$$h(t) \leftrightarrow H(\omega). \tag{A.49}$$

Applying the convolution theorem of equation (A.19) to equation (A.48) gives

$$\begin{aligned} g(t) &= v(t) * h(t) \\ &= \int_{-\infty}^{\infty} v(\tau)\, h(t - \tau)\, d\tau, \end{aligned} \tag{A.50}$$

so that the output waveform is the convolution of the input waveform with the device impulse response. In the particular case $v(t) = \delta(t)$, the output waveform is equal to $h(t)$ [by equation (A.23)], and hence $h(t)$ is the device output waveform when a delta function is applied to the input. As expected, $h(t)$ is real; this follows from the symmetry of $H(\omega)$, equation (A.47), and equation (A.6). A practical constraint is that $h(t) = 0$ for $t < 0$, since otherwise the device is not causal.

A particular example is an ideal delay line, which delays the input waveform by an amount τ_0, say, without distortion. In this case $g(t) = v(t - \tau_0)$ and the device responses are

$$\begin{aligned} h(t) &= \delta(t - \tau_0), \\ H(\omega) &= \exp(-j\omega\tau_0). \end{aligned}$$

An important conclusion is that if $H(\omega)$ has a phase term proportional to ω, this causes no distortion of the waveform.

If the input waveform is a random noise waveform, the analysis must be treated statistically [469, p. 312]. We assume here that the input noise is stationary, that is, its statistical properties are independent of time. It follows that the output noise will also be stationary, and a power spectral density can be defined for both the input and output noise waveforms. The power of the input noise waveform is denoted P_i, and is defined by

$$P_i = E\{[v(t)]^2\}, \tag{A.51}$$

where $E\{\ \}$ denotes the statistical expectation value and $v(t)$ is now a statistical ensemble of functions. The expectation value is independent of t because the noise is stationary. A similar definition gives the output noise power P_0, with $v(t)$ replaced by $g(t)$. The input noise has a power spectral density $N_i(\omega)$, and the output noise has a power spectral density $N_0(\omega)$. These spectra are assumed to exist only for positive or zero frequencies. They are related to the noise powers by

$$P_i = \frac{1}{2\pi}\int_0^\infty N_i(\omega)\,\mathrm{d}\omega, \tag{A.52}$$

$$P_0 = \frac{1}{2\pi}\int_0^\infty N_0(\omega)\,\mathrm{d}\omega, \tag{A.53}$$

and the two spectral densities are related by

$$N_0(\omega) = N_i(\omega)|H(\omega)|^2. \tag{A.54}$$

Physically, $N_0(\omega)$ can be taken as the power per Hz of bandwidth for the output noise spectral components in the immediate vicinity of frequency ω. The same statement can be made about $N_i(\omega)$. This interpretation follows from equations (A.52)–(A.54) by taking the linear filter to be a narrow-band device.

In many cases the input waveform is taken to be *white* noise, which has a spectral density $N_i(\omega)$ independent of frequency. Strictly, this implies that the noise power P_i will be infinite. In all practical cases however, the spectral density will decay at high frequencies, and the power will be finite. White noise can be defined such that its spectral density is constant up to some high frequency, beyond which the filter response $H(\omega)$ can be taken to be negligible. Equations (A.53) and (A.54) can then be used, with $N_i(\omega)$ independent of ω.

A.3. Matched Filtering

We now consider a waveform $s(t)$ of finite length, accompanied by wide-band noise. If this is applied to a linear filter, the output signal-to-noise ratio will depend on the response of the filter. It is shown here that the output signal-to-noise ratio is maximised if the filter is designed such that its impulse response has a specific form, essentially the time-reverse of $s(t)$. Such a filter is called a *matched* filter [277–279, 282]; the terminology refers to the fact that the filter is matched to a specified input waveform $s(t)$, and not to the more familiar meaning referring to the impedances at the terminals. Matched filters are used in pulse-compression radar systems (Section 9.1), where the waveform $s(t)$ represents the received signal reflected from a target, and are also used in spread-spectrum communications (Section 10.1). For present purposes it can be assumed that the noise at the filter input is white, though this is not always the case.

The filter output waveform is a linear sum of a waveform $g(t)$, due to the input waveform $s(t)$, plus random noise due to the noise applied at the input. The power of the output noise is denoted P_0. The output signal power is $[g(t)]^2$, and since this varies with time its maximum value is used when defining the output signal-to-noise power ratio. Denoting the latter by SNR_0, we define

$$\mathrm{SNR}_0 = \frac{[g(t)]^2_{\max}}{P_0}. \tag{A.55}$$

Note that for an oscillatory waveform this refers to the *peak* signal power, not the r.m.s. value; the noise power P_0 is average power. Now, if $H(\omega)$ is the frequency response of the filter and $S(\omega)$ the spectrum of the input waveform $s(t)$, the output waveform $g(t)$ has a spectrum $G(\omega) = S(\omega)H(\omega)$, from equation (A.48). The output signal power can therefore be written

$$[g(t)]^2 = \left|\frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)H(\omega)\, e^{j\omega t}\, d\omega\right|^2. \tag{A.56}$$

The input noise is taken to be stationary and white, with spectral power density N_i per Hz of bandwidth, so that N_i is independent of frequency. From equation (A.53), the output noise power is

$$P_0 = \frac{N_i}{4\pi}\int_{-\infty}^{\infty} |H(\omega)|^2\, d\omega. \tag{A.57}$$

We now apply Schwartz's inequality, which states that, if $A(\omega)$ and $B(\omega)$ are complex functions of ω, then

$$\left|\int_a^b A^*(\omega)B(\omega)\, d\omega\right|^2 \leqslant \int_a^b |A(\omega)|^2\, d\omega \int_a^b |B(\omega)|^2\, d\omega, \tag{A.58}$$

where the equality applies only when $B(\omega) = kA(\omega)$, and k is an arbitrary constant. Assume initially that the signal power $[g(t)]^2$ is maximised at some time t_0, say, so that the output SNR is given by equations (A.55)–(A.57) with $t = t_0$. Using equation (A.58), with $A^*(\omega) = S(\omega) \exp(j\omega t_0)$ and $B(\omega) = H(\omega)$, we find

$$\mathrm{SNR}_0 \leqslant \frac{1}{\pi N_i}\int_{-\infty}^{\infty} |S(\omega)|^2\, d\omega. \tag{A.59}$$

We define E_s, the energy of the input signal, by

$$E_s = \int_{-\infty}^{\infty} [s(t)]^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |S(\omega)|^2\, d\omega, \tag{A.60}$$

where the equality of the two forms is an expression of the energy theorem, given in

Section A.1. We thus have

$$\mathrm{SNR}_0 \leqslant 2E_s/N_i. \tag{A.61}$$

Thus the maximum value of SNR_0 is simply $2E_s/N_i$. Note that this depends only on the signal energy and the noise spectral density, and not on the form of the input signal. The maximum output SNR is obtained when the equality in equation (A.61) applies, that is, when $B(\omega) = kA(\omega)$, and the filter response $H(\omega)$ then satisfies the condition

$$H(\omega) = kS^*(\omega) \exp(-j\omega t_0). \tag{A.62}$$

A filter satisfying this criterion is a matched filter, matched to the input waveform $s(t)$. The impulse response $h(t)$ of the matched filter is the inverse Fourier transform of $H(\omega)$. Using standard relations from Fourier analysis, this is given by

$$h(t) = ks(t_0 - t). \tag{A.63}$$

This is simply the time-reverse of the signal, delayed by an amount t_0 and multiplied by an arbitrary constant k. Clearly, k must in practice be real.

It was assumed in the above argument that the output power $[g(t)]^2$ was maximised at time $t = t_0$. However, since the maximum value of SNR_0, given by equation (A.61), is independent of t_0, the result is valid irrespective of when the output power is maximised. In practice there is a constraint on t_0 because the filter must be causal, that is, its impulse response $h(t)$ must be zero for $t < 0$. If, for example, the signal $s(t)$ has duration T and commences at $t = 0$, causality requires that $t_0 \geqslant T$.

As for any linear filter, the output waveform $g(t)$ can be written as the convolution $g(t) = s(t) * h(t)$. Here, this can conveniently be expressed in terms of the *correlation function* of $s(t)$, which is defined as the convolution of $s(t)$ with $s(-t)$. Denoting the correlation function by $c(t)$, we thus have

$$c(t) \equiv s(t) * s(-t) = \int_{-\infty}^{\infty} s(\tau)s(\tau - t)\, \mathrm{d}\tau. \tag{A.64}$$

Using equation (A.63) we find

$$g(t) = kc(t - t_0), \tag{A.65}$$

so that $g(t)$ is essentially the delayed correlation function of $s(t)$. By substituting $\tau = \tau' + t$ in equation (A.64) it is readily seen that $c(-t) = c(t)$, so that $c(t)$ symmetric about $t = 0$. Consequently, $g(t)$ is symmetric about the time t_0 where its power is maximised. The spectrum of the output waveform is $G(\omega) = S(\omega)H(\omega)$, and from equation (A.62) we have

$$G(\omega) = k|S(\omega)|^2 \, \mathrm{e}^{-j\omega t_0} = \frac{1}{k}|H(\omega)|^2 \, \mathrm{e}^{-j\omega t_0}. \tag{A.66}$$

A.4. Non-Uniform Sampling

The theory of uniform sampling was given briefly in Section 8.1.2. It was shown that, if a bandpass waveform is sampled using a uniformly-spaced sequence of delta functions, the original waveform can be recovered from the sampled waveform by low-pass filtering, provided the sampling frequency is at least as large as the Nyquist frequency. Here we consider a waveform sampled using delta functions whose spacing varies, and show that a similar result can be obtained. This result is needed for the analysis of chirp filters in Chapter 9. The analysis is based on that given by Tancrell and Holland [79] and others [470, 471].

It is first necessary to establish a relationship concerning delta functions. Consider a monotonic function $u(t)$ which has one zero at $t = t_0$, and whose differential $\dot{u}(t)$ is non-zero at $t = t_0$. We then have [467, p. 95]

$$\delta[u(t)] = \frac{\delta(t - t_0)}{|\dot{u}(t)|}. \tag{A.67}$$

This can be proved by substituting $\delta[u(t)]$ for $\delta(t - a)$ in equation (A.22) and taking u as the independent variable.

Now consider sampling a finite-length continuous waveform $v(t)$. We consider a general case first, and later specialise by taking $v(t)$ to be a chirp waveform. The sampled version of $v(t)$ is $v_s(t)$, a sequence of delta functions at times t_n, given by

$$v_s(t) = v(t) \sum_{n=-\infty}^{\infty} \delta(t - t_n). \tag{A.68}$$

Here the sampling times t_n are to have non-uniform spacing. It is assumed that their values can be obtained from a smooth monotonic non-linear function $\theta(t)$, by solving the equation

$$\theta(t_n) = n\Delta, \quad n = 0, \pm 1, \pm 2, \ldots, \tag{A.69}$$

where Δ is a positive constant. The times t_n thus correspond to uniform increments of $\theta(t)$. With $u(t) = \theta(t) - n\Delta$, we obtain from equation (A.67)

$$\sum_{n=-\infty}^{\infty} \delta(t - t_n) = |\dot{\theta}(t)| \sum_{n=-\infty}^{\infty} \delta[\theta(t) - n\Delta]. \tag{A.70}$$

Here the delta-functions on the right can be expressed as a sum of complex exponentials using equation (A.43), giving

$$v(t) \sum_{n=-\infty}^{\infty} \delta(t - t_n) = v(t) \frac{|\dot{\theta}(t)|}{\Delta} \sum_{m=-\infty}^{\infty} \exp\,[j2\pi m\theta(t)/\Delta] \tag{A.71}$$

and this is equal to $v_s(t)$, equation (A.68). This shows that the sampled waveform can be expressed as a fundamental, with $m = 0$, plus a series of "harmonics" with other values of m. The fundamental has the same form as the original waveform $v(t)$, except for an amplitude distortion produced by the term $|\dot{\theta}(t)|$. Each "harmonic" is essentially the original waveform multiplied by a chirp waveform. If the original waveform $v(t)$ is band-limited and has finite length, and if the increment Δ is small enough, the frequency band occupied by the fundamental will not overlap the bands occupied by the harmonics. The fundamental component $v(t)|\dot{\theta}(t)|/\Delta$ may then be obtained from the sampled waveform by using a low-pass filter to reject the harmonics.

In surface-wave chirp filters, the waveform $v(t)$ to be sampled represents the required impulse response, and is itself a chirp waveform. Sampling is nearly always done in synchronism with the waveform, that is, at corresponding points in each cycle. This implies that the function $\theta(t)$ which defines the sampling points, equation (A.69), must also be the time-domain phase of the waveform, apart from an additive constant. Thus the original chirp waveform can be written as

$$v(t) = a(t) \cos [\theta(t) + \phi_0], \tag{A.72}$$

where ϕ_0 is an arbitrary constant. The envelope $a(t)$ will have finite length and is taken to be a smooth function, such that $v(t)$ is a band-pass waveform; thus the spectrum of $v(t)$ is non-zero only for $\omega_1 < |\omega| < \omega_2$, where ω_1 and ω_2 are two positive frequencies. We assume here that an integer number of samples is taken for each cycle of the waveform, as is done in practical device design. The number of samples per cycle is denoted by the integer S_e, as in Chapter 9, and in practice $S_e \geqslant 2$. The increment Δ is therefore $2\pi/S_e$, and the sampling times t_n are the solutions of $\theta(t_n) = 2\pi n/S_e$. From equations (A.71) and (A.72) the sampled waveform is

$$\begin{aligned} v_s(t) &= v(t) \sum_{n=-\infty}^{\infty} \delta(t - t_n) \\ &= \frac{S_e a(t)|\dot{\theta}(t)|}{2\pi} \Bigg[\cos [\theta(t) + \phi_0] \\ &\quad + \sum_{m=1}^{\infty} \{\cos [(mS_e + 1)\theta(t) + \phi_0] + \cos [(mS_e - 1)\theta(t) - \phi_0]\} \Bigg]. \end{aligned} \tag{A.73}$$

In practice we are mainly interested in the fundamental component of $v_s(t)$, which is denoted $\tilde{v}_s(t)$. For $S_e > 2$ the fundamental is obtained by omitting the terms dependent on m, giving

$$\tilde{v}_s(t) = \frac{1}{2\pi} S_e|\dot{\theta}(t)|a(t) \cos [\theta(t) + \phi_0], \quad \text{for } S_e > 2, \tag{A.74}$$

which is essentially the original waveform, equation (A.72), multiplied by $|\dot{\theta}(t)|$. The harmonic terms which have been omitted have the same form except that the phase $\theta(t)$ in the cosine is replaced by $M\theta(t)$, with the integer $M \geqslant 2$. For $S_e = 2$ the term in equation (A.73) involving $(mS_e - 1)\theta(t)$ contributes to the fundamental when $m = 1$, and we find

$$\tilde{v}_s(t) = \frac{1}{\pi} S_e|\dot{\theta}(t)|a(t) \cos [\theta(t)] \cos \phi_0, \quad \text{for } S_e = 2. \tag{A.75}$$

If ϕ_0 is a multiple of 2π, this is essentially the original waveform multiplied by $|\dot{\theta}(t)|$; in this case the sampling times given by $\theta(t_n) = 2\pi n/S_e$ are at the maxima and minima of the original waveform. For other values of ϕ_0 a constant phase change is introduced, but this is generally inconsequential. However, ϕ_0 must not be an odd multiple of $\pi/2$ because the samples are then at the zeros of the original waveform, and this gives $v_s(t) = 0$.

A.5. Some Properties of Bandpass Waveforms

A waveform $v(t)$ is referred to as a bandpass waveform if its spectrum $V(\omega)$ is finite only in a specified frequency range excluding zero, that is, it is finite only for $\omega_1 < |\omega| < \omega_2$ where ω_1 and ω_2 are two positive frequencies. The impulse response of a linear surface-wave filter is always a bandpass waveform. Here we consider some properties of bandpass waveforms relating to the design of surface-wave transducers, discussed in Section 8.1.3.

It is convenient to represent a bandpass waveform $v(t)$ in terms of its complex envelope $\hat{v}(t)$. To do this, consider the positive-frequency part of the spectrum $V(\omega)$, shifted downward in frequency by an amount ω_r. This base band version of $V(\omega)$ is denoted by $2\hat{V}(\omega)$. The positive-frequency part of $V(\omega)$ is therefore given by

$$V(\omega) = \tfrac{1}{2}\hat{V}(\omega - \omega_r), \quad \text{for } \omega \geqslant 0. \tag{A.76}$$

Here ω_r is a positive reference frequency whose value is arbitrary except that it is taken to be between ω_1 and ω_2. For an amplitude-modulated waveform, ω_r is usually taken to be the carrier frequency. It is assumed that $v(t)$ is real, which implies that $V(-\omega) = V^*(\omega)$ and hence $V(\omega) = \tfrac{1}{2}\hat{V}^*(-\omega - \omega_r)$ for $\omega \leqslant 0$. The complex envelope of $v(t)$ is $\hat{v}(t)$, defined as the inverse Fourier transform of $\hat{V}(\omega)$:

$$\hat{v}(t) \leftrightarrow \hat{V}(\omega). \tag{A.77}$$

The waveform $v(t)$ can be related to this by transforming $V(\omega)$, making use of the shifting theorem of equation (A.11), giving

$$v(t) = \tfrac{1}{2}\hat{v}(t)\,e^{j\omega_r t} + \tfrac{1}{2}\hat{v}^*(t)\,e^{-j\omega_r t}, \tag{A.78}$$

where the first term on the right arises from the positive-frequency part of $V(\omega)$, and the second term from the negative-frequency part. We also define $\hat{a}(t)$ and $\hat{\theta}(t)$ as the amplitude and phase of $\hat{v}(t)$, so that

$$\hat{v}(t) = \hat{a}(t)\exp[j\hat{\theta}(t)]. \tag{A.79}$$

The waveform $v(t)$ can then be written as

$$v(t) = \hat{a}(t)\cos[\omega_r t + \hat{\theta}(t)]. \tag{A.80}$$

Thus, $\hat{a}(t)$ is the envelope of the waveform $v(t)$.

Linear Phase in the Frequency Domain. In surface-wave transducer design, where a waveform $v(t)$ is sampled in order to determine the geometry, it is often a requirement that the spectrum $V(\omega)$ should have a phase linear with frequency. It is therefore useful to consider what constraints this imposes on $v(t)$. We define $\phi(\omega)$ as the phase of $V(\omega)$, so that $V(\omega) = |V(\omega)| \exp[j\phi(\omega)]$, and take $\phi(\omega)$ to have the form

$$\phi(\omega) = \theta_c - (\omega - \omega_r)t_0, \quad \text{for } \omega > 0, \tag{A.81}$$

where θ_c and t_0 are arbitrary constants, and ω_r is the reference frequency for the complex envelope. Using equation (A.76), we have in this case

$$\hat{V}(\omega) = |\hat{V}(\omega)| \exp[j(\theta_c - \omega t_0)].$$

The inverse transform of $\hat{V}(\omega)$ is the complex envelope $\hat{v}(t)$. Using the shifting theorem, equation (A.10), we find that the transform of $|\hat{V}(\omega)|$ is given by

$$|\hat{V}(\omega)| \leftrightarrow e^{-j\theta_c}\,\hat{v}(t + t_0). \tag{A.82}$$

Here the left side is real. Since the transform $g(t)$ of some real function $G(\omega)$ gives $g(-t) = g^*(t)$, it can be concluded that

$$\hat{v}(t_0 - t) = e^{2j\theta_c}\,\hat{v}^*(t_0 + t).$$

Writing the complex envelope as $\hat{v}(t) = \hat{a}(t)\exp[j\hat{\theta}(t)]$, this gives

$$\hat{a}(t_0 - t)/\hat{a}(t_0 + t) = \exp\{j[2\theta_c - \hat{\theta}(t_0 + t) - \hat{\theta}(t_0 - t)]\}. \tag{A.83}$$

Since $\hat{a}(t)$ is real the right side of this equation must be equal to ± 1 , and can be written as $\exp(jn\pi)$. We therefore have

$$\hat{\theta}(t_0 - t) = 2\theta_c - n\pi - \hat{\theta}(t_0 + t) \tag{A.84}$$

and

$$\hat{a}(t_0 - t) = e^{jn\pi}\,\hat{a}(t_0 + t) = \pm\hat{a}(t_0 + t). \tag{A.85}$$

Thus the envelope $\hat{a}(t)$ is either symmetric or anti-symmetric about $t = t_0$, which is the slope of the phase in the frequency domain, equation (A.81). The phase $\hat{\theta}(t)$ is equal to the constant $\theta_c - n\pi/2$, plus a function anti-symmetric about t_0.

Amplitude Modulation. Another important case for surface-wave transducer design occurs when $v(t)$ is an amplitude-modulated waveform, with no phase modulation. This implies some constraint on the spectrum $V(\omega)$. An amplitude-modulated waveform has the form

$$v(t) = \hat{a}(t)\cos(\omega_r t + \theta_c), \tag{A.86}$$

where θ_c is a constant and the reference frequency ω_r has been chosen to be equal to the carrier frequency. The envelope $\hat{a}(t)$ must of course be real. In this case $\hat{v}(t) = \hat{a}(t)\exp(j\theta_c)$, and hence $\hat{V}(\omega) = \hat{A}(\omega)\exp(j\theta_c)$, where $\hat{A}(\omega)$ is the transform of $\hat{a}(t)$. Since $\hat{a}(t)$ is real we have $\hat{A}(-\omega) = \hat{A}^*(\omega)$, and hence

$$\hat{V}(-\omega) = \hat{V}^*(\omega)\exp(2j\theta_c). \tag{A.87}$$

For positive frequencies, the spectrum $V(\omega)$ of $v(t)$ is given by equation (A.76). Using equation (A.87) we have

$$V(\omega_r - \omega) = V^*(\omega_r + \omega) \exp(2j\theta_c), \quad \text{for } |\omega| < \omega_r. \tag{A.88}$$

If $A(\omega)$ and $\phi(\omega)$ are the spectral amplitude and phase, so that $V(\omega) = A(\omega) \exp[j\phi(\omega)]$, we find, for $|\omega| < \omega_r$,

$$\phi(\omega_r - \omega) = 2\theta_c - n\pi - \phi(\omega_r + \omega) \tag{A.89}$$

and

$$A(\omega_r - \omega) = A(\omega_r + \omega)\, e^{jn\pi} = \pm A(\omega_r + \omega). \tag{A.90}$$

Thus $A(\omega)$ is either symmetric or anti-symmetric about the carrier frequency ω_r. The phase $\phi(\omega)$ equals a constant $\theta_c - n\pi/2$, plus a function anti-symmetric about ω_r.

Sampling of Amplitude-Modulated Waveforms. It was shown in Section 8.1.2 that a band-limited waveform can be uniformly sampled in a manner such that the original waveform can be recovered by low-pass filtering. In general, a necessary condition for this is that the sampling frequency ω_s must exceed the Nyquist frequency $2\omega_2$, where ω_2 is defined such that the original waveform has negligible spectral energy for $\omega > \omega_2$. Here we show that, for the special case of an amplitude-modulated bandpass waveform, the sampling frequency may be *below* the Nyquist frequency.

Suppose that $v(t)$ is some arbitrary bandpass waveform whose spectrum $v(t)$ is negligible except in the intervals given by $\omega_1 < |\omega| < \omega_2$. Sampling $v(t)$ at times $t = n\tau_s$ gives the sampled version $v_s(t)$, so that

$$v_s(t) = v(t) \sum_{n=-\infty}^{\infty} \delta(t - n\tau_s),$$

as in equation (8.8) of Section 8.1.2. The spectrum of $v_s(t)$ is $V_s(\omega)$, and from equation (8.9) this is given by

$$V_s(\omega) = \frac{\omega_s}{2\pi} \sum_{m=-\infty}^{\infty} V(\omega - m\omega_s), \tag{A.91}$$

where $\omega_s = 2\pi/\tau_s$. If the sampling frequency ω_s is below the Nyquist frequency, there will generally be some overlap of the original spectrum $V(\omega)$ and the image spectra

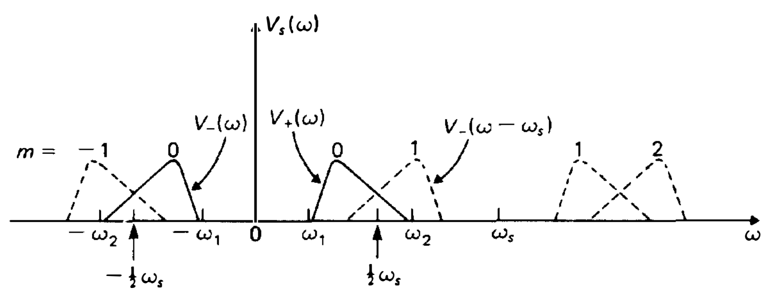

FIGURE A.2. Effect of sampling a bandpass waveform, with sampling frequency below the Nyquist frequency.

due to the sampling, that is, aliasing will occur. This is shown in Figure A.2, where ω_s is assumed to be between $2\omega_1$ and $2\omega_2$ and the terms with $m \neq 0$ are shown by broken lines.

We define $U(\omega)$ as the part of the spectrum $V_s(\omega)$ below frequency ω_s, so that

$$\begin{aligned} U(\omega) &= V_s(\omega), \quad \text{for } |\omega| < \omega_s, \\ &= 0, \quad\quad\;\; \text{for } |\omega| > \omega_s. \end{aligned} \tag{A.92}$$

This function can be obtained from $V_s(\omega)$ by low-pass filtering. It is convenient to write

$$V(\omega) = V_+(\omega) + V_-(\omega), \tag{A.93}$$

where $V_+(\omega)$ is the positive-frequency part of $V(\omega)$ and $V_-(\omega)$ is the negative-frequency part. Using equation (A.91), the positive-frequency part of $U(\omega)$ is given by

$$\begin{aligned} 2\pi U(\omega)/\omega_s &= V_+(\omega) + V_-(\omega - \omega_s), \\ &= V_+(\omega) + V_+^*(\omega_s - \omega), \quad \text{for } \omega > 0. \end{aligned} \tag{A.94}$$

Here the second form follows because $v(t)$ is real, which implies that $V_-(\omega) = V_+^*(-\omega)$. Now, the original waveform $v(t)$ may be recovered from the sampled waveform if the filtered spectrum $U(\omega)$ is proportional to the original spectrum $V(\omega)$. We therefore consider the case

$$2\pi U(\omega)/\omega_s = KV_+(\omega), \quad \text{for } \omega > 0, \tag{A.95}$$

where K is a constant which may in general be complex. For this to be consistent with equation (A.94) there must be a constraint on $V_+(\omega)$, and therefore on the original waveform $v(t)$. Writing $\omega = \frac{1}{2}\omega_s - \Delta\omega$ we have

$$(K - 1)V_+(\tfrac{1}{2}\omega_s - \Delta\omega) = V_+^*(\tfrac{1}{2}\omega_s + \Delta\omega). \tag{A.96}$$

Considering the case $\Delta\omega = 0$, and assuming that $V_+(\frac{1}{2}\omega_s) \neq 0$, this equation shows that $|K - 1| = 1$. We can therefore write $K - 1 = \exp(-2j\alpha_c)$, where the constant α_c is the phase of $V_+(\frac{1}{2}\omega_s)$, and equation (A.96) becomes

$$V_+(\tfrac{1}{2}\omega_s - \Delta\omega) = V_+^*(\tfrac{1}{2}\omega_s + \Delta\omega)\exp(2j\alpha_c). \tag{A.97}$$

This equation has the same form as equation (A.88), with ω_r replaced by $\frac{1}{2}\omega_s$ and θ_c replaced by α_c. It follows that $v(t)$ must be an amplitude-modulated waveform, with carrier frequency $\frac{1}{2}\omega_s$. Comparison with equation (A.86) shows that

$$v(t) = \hat{a}(t)\cos(\tfrac{1}{2}\omega_s t + \alpha_c), \tag{A.98}$$

where the envelope $\hat{a}(t)$ is not determined. We have thus shown that, if an amplitude-modulated waveform is sampled with a sampling frequency ω_s equal to twice the carrier frequency, the sampled waveform has the same spectrum as the original waveform in the region $|\omega| < \omega_s$, except for a complex constant. Equation (A.95) shows that the constant is $K\omega_s/(2\pi)$, and since $K - 1 = \exp(-2j\alpha_c)$ we have

$$K = 2\exp(-j\alpha_c)\cos\alpha_c. \tag{A.99}$$

If the sampled waveform is filtered by an ideal low-pass filter, such that all frequency components with $|\omega| > \omega_s$ are rejected, the resulting waveform $u(t)$ is the inverse transform of $U(\omega)$, and this is found to give

$$u(t) = \frac{\omega_s}{\pi} \hat{a}(t) \cos(\tfrac{1}{2}\omega_s t) \cos \alpha_c, \tag{A.100}$$

which is the same as the original waveform $v(t)$, equation (A.98), except for changes of amplitude and phase.

It should be noted that α_c must not be an odd multiple of $\pi/2$, since this gives $u(t) = 0$. It also gives $K = 0$ and therefore, from equation (A.95), $U(\omega) = 0$. The reason for this is that the sampling points have been taken to be at $t = n\tau_s = 2n\pi/\omega_s$, and for $\alpha_c = \pi/2$ the waveform $v(t)$ is zero at these points. The waveform is usually sampled at the maxima and minima. In this case α_c is a multiple of π and $K = 2$, so that $u(t)$ has the same phase as $v(t)$ and the amplitude of $u(t)$ is maximised.