Chapter 3

Electrical Excitation at a Plane Surface

For a piezoelectric material, the surface waves described in Chapter 2 can be generated and detected electrically by means of metal electrodes on the surface. This principle is used in interdigital transducers and multistrip couplers. A basic concept used in the analysis of these components is the effective permittivity, which gives a description of the electrical behaviour of the surface taking account of the acoustic behaviour of the material. A Green's function derived from the effective permittivity is also used. This chapter gives the basic theory. Sections 3.1 to 3.4 describe the effective permittivity and the Green's function for excitation of a half-space, including some approximations which will be used for analysis of transducers and multistrip couplers in the following chapters. In Section 3.5 some other applications of the basic concepts are described briefly.

It is assumed throughout this chapter that the electrodes do not cause any mechanical perturbations.

3.1. NON-PIEZOELECTRIC HALF-SPACE

We first consider electrical excitation at the surface of a half-space, assuming the material to be non-piezoelectric. This will be referred to as the electrostatic case. In the cases of practical interest the material is always piezoelectric, of course. However, the electrostatic solution is of considerable importance because it can be used as a first-order approximation in the piezoelectric case, and it will be used extensively in later chapters. Some previous analysis is given by Ingebrigtsen [62] and by Hartmann and Secrest [63].

Coordinate axes are defined as in Figure 2.1, so that a homogeneous anisotropic dielectric occupies the half-space $x_3 < 0$, with a vacuum in the region $x_3 > 0$. The potential is assumed to be invariant in the x_2-direction. We shall be concerned with cases where electrodes are present at the surface, so that there will in general be free charges present in the plane $x_3 = 0$. However, the electrodes are not considered explicitly at this stage; they will be allowed for later by applying appropriate boundary conditions at the surface. Since the potential does not vary in the x_2-direction, the

edges of the electrodes must be parallel to the x_2-axis.

It is assumed that there are no free charges except at the plane $x_3 = 0$, so that the potential $\Phi(x_1, x_3)$ must satisfy Laplace's equation in the vacuum and in the dielectric. In the dielectric we have div $\mathbf{D} = 0$, and $\mathbf{E} = -$ grad Φ, assuming the electric field to be quasi-static. Using $\partial/\partial x_2 = 0$ and $\mathbf{D} = \boldsymbol{\varepsilon}.\ \mathbf{E}$, we find

$$\varepsilon_{11} \frac{\partial^2 \Phi}{\partial x_1^2} + 2\varepsilon_{13} \frac{\partial^2 \Phi}{\partial x_1 \partial x_3} + \varepsilon_{33} \frac{\partial^2 \Phi}{\partial x_3^2} = 0 \tag{3.1}$$

for $x_3 < 0$. In the vacuum, $x_3 > 0$, the same relation applies, with ε_{11} and ε_{33} replaced by ε_0 and ε_{13} set to zero, so that

$$\frac{\partial^2 \Phi}{\partial x_1^2} + \frac{\partial^2 \Phi}{\partial x_3^2} = 0 \tag{3.2}$$

for $x_3 > 0$.

We consider first a harmonic solution in which the potential varies as $\exp(j\beta x_1)$. The potential is required to vanish at $x_3 = \pm \infty$ and must also be continuous at $x_3 = 0$. Taking β to be real, a harmonic solution that satisfies these requirements and also satisfies equations (3.1) and (3.2) is given by

$$\tilde{\Phi}(x_1, x_3) = \exp[j\beta x_1 - |\beta| x_3], \quad \text{for } x_3 > 0, \tag{3.3a}$$

$$\tilde{\Phi}(x_1, x_3) = \exp[j\beta(x_1 - x_3 \cdot \varepsilon_{13}/\varepsilon_{33}) + |\beta| x_3 \cdot \varepsilon_p/\varepsilon_{33}], \quad \text{for } x_3 < 0 \tag{3.3b}$$

where

$$\varepsilon_p = (\varepsilon_{11}\varepsilon_{33} - \varepsilon_{13}^2)^{1/2} \tag{3.4}$$

and the tilde is used to indicate the harmonic solution. With these expressions for the potential, the component of the displacement normal to the surface is, in the vacuum,

$$\tilde{D}_3(x_1, x_3) = \varepsilon_0 |\beta| \cdot \tilde{\Phi}(x_1, x_3), \quad \text{for } x_3 > 0 \tag{3.5a}$$

and, in the dielectric,

$$\tilde{D}_3(x_1, x_3) = -\varepsilon_p |\beta| \cdot \tilde{\Phi}(x_1, x_3), \quad \text{for } x_3 < 0. \tag{3.5b}$$

The displacements can also be written in terms of the electric field component parallel to the surface:

$$\tilde{D}_3(x_1, x_3) = j\varepsilon_0 \operatorname{sgn}(\beta) \cdot \tilde{E}_1(x_1, x_3), \quad \text{for } x_3 > 0, \tag{3.6a}$$

$$\tilde{D}_3(x_1, x_3) = -j\varepsilon_p \operatorname{sgn}(\beta) \cdot \tilde{E}_1(x_1, x_3), \quad \text{for } x_3 < 0, \tag{3.6b}$$

where $\operatorname{sgn}(\beta) = +1$ for $\beta > 0$ and $\operatorname{sgn}(\beta) = -1$ for $\beta < 0$.

Any discontinuity in $\tilde{D}_3$ at the surface $x_3 = 0$ implies, by Gauss's law, that there must be free charges there. Define $\tilde{D}_3(+)$ as the value of $\tilde{D}_3$ in the vacuum at $x_3 = 0$, and $\tilde{D}_3(-)$ as the value of $\tilde{D}_3$ in the dielectric at $x_3 = 0$. The density of free charges, denoted $\tilde{\sigma}(x_1)$, is then given by

$$\tilde{\sigma}(x_1) = \tilde{D}_3(+) - \tilde{D}_3(-),$$

where $\tilde{\sigma}(x_1)$ includes charges on both sides of the electrodes. Using equations (3.5),

$$\tilde{\sigma}(x_1) = (\varepsilon_0 + \varepsilon_p)|\beta|\tilde{\Phi}(x_1, 0). \tag{3.7}$$

The quantity $(\varepsilon_0 + \varepsilon_p)$ is an effective permittivity which characterises the interface between the vacuum and the dielectric. It gives the discontinuity in $\tilde{D}_3$ in terms of the parallel field $\tilde{E}_1$ at the surface:

$$\tilde{D}_3(+) - \tilde{D}_3(-) = j \operatorname{sgn}(\beta)(\varepsilon_0 + \varepsilon_p)\tilde{E}_1(x_1, 0). \tag{3.8}$$

We now consider a more general solution. It is convenient to write this in terms of the potential at the surface which, for the harmonic solution, is defined as $\tilde{\phi}(x_1) \equiv \tilde{\Phi}(x_1, 0)$. Thus equation (3.7) becomes

$$\tilde{\sigma}(x_1) = (\varepsilon_0 + \varepsilon_p)|\beta|\tilde{\phi}(x_1), \tag{3.9}$$

where $\tilde{\sigma}(x_1)$ and $\tilde{\phi}(x_1)$ are both proportional to $\exp(j\beta x)$. The general solution, with charge density $\sigma(x_1)$ and surface potential $\phi(x_1)$, is obtained by Fourier synthesis. We have

$$\sigma(x_1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\sigma}(\beta) \exp(j\beta x_1)\, \mathrm{d}\beta, \tag{3.10}$$

and

$$\phi(x_1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\phi}(\beta) \exp(j\beta x_1)\, \mathrm{d}\beta, \tag{3.11}$$

where $\bar{\sigma}(\beta)$ and $\bar{\phi}(\beta)$ are respectively the Fourier transforms of $\sigma(x_1)$ and $\phi(x_1)$. The general solution is an infinite sum of harmonic solutions with different values of β. For each β, equation (3.9) applies, and we thus have

$$\bar{\sigma}(\beta) = (\varepsilon_0 + \varepsilon_p)|\beta|\bar{\phi}(\beta). \tag{3.12}$$

Thus the charge density $\sigma(x_1)$ may be obtained if any surface potential $\phi(x_1)$ is specified.

It should be noted that $\bar{\phi}(\beta)$ must be zero at $\beta = 0$, because if this were not so the potential would be finite at $x_3 = \pm\infty$, as can be seen by considering equations (3.3) for $\beta = 0$. It follows from equation (3.12) that $\bar{\sigma}(\beta)$ is also zero at $\beta = 0$, and from the definition of the Fourier transform this implies that

$$\int_{-\infty}^{\infty} \sigma(x_1)\, \mathrm{d}x_1 = 0, \tag{3.13}$$

that is, the sum total of all the charges at the surface $x_3 = 0$ is equal to zero.

Corresponding relationships in the spatial domain can be obtained using Fourier analysis. The tangential electric field $E_1(x_1)$ at the surface is the negative differential of $\phi(x_1)$, and hence its Fourier transform is $\bar{E}_1(\beta) = -j\beta\bar{\phi}(\beta)$. Equation (3.12) thus gives

$$(\varepsilon_0 + \varepsilon_p)\bar{E}_1(\beta) = -j \operatorname{sgn}(\beta)\bar{\sigma}(\beta)$$

This is transformed to the x_1-domain by using the convolution theorem, equation (A.19), noting that the inverse transform of $\operatorname{sgn}(\beta)$ is $j/(\pi x)$ as shown by equation

(A.35). This gives

$$(\varepsilon_0 + \varepsilon_p)E_1(x_1) = \sigma(x_1) * \frac{1}{\pi x_1}. \tag{3.14}$$

The surface potential $\phi(x_1)$ can be obtained by integrating $E_1(x_1)$, and with the aid of equation (3.13) this gives

$$(\varepsilon_0 + \varepsilon_p)\phi(x_1) = -\sigma(x_1) * \ln |x_1|/\pi. \tag{3.15}$$

To obtain solutions for $\phi(x_1)$ and $\sigma(x_1)$, it is also necessary to use the boundary conditions that $\phi(x_1)$ must be constant on any electrode and $\sigma(x_1)$ must be zero at all unmetallised locations. Methods for obtaining solutions will be considered later, in Chapter 4.

3.2. PIEZOELECTRIC HALF-SPACE

In Section 3.1 it was found that the potential and charge density at the surface of a non-piezoelectric half-space are related by an effective permittivity $(\varepsilon_0 + \varepsilon_p)$. Here we consider the effective permittivity for a piezoelectric half-space. The method was first given by Ingebrigtsen [62] and developed later by Greebe *et al.* [64] and by Milsom *et al.* [65, 66]. The theory given here follows most closely that of Milsom. Other approaches, which will not be considered here, are the perturbation theory [67, 68] and normal mode theory [69, 70]; these give results which are essentially the same as the results of the effective permittivity approach.

It is assumed that the potential and the acoustic displacements are proportional to $\exp(j\omega t)$, with the frequency ω positive. As in Section 3.1 we consider initially a harmonic solution with variables proportional to $\exp(j\beta x_1)$, with β real, and generalise later using Fourier synthesis. The procedure is similar to that used for calculating surface wave velocities, Section 2.3.2, but here the electric boundary condition at the surface is not specified. This enables a solution to be obtained for any value of β. Another difference is that positive values of β refer to wave motion propagating in the $-x_1$ direction instead of the $+x_1$ direction. This is done for convenience when using Fourier synthesis.

As in Section 2.3.2, we consider partial waves in which the displacements $\mathbf{u}'$ and potential Φ' have the form

$$\begin{aligned} \mathbf{u}' &= \mathbf{u}_0' \exp(j\gamma x_3) \exp[j(\omega t + \beta x_1)], \\ \Phi' &= \Phi_0' \exp(j\gamma x_3) \exp[j(\omega t + \beta x_1)], \end{aligned} \tag{3.16}$$

where $\mathbf{u}_0'$ and Φ_0' are constants and γ is the x_3-component of the wave vector, which by definition has no x_2-component. These expressions are required to satisfy the equations of motion, equations (2.14), for an infinite medium, with the material tensors rotated into the frame of the axes x_1, x_2, x_3. Substitution into equations (2.14) gives four linear homogeneous equations in the four variables $\mathbf{u}_0'$, Φ_0', and for non-trivial solutions the determinant of coefficients is set to zero. The determinant is an eight-order polynomial in γ and thus gives eight roots, and for each root the equations also give the relative values of $\mathbf{u}_0'$ and Φ_0'. Four of the roots are unacceptable however, because they do not correspond to excitation at the surface. Care is needed

in choosing acceptable roots, because the solution will not in general be a surface wave solution. Complex or imaginary values of γ are acceptable if the imaginary part is negative, so that $\mathbf{u}'$ and Φ' decay away from the surface. Real values of γ give plane wave solutions, and are acceptable only if they carry energy away from the surface. Usually, this requires γ to have its sign opposite to that of β. This is not always the case, however, because for an anisotropic material the power flow direction is not in general collinear with the wave vector, an effect known as beam steering. The power flow direction may be found by examining the variation of phase velocity with propagation direction, as will be shown later for surface waves in Chapter 6.

The four acceptable partial wave solutions are written

$$\begin{aligned} \mathbf{u}'_m &= \mathbf{u}'_{0m} \exp(j\gamma_m x_3) \exp[j(\omega t + \beta x_1)], \\ \Phi'_m &= \Phi'_{0m} \exp(j\gamma_m x_3) \exp[j(\omega t + \beta x_1)], \quad m = 1, 2, 3, 4. \end{aligned} \tag{3.17}$$

The total solution in the half-space has displacements $\tilde{\mathbf{u}}$ and $\tilde{\Phi}$, where the tilde indicates that the solution is harmonic, with variables proportional to $\exp(j\beta x_1)$. The total solution is taken to be a linear combination of the partial waves, so that

$$\begin{aligned} \tilde{\mathbf{u}} &= \sum_{m=1}^{4} A_m \mathbf{u}'_m, \\ \tilde{\Phi} &= \sum_{m=1}^{4} A_m \Phi'_m. \end{aligned} \tag{3.18}$$

The relative values of the constants A_m are determined by the boundary condition that there must be no force on the free surface $x_3 = 0$, that is,

$$T_{13} = T_{23} = T_{33} = 0, \quad \text{at } x_3 = 0, \tag{3.19}$$

with the stresses given by equation (2.9). The electrical boundary conditions are not specified here. We thus have three equations relating the four constants A_m, and hence the relative values of these constants can be found. The relative values of the displacements $\tilde{\mathbf{u}}$ and potential $\tilde{\Phi}$ for the harmonic solution can then be obtained from equation (3.18), giving a solution for any value of β.

For problems concerning electrical excitation at the surface, the variables of interest, constrained by boundary conditions, are the potential and the normal component of the electric displacement $\tilde{\mathbf{D}}$. The electric displacement can be calculated from the potential and the acoustic displacements by using equation (2.10). At the surface, the normal component $\tilde{D}_3$ in the piezoelectric is denoted $\tilde{D}_3(-)$. The potential at the surface is denoted $\tilde{\phi}(x_1)$, so that $\tilde{\phi}(x_1) = \tilde{\Phi}(x_1, 0)$. The ratio $\tilde{D}_3(-)/\tilde{\phi}(x_1)$ is determined by the solution described above, and will in general be a function of β.

In the vacuum, $x_3 > 0$, the potential $\tilde{\Phi}(x_1, x_3)$ must satisfy Laplace's equation $\nabla^2\tilde{\Phi} = 0$. Since $\tilde{\Phi}$ is proportional to $\exp(j\beta x_1)$ and it must vanish at $x_3 = \infty$, the x_3 dependence is $\exp(-|\beta|x_3)$, and so

$$\tilde{\Phi}(x_1, x_3) = \tilde{\phi}(x_1) \exp(-|\beta|x_3), \tag{3.20}$$

for $x_3 > 0$. At the surface $x_3 = 0$, the normal displacement in the vacuum is denoted $\tilde{D}_3(+)$, and this is given by

$$\tilde{D}_3(+) = \varepsilon_0|\beta|\tilde{\phi}(x_1). \tag{3.21}$$

The surface potential $\tilde{\phi}(x_1)$ must of course be the same on both sides of the boundary. However the normal component of displacement can be different. The discontinuity is related to the potential by the *effective permittivity* $\varepsilon_s(\beta)$, defined by

$$\varepsilon_s(\beta) = \frac{\tilde{D}_3(+) - \tilde{D}_3(-)}{|\beta|\tilde{\phi}(x_1)}. \tag{3.22}$$

Here the x_1-dependence cancels on the right side, so that $\varepsilon_s(\beta)$ is not dependent on x_1. Thus the effective permittivity gives the electrical behaviour of the interface between the vacuum and the piezoelectric half-space.

If $\tilde{D}_3(+)$ and $\tilde{D}_3(-)$ differ, there must be free charges present at the surface, implying the presence of electrodes. $\tilde{D}_3(+)$ will then be equal to the charge density on the vacuum side of the electrodes, and $\tilde{D}_3(-)$ is equal to the negative of the charge density on the piezoelectric side. Thus, if the total charge density at x_1, including both sides, is denoted $\tilde{\sigma}(x_1)$, we have

$$\varepsilon_s(\beta) = \frac{\tilde{\sigma}(x_1)}{|\beta|\tilde{\phi}(x_1)}, \tag{3:23}$$

where $\tilde{\sigma}(x_1)$ and $\tilde{\phi}(x_1)$ are both proportional to exp $[j(\omega t + \beta x_1)]$. If $\tilde{D}_3(+) = \tilde{D}_3(-)$ there may be electrodes present with equal and opposite charges on the two sides, giving $\tilde{\sigma}(x_1) = 0$. Alternatively there may be no electrodes and hence no free charges. Some authors exclude the charges on the vacuum side when defining $\varepsilon_s(\beta)$, that is, they omit the term $\tilde{D}_3(+)$ in equation (3.22). This reduces the value of $\varepsilon_s(\beta)$ by an amount ε_0, as can be seen from equation (3.21).

In the above equations the potential $\tilde{\phi}(x_1)$ and charge density $\tilde{\sigma}(x_1)$ are proportional to exp $(j\omega t)$, and the frequency ω was taken to be constant throughout. If ω is changed, the value of $\varepsilon_s(\beta)$ changes, so $\varepsilon_s(\beta)$ is a function of ω as well as β. However, since $\varepsilon_s(\beta)$ is essentially the ratio of $\tilde{D}_3$ to $\tilde{E}_1$, as shown by equation (3.22), it can be seen that it remains unchanged if ω and β are changed in proportion. Thus $\varepsilon_s(\beta)$ is a function of the normalised variable β/ω. This has dimensions the same as the reciprocal of velocity, and is often termed the "slowness". In this chapter the analysis applies for constant frequency, and for brevity the effective permittivity is written as $\varepsilon_s(\beta)$, without showing the frequency dependence explicity.

A more general solution, with surface potential $\phi(x_1)$ and charge density $\sigma(x_1)$, is readily obtained by Fourier synthesis. The method is the same as in the electrostatic case of Section 3.1, so the result follows directly. Thus, comparing with equation (3.12), the general solution obtained from equation (3.23) gives

$$\varepsilon_s(\beta) = \frac{\bar{\sigma}(\beta)}{|\beta|\bar{\phi}(\beta)}, \tag{3.24}$$

where $\bar{\sigma}(\beta)$ and $\bar{\phi}(\beta)$ are respectively the Fourier transforms of $\sigma(x_1)$ and $\phi(x_1)$. Thus, given some general potential function $\phi(x_1)$, the corresponding charge density may be obtained by transforming to obtain $\bar{\phi}(\beta)$, using $\varepsilon_s(\beta)$ to obtain $\bar{\sigma}(\beta)$, and then transforming back to the x_1-domain.

For the general solution the potential $\phi(x_1)$ and charge density $\sigma(x_1)$ are

proportional to exp ($j\omega t$), with the frequency ω regarded as a constant in the above equations. In solving a particular problem it is usually found that the potential and charge density are functions of frequency, so their transforms $\bar{\phi}(\beta)$ and $\bar{\sigma}(\beta)$ will also be functions of frequency. In the Fourier transform, the frequency ω is held constant during the integration. The relationship given by the effective permittivity, equation (3.24), applies for all values of ω.

3.3. SOME PROPERTIES OF THE EFFECTIVE PERMITTIVITY

The effective permittivity described above is a powerful tool for solving problems concerning a one-dimensional set of electrodes on the surface of a piezoelectric half-space. For variables proportional to exp ($j\omega t$), the surface potential $\phi(x_1)$ and charge density $\sigma(x_1)$ are related by the effective permittivity, and the solution is then determined if appropriate boundary conditions are applied. Usually, $\phi(x_1)$ is specified at the electrode locations, while $\sigma(x_1)$ must be zero on all unmetallised regions. Acoustic wave excitation is allowed for implicitly by the definition of the effective permittivity. This includes all forms of acoustic wave that can be excited; thus, in addition to the usual excitation of piezoelectric Rayleigh waves, the effective permittivity will when appropriate include the effects of Bleustein–Gulyaev waves, pseudo-surface waves and bulk waves. In fact, many of the properties of these waves in the material under consideration may be deduced by examining the effective permittivity. However, it should be noted that the permittivity does not show the effect of any acoustic waves which are not piezoelectrically coupled at the surface. Such waves, which may occur in a piezoelectric material, cannot of course be excited by electrodes on the surface; nevertheless, they may be present in a practical device owing to mode conversion at a discontinuity, for example an edge of the substrate.

An important limitation of the method follows from the assumption, used in the derivation, that there are no mechanical forces on the surface. This implies that any electrodes on the surface must be sufficiently thin that they can be assumed to cause negligible mechanical perturbations, that is, mechanical loading is neglected. In practice, this is usually a good approximation.

Generally, the effective permittivity is a complicated function of β, and must be found numerically using the method described in Section 3.2 above. There are however a number of important properties which are readily deduced. Firstly, the function is symmetrical, so that $\varepsilon_s(-\beta) = \varepsilon_s(\beta)$. This follows from the general reciprocity relation, as shown in Appendix B, Section B.4. Secondly, if $\varepsilon_s(\beta)$ is complex this indicates that energy is being radiated away from the surface into the bulk of the material, in the form of acoustic waves. This can be seen from the definition involving the harmonic solution, equation (3.23). The quantity $\tilde{\sigma}(x_1)$ is the charge density on some set of electrodes with edges parallel to the x_2-axis. Since the charge density is time-variant there must in general be currents entering these electrodes from outside. For unit length in the x_2-direction, the current in an interval $\mathrm{d}x_1$ is $J(x_1)\,\mathrm{d}x_1$, where $J(x_1) = j\omega\tilde{\sigma}(x_1)$ is the current density. Now, if there is no power transferred into the system from outside, the current density $J(x_1)$ and the potential $\tilde{\phi}(x_1)$ must be in phase

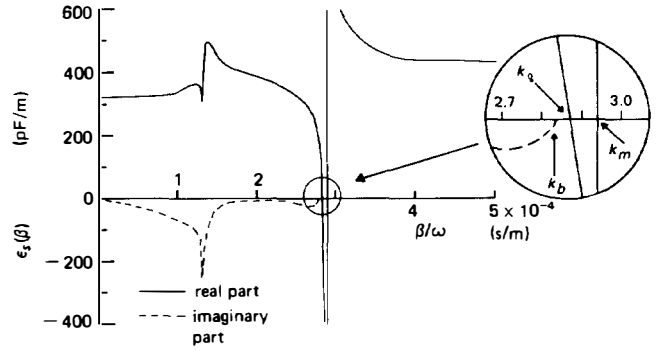

FIGURE 3.1. Effective permittivity for Y, Z lithium niobate. After Milsom *et al.* [65], copyright © 1977 IEEE.

quadrature, and hence, from equation (3.23), $\varepsilon_s(\beta)$ must be real. In the steady-state condition, a net transfer of power can only occur if bulk acoustic waves are radiated away from the surface, so a complex value of $\varepsilon_s(\beta)$ indicates that bulk wave radiation is occurring. It is usually found that $\varepsilon_s(\beta)$ is complex for some values of β and real for other values.

The form of $\varepsilon_s(\beta)$ depends markedly on the type of acoustic wave involved. Usually we shall be concerned with excitation of piezoelectric Rayleigh waves. However, bulk wave excitation has been investigated quite extensively because it occurs in most surface-wave devices to some extent, and in fact some devices use bulk waves as the main form of acoustic propagation. For excitation of Bleustein–Gulyaev waves the effective permittivity can be expressed as an analytic formula [64, 71].

In the description to follow it is assumed that the surface wave involved is a piezoelectric Rayleigh wave, as is usually the case in practice. An example is Y, Z lithium niobate, and the effective permittivity is shown for this material in Figure 3.1. The permittivity has a zero and a pole at two values of β close together. From Equation (3.23), the zero corresponds to a surface wave solution for a free surface, since the charge density is zero. The wavenumber here is denoted k_0, which is taken to be positive, so that the zeros of $\varepsilon_s(\beta)$ occur at $\beta = \pm k_0$. The pole of $\varepsilon_s(\beta)$ indicates a surface-wave solution for a metallised surface, since it gives a finite charge density and zero potential. In this case the wavenumber is $k_m > 0$, so that the poles occur at $\beta = \pm k_m$. The surface wave velocities for these two cases are v_0 and v_m, so that $k_0 = \omega/v_0$ and $k_m = \omega/v_m$. At a lower value of β, denoted by k_b in Figure 3.1, the permittivity becomes complex, and it remains complex for all smaller values of β. In this region, bulk wave excitation is occurring. For most practical purposes it is not necessary to consider bulk wave excitation in any detail.

An important parameter is the differential of the permittivity at the free-surface wavenumber k_0. This quantity is directly related to the amplitude of the surface waves generated by a transducer. The relationship is derived in Appendix B, Section B.6. The same quantity also relates the power flow of a surface wave to its associated surface potential, as shown below.

Surface-wave Power Flow. The power flow is derived by a method similar to that of Ingebrigtsen [62]. We consider the harmonic solution for the half space, with the surface potential $\tilde{\phi}(x_1)$ and charge density $\tilde{\sigma}(x_1)$ both proportional to $\exp(j\beta x_1)$. If β is assumed to have a small imaginary part the amplitude of the wave rises or falls exponentially with x_1, and the surface wave power density may be obtained by considering the electrical power applied to the system from outside.

We consider the power transferred in a width W in the x_2-direction. If $J(x_1)$ is the current density at the surface, the power flowing into the system, in a small interval Δx_1, is given by

$$\Delta P(x_1) = \tfrac{1}{2} W \operatorname{Re}[\tilde{\phi}^*(x_1) J(x_1)] \Delta x_1, \tag{3.25}$$

where the asterisk indicates a complex conjugate and the current density $J(x_1) = j\omega\tilde{\sigma}(x_1)$. The potential and charge density are written, omitting a term $\exp(j\omega t)$, as

$$\tilde{\phi}(x_1) = \tilde{\phi}_0 \exp(j\beta x_1),$$

$$\tilde{\sigma}(x_1) = \tilde{\sigma}_0 \exp(j\beta x_1),$$

where $\tilde{\phi}_0$ and $\tilde{\sigma}_0$ are constant[illegible], this gives

$$\Delta P(x_1) = -\tfrac{1}{2}\omega W \exp(-2\beta_i x_1) \operatorname{Im}[\tilde{\phi}_0^* \tilde{\sigma}_0] \Delta x_1. \tag{3.26}$$

Now, for real β we have $\tilde{\sigma}_0 = f(\beta)\tilde{\phi}_0$ with, from equation (3.23),

$$f(\beta) = |\beta| \varepsilon_s(\beta). \tag{3.27}$$

It is assumed that $f(\beta)$ can be continued analytically for complex values of β near the real axis, that is, for small β_i. Equation (3.26) thus becomes

$$\Delta P(x_1) = -\tfrac{1}{2}\omega W |\tilde{\phi}_0|^2 \exp(-2\beta_i x_1) \operatorname{Im}[f(\beta)] \Delta x_1. \tag{3.28}$$

Using the Cauc [illegible] = 0, we have for small β_i

$$\left[\frac{\partial f(\beta)}{\partial \beta}\right] \tag{3.29}$$

Now, since the potential $\tilde{\phi}(x_1)$ is proportional to $\tilde{\phi}_0 \exp(-\beta_i x_1)$ the power of the wave, $P_s(x_1)$, must have the form

$$P_s(x_1) = C |\tilde{\phi}_0|^2 W \exp(-2\beta_i x_1), \tag{3.30}$$

where C is a constant. The change of $P_s(x_1)$ over a distance Δx_1 is equated with $\Delta P(x_1)$ of equation (3.28), giving

$$C = -\tfrac{1}{4}\omega \left[\frac{\partial f(\beta)}{\partial \beta}\right]_{\beta_i = 0}, \tag{3.31}$$

where equation (3.29) has also been used.

In the limit $\beta_i \to 0$ the power of the wave, P_s, is independent of x_1. We consider a wave propagating on a free surface, so that $\beta = k_0$. For this case, using equation

(3.27) and noting that $\varepsilon_s(k_0) = 0$, we have

$$P_s = -\tfrac{1}{4}\omega W|\tilde{\phi}_0|^2 k_0 \left[\frac{\mathrm{d}\varepsilon_s(\beta)}{\mathrm{d}\beta}\right]_{k_0}. \tag{3.32}$$

It is convenient to define a real positive constant Γ_s by

$$\frac{1}{\Gamma_s} = -k_0\left[\frac{\mathrm{d}\varepsilon_s(\beta)}{\mathrm{d}\beta}\right]_{k_0}. \tag{3.33}$$

This is independent of the frequency ω; it depends only on the properties of the material and the orientation. The power flow of the surface wave is thus

$$P_s = \tfrac{1}{4}\omega W|\tilde{\phi}_0|^2/\Gamma_s. \tag{3.34}$$

Milsom *et al.* [65] give an alternative derivation.

Ingebrigtsen's Approximation. A convenient approximate form for $\varepsilon_s(\beta)$, suitable when the main acoustic wave present is a piezoelectric Rayleigh wave, has been given by Ingebrigtsen [62, 72]. Since $\varepsilon_s(\beta)$ has a zero at $\beta = k_0$, it must be proportional to $(\beta - k_0)$ when β is close to k_0, by Taylor's theorem. Similarly, $1/\varepsilon_s(\beta)$ is zero at $\beta = k_m$, the wavenumber for the metallised case, so $\varepsilon_s(\beta)$ is proportional to $(\beta - k_m)^{-1}$ for β close to k_m. Thus $\varepsilon_s(\beta)$ must be proportional to $(\beta - k_0)/(\beta - k_m)$. This is modified a little to make it an even function of β, giving the approximate formula

$$\varepsilon_s(\beta) \approx A\frac{\beta^2 - k_0^2}{\beta^2 - k_m^2}, \tag{3.35}$$

where A is a constant. Differentiating this equation with respect to β, the constant Γ_s, defined by equation (3.33), is given by

$$\Gamma_s = \frac{v_0^2/v_m^2 - 1}{2A} \approx \frac{1}{A}\frac{v_0 - v_m}{v_0}, \tag{3.36}$$

where the approximate form is obtained by using $v_0 \approx v_m$.

The value of the constant A cannot be given unambiguously, because equation (3.35) is only an approximate form for the permittivity. However, for a non-piezoelectric material the permittivity is equal to $(\varepsilon_0 + \varepsilon_p)$, as shown by equation (3.7). In this case $k_m = k_0$ and hence equation (3.35) gives $A = \varepsilon_0 + \varepsilon_p$, with ε_p defined by

$$\varepsilon_p = (\varepsilon_{11}\varepsilon_{33} - \varepsilon_{13}^2)^{1/2}. \tag{3.37}$$

For a piezoelectric material, an appropriate value for A may be obtained by comparing the power flow formula, equation (3.34), with the corresponding result

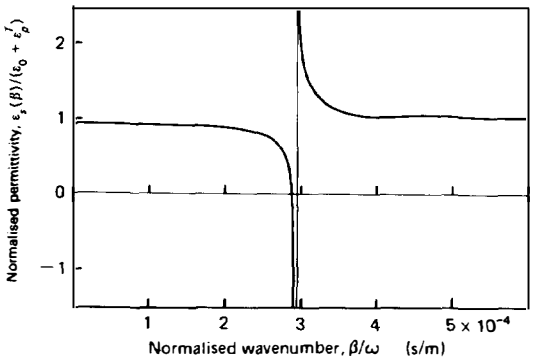

FIGURE 3.2. Effective permittivity for Y, Z lithium niobate, using Ingebrigtsen's approximation.

obtained by perturbation theory, given in Chapter 2, equation (2.48). Using equation (3.36) for Γ_s, this gives

$$A \approx \varepsilon_0 + \varepsilon_p^T,$$

where ε_p^T is defined as in equation (3.37), with the permittivity elements ε_{ij} measured at constant stress. This value for A is adequate for most practical purposes, and has the merit that it is easily evaluated for many materials, using readily available data. A further justification is given in Section 3.5 below. Thus, Ingebrigtsen's approximation, equation (3.35), becomes

$$\varepsilon_s(\beta) \approx (\varepsilon_0 + \varepsilon_p^T) \frac{\beta^2 - k_0^2}{\beta^2 - k_m^2} \tag{3.38}$$

and the constant Γ_s is

$$\Gamma_s \approx \frac{1}{\varepsilon_0 + \varepsilon_p^T} \frac{v_0 - v_m}{v_0}. \tag{3.39}$$

A relation almost identical to equation (3.38) can be obtained by using normal mode theory [70].

The permittivity is plotted on Figure 3.2, using equation (3.38) with data appropriate for lithium niobate ($v_0 = 3488\,\text{m/s}$, $v_m = 3404\,\text{m/s}$). The exact solution for this case is shown in Figure 3.1.

The approximate form of the permittivity is convenient for analysis of practical devices, because it depends on only three variables, and numerical data for these are readily available. The main limitation is that bulk wave excitation is excluded, since the function is real for all values of β. It should be noted that the approximation is only valid for piezoelectric Rayleigh waves; for example, it cannot be used for leaky waves or Bleustein–Gulyaev waves.

3.4. GREEN'S FUNCTION

The effective permittivity $\varepsilon_s(\beta)$ relates the charge density to the surface potential in the β-domain. However, here we are concerned with problems in which the boundary conditions are expressed in the x_1-domain. The relationship in this domain can be expressed by using a Green's function [65, 66].

From equation (3.24) the surface potential and charge density, in the β-domain, are related by

$$\bar{\phi}(\beta) = \bar{G}(\beta, \omega)\bar{\sigma}(\beta), \tag{3.40}$$

where

$$\bar{G}(\beta, \omega) = [|\beta|\varepsilon_s(\beta)]^{-1}. \tag{3.41}$$

In the x_1-domain the surface potential $\phi(x_1)$ and the charge density $\sigma(x_1)$ are the inverse transforms of $\bar{\phi}(\beta)$ and $\bar{\sigma}(\beta)$. Generally, these functions also depend on the frequency ω, but here ω is taken to be a constant. Now, by the convolution theorem of Fourier analysis [Appendix A, equation (A.19)] we may transform equation (3.40) from the β-domain to the x_1-domain by transforming the two functions on the right side individually and the convolving. Thus, if $G(x_1, \omega)$ is the inverse transform of $\bar{G}(\beta, \omega)$, we have

$$\begin{aligned} \phi(x_1) &= G(x_1, \omega) * \sigma(x_1) \\ &= \int_{-\infty}^{\infty} G(x_1 - x_1', \omega)\sigma(x_1')\, dx_1', \end{aligned} \tag{3.42}$$

where the asterisk indicates convolution. The function $G(x_1, \omega)$ is the *Green's function*. It can be interpreted as the surface potential produced by a line charge at $x_1 = 0$, as can be seen by putting $\sigma(x_1) = \delta(x_1)$ in equation (3.42). Transforming equation (3.41) we have

$$G(x_1, \omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\exp(j\beta x_1)}{|\beta|\varepsilon_s(\beta)}\, d\beta. \tag{3.43}$$

Using reciprocity, it is shown in Appendix B that this is an even function of x_1, that is $G(-x_1, \omega) = G(x_1, \omega)$.

Green's Function for Piezoelectric Rayleigh Waves. We consider the form of the Green's function for the particular case of a substrate supporting propagation of piezoelectric Rayleigh waves, as in for example Y, Z lithium niobate. In addition to surface waves, excitation of bulk waves can occur, as shown by the fact that $\varepsilon_s(\beta)$ is complex for some values of β (Figure 3.1). Milsom *et al.* [65] have shown that the Green's function $G(x_1, \omega)$ may be regarded as a sum of three terms, giving contributions due to surface wave excitation, electrostatic effects, and bulk wave excitation.

The surface wave term is deduced by considering the generation of surface waves by a transducer, consisting of a set of electrodes occupying a finite region of x_1, with voltages applied to them. The charge density on the electrodes is $\sigma(x_1)$, with Fourier

transform $\bar{\sigma}(\beta)$. The potential associated with these waves is derived in Appendix B, Section B.6, and is given by

$$\phi(x_1) = j\Gamma_s\bar{\sigma}(\mp k_0, \omega) \exp(\mp jk_0x_1), \tag{3.44}$$

where the upper signs refer to waves radiated in the $+x_1$ direction, and the lower signs to waves radiated in the $-x_1$ direction. This potential is considered to arise from a surface-wave component $G_s(x_1, \omega)$ of the Green's function. By equation (3.42), this is the surface-wave potential obtained when $\sigma(x_1) = \delta(x_1)$, that is, for $\bar{\sigma}(\beta) = 1$. Thus, from equation (3.44)

$$G_s(x_1, \omega) = j\Gamma_s \exp(-jk_0|x_1|). \tag{3.45}$$

The surface-wave component of $G(x_1, \omega)$ arises because $\varepsilon_s(\beta)$ is zero at $\beta = k_0$, so that the integrand of equation (3.43) has a pole at this point. There is also a pole at $\beta = 0$. If the variation of $\varepsilon_s(\beta)$ is ignored in this region, then from equation (3.40), $\bar{\phi}(\beta)$ is proportional to $\bar{\sigma}(\beta)/|\beta|$. A relation of this form applies for the electrostatic case, as shown by equation (3.12) of Section 3.1. In the x_1-domain this contribution to the potential is given by an electrostatic Green's function $G_e(x_1)$, and by comparing with equation (3.15) this can be written

$$G_e(x_1) = -\frac{\ln|x_1|}{\pi(\varepsilon_0 + \varepsilon_p^T)}. \tag{3.46}$$

The use of the constant $(\varepsilon_0 + \varepsilon_p^I)$ will be justified later in this section, though equation (3.15) shows that it is correct if the material is not piezoelectric. It should be noted that $G_e(x_1)$ is independent of the frequency ω.

In addition to surface wave effects and electrostatic effects, the total Green's function $G(x_1, \omega)$ must also include the effects of bulk waves. It is assumed that these can be accounted for by adding a further term $G_b(x_1, \omega)$, so that the total Green's function of equation (3.43) is given by

$$G(x_1, \omega) = G_e(x_1) + G_s(x_1, \omega) + G_b(x_1, \omega), \tag{3.47}$$

with $G_e(x_1)$ and $G_s(x_1, \omega)$ given by equations (3.46) and (3.45) respectively. The bulk wave term $G_b(x_1, \omega)$ is more difficult to obtain analytically, though it can be obtained numerically from the effective permittivity [65]. For most purposes the details of this function will not be required.

Approximate Form for the Green's Function. An approximate Green's function, convenient for solving many practical problems, is obtained by omitting the bulk wave contribution from the exact expression of equation (3.47). Thus, using equations (3.46) and (3.45) for the electrostatic and surface wave terms, the approximate Green's function is

$$\begin{aligned} G(x_1, \omega) &\approx G_e(x_1) + G_s(x_1, \omega) \\ &= -\frac{\ln|x_1|}{\pi(\varepsilon_0 + \varepsilon_p^T)} + j\Gamma_s \exp(-jk_0|x_1|). \end{aligned} \tag{3.48}$$

To confirm the validity of this, we deduce the corresponding effective permittivity and compare it with Ingebrigtsen's approximation. From equation (3.41), the permittivity is the reciprocal of $|\beta|$. $\bar{G}(\beta, \omega)$, where $\bar{G}(\beta, \omega)$ is the transform of the right side of equation (3.48). The transform of the exponential term is given by equation (A.41), and the transform on $\ln |x_1|$ can be taken as $-\pi/|\beta|$, which follows by comparing equation (3.15) with equation (3.12). We thus find

$$\bar{G}(\beta, \omega) \approx [(\varepsilon_0 + \varepsilon_p^T)\,|\beta|]^{-1} + j\Gamma_s[\pi\delta(\beta - k_0) + \pi\delta(\beta + k_0) + 2jk_0/(\beta^2 - k_0^2)]. \tag{3.49}$$

This function is infinite at $\beta = \pm k_0$, so that $\varepsilon_s(\beta)$ is zero at these points, as required. It is also necessary that $\bar{G}(\beta, \omega)$ should be zero at $\beta = \pm k_m$, so that $\varepsilon_s(\beta)$ is infinite at these points. This condition requires an appropriate value for the constant Γ_s. Setting $\bar{G}(\pm k_m, \omega) = 0$ in equation (3.49) gives

$$\Gamma_s = \frac{k_m^2 - k_0^2}{2(\varepsilon_0 + \varepsilon_p^T)\,k_0 k_m} \approx \frac{1}{\varepsilon_0 + \varepsilon_p^T}\frac{v_0 - v_m}{v_0}, \tag{3.50}$$

in good agreement with the expression deduced previously, equation (3.39).

In equation (3.49), the term $1/(\beta^2 - k_0^2)$ makes $\bar{G}(\beta, \omega)$ infinite at $\beta = \pm k_0$ and hence the reciprocal of $\bar{G}(\beta, \omega)$ is zero at these points. It follows that the reciprocal of $\bar{G}(\beta, \omega)$ is not affected by the presence of the delta-functions, which can therefore be omitted when calculating $\varepsilon_s(\beta)$. Using equation (3.50) for Γ_s, the effectivity permittivity is

$$\varepsilon_s(\beta) = \frac{1}{|\beta|\bar{G}(\beta, \omega)} = (\varepsilon_0 + \varepsilon_p^T)\frac{\beta^2 - k_0^2}{(|\beta| - k_m)\,(|\beta| + k_0^2/k_m)}. \tag{3.51}$$

This is almost identical with Ingebrigtsen's approximation, equation (3.38); it becomes exactly the same if the term k_0^2/k_m is replaced by k_m, which is quantitatively similar. The agreement confirms the validity of the approximate Green's function, equation (3.48), and also justifies the use of the constant $(\varepsilon_0 + \varepsilon_p^T)$ in the electrostatic Green's function, equation (3.46).

3.5. OTHER APPLICATIONS OF THE EFFECTIVE PERMITTIVITY

The above development shows how the effective permittivity $\varepsilon_s(\beta)$ may be used to relate the surface charge density and potential for a piezoelectric half-space, with the region above the piezoelectric assumed to be a vacuum. In Chapter 4 this method will be used for the analysis of surface-wave transducers on a half-space. However, the concept of the effective permittivity may be generalised to analyse a number of other problems, and here we digress briefly to discuss these.

We first consider the coupling between a piezoelectric half-space and a plane at a height h above the surface [62, 67, 68, 70]. As shown in Figure 3.3(a), the surface of

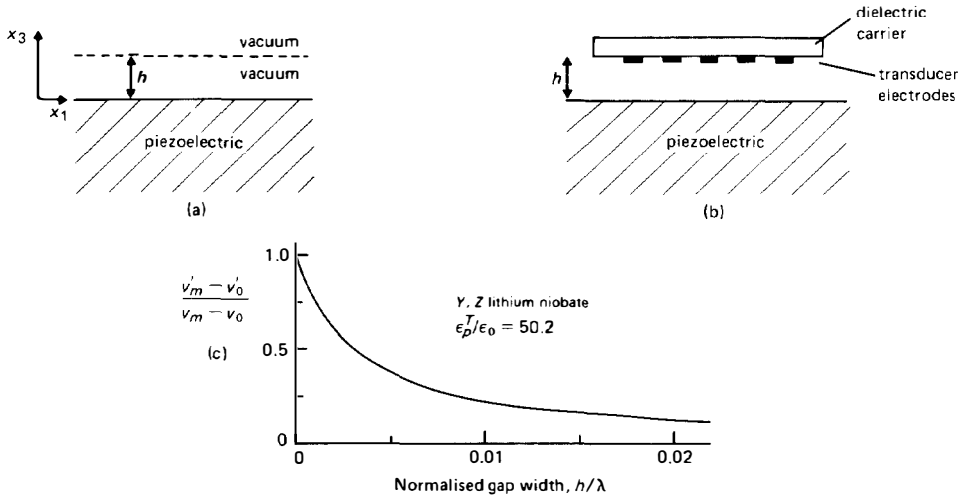

FIGURE 3.3. Piezoelectric coupling across a gap.

the piezoelectric is at $x_3 = 0$, and free charges are allowed to exist only at the plane $x_3 = h$. Considering a harmonic solution, with the relevant quantities proportional to exp $(j\beta x_1)$, we can define an effective permittivity $\varepsilon_s'(\beta)$ for the plane $x_3 = h$:

$$\varepsilon_s'(\beta) = \frac{\tilde{\sigma}}{|\beta|\tilde{\Phi}(h)}, \tag{3.52}$$

where $\tilde{\Phi}(h)$ and $\tilde{\sigma}$ are the potential and charge density at $x_3 = h$. This function may be expressed in terms of the surface permittivity $\varepsilon_s(\beta)$, defined in equation (3.23) above. At the plane $x_3 = 0$, the value of $\tilde{D}_3$ in the piezoelectric is, from equations (3.21) and (3.22),

$$\tilde{D}_3(-) = \tilde{\phi}|\beta|[\varepsilon_0 - \varepsilon_s(\beta)], \tag{3.53}$$

where $\tilde{\phi}$ is the potential at $x_3 = 0$. This relation is valid irrespective of the electrical conditions above the surface $x_3 = 0$. In the vacuum region $0 < x_3 < h$ the potential $\tilde{\Phi}$ has terms proportional to exp $(\pm \beta x_3)$, from Laplace's equation, and in the region $x_3 > h$ the potential is proportional to exp $(-|\beta|x_3)$. The potential must be continuous everywhere, and $\tilde{D}_3$ must be continuous at $x_3 = 0$. The discontinuity of $\tilde{D}_3$ at $x_3 = h$ gives the charge density, $\tilde{\sigma}$. Using these relations, the effective permittivity at $x_3 = h$ is found to be

$$\varepsilon_s'(\beta) = \frac{\varepsilon_s(\beta)[1 + \tanh(|\beta|h)]}{1 - [1 - \varepsilon_s(\beta)/\varepsilon_0]\tanh(|\beta|h)}. \tag{3.54}$$

This function may be used to analyse excitation by a transducer located at the plane $x_3 = h$, as in Figure 3.3(b). In practice the transducer would be supported by

a dielectric, but this does not appreciably alter the effective permittivity. As for excitation at the surface, the permittivity has a zero at $\beta = k_0'$, say, and a pole at $\beta = k_m'$, say, with corresponding surface wave velocities $v_0' = \omega/k_0'$ and $v_m' = \omega/k_m'$. The difference between these velocities is a measure of the coupling strength. Equation (3.54) shows that $\varepsilon_s'(\beta)$ is zero when $\varepsilon_s(\beta)$ is zero, so that $v_0' = v_0$. The metallised-surface velocity v_m' depends on h, approaching v_0' for large h. Assuming that the surface wave is a piezoelectric Rayleigh wave, Ingebrigtsen's approximation, equation (3.38), may be used for $\varepsilon_s(\beta)$ in equation (3.54) to evaluate v_m'. Noting that the four velocities v_m', v_0', v_m and v_0 are numerically similar, this gives

$$\frac{v_m' - v_0'}{v_m - v_0} \approx \left[1 - \frac{1 + \varepsilon_p^T/\varepsilon_0}{1 - \coth(k_0 h)}\right]^{-1}. \tag{3.55}$$

Here the left side is a measure of the coupling strength for the plane at $x_3 = h$, dividied by the coupling strength obtained at the surface $x_3 = 0$. The function is shown in Figure 3.3(c), assuming a Y, Z lithium niobate half-space, with the height h normalised to the wavelength $\lambda = 2\pi/k_0$. The coupling strength decreases very rapidly with h. It can be concluded that, to be practically effective, a transducer held above the surface would need to be very close, with a gap width much less than the wavelength. In practice, such a small gap is usually awkward to obtain, and so transducers are usually deposited directly on the substrate. However, coupling across a gap has the advantage that the transducer is movable, and this has been exploited as a rapid method of assessing the surface-wave properties of materials [73].

The quantitative results obtained from equation (3.55) have been found to give excellent agreement with accurate calculations, obtained without the use of approximations [67, 70]. This gives further confirmation of the validity of Ingebrigtsen's approximation, equation (3.38), and in particular confirms the use of the coefficient $(\varepsilon_0 + \varepsilon_p^T)$.

Another type of problem which can be analysed using the effective permittivity is that of coupling to a semiconductor above the piezoelectric surface [64, 70, 71, 74]. Usually the semiconductor is considered to be separated from the surface by a small gap, as in Figure 3.4(a). As before, the ratio of $\tilde{D}_3/\tilde{\Phi}$ at the piezoelectric surface is determined by the effective permittivity, while the same ratio at the semiconductor surface is determined by the semiconductor equations. Taken together, these relations give the wavenumber β, which in this case is complex. As in the case of a transducer held above the surface, the gap has to be very small if significant interactions are to be obtained. Coupling to a semiconductor has been exploited in several ways, which are reviewed by Kino [75]. Amplification of the surface wave can be obtained by applying a drift field in the x_1-direction, so that the carrier drift velocity in the semiconductor exceeds the surface wave velocity. Typically, a gain of 50 dB/cm can be obtained, though a large drift field, typically a few kV/cm, is required. Coupling to a semiconductor is also used in a type of convolver, where a non-linear effect in the semiconductor is used to mix two surface-wave signals.

The effective permittivity concept can also be applied to the analysis of transducers using piezoelectric layers. As illustrated in Figure 3.4(b), a piezoelectric film is

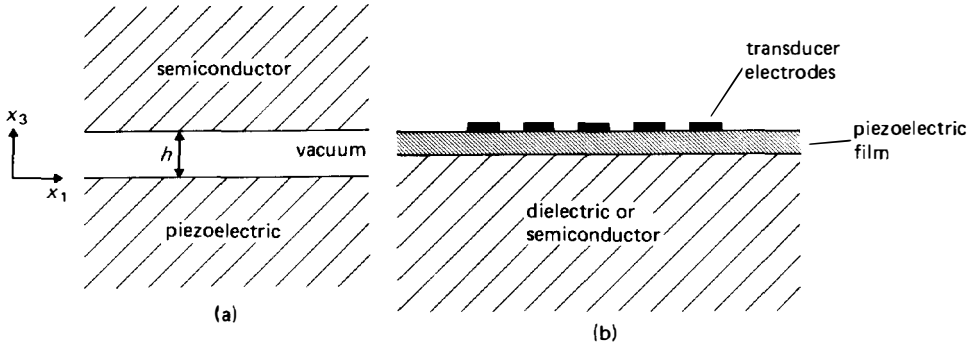

FIGURE 3.4. Other excitation problems.

deposited on a non-piezoelectric substrate, and this enables an interdigitial transducer to be used to generate surface waves. The transducer may be on top of the film, as in Figure 3.4(b), or at the interface between the film and the substrate. Experimentally, the commonest material for the film is zinc oxide and, with a glass substrate, this is used in some bandpass filters for television receivers [76]. Alternatively, a zinc oxide film can be used to generate surface waves on a silicon substrate, enabling a surface wave device to be integrated with circuitry on the same substrate [77].

For the layered system, an effective permittivity can be defined in the usual way, relating the charge density to the potential at the plane of the transducer. As before, two key parameters are the surface wave velocities obtained when there is no metallisation at the transducer plane, and when a uniform conducting sheet is present at this plane. The velocity difference gives a measure of the coupling strength. For simplicity, Ingebrigtsen's approximation [equation (3.38)] can be used for the permittivity, with the wavenumbers k_0 and k_m obtained from the two surface wave velocities. Much of the transducer analysis of Chapter 4 can then be used. However, there is the complication that for the layered system the surface wave is dispersive, as already noted in Section 2.3.6, so that k_0 and k_m are not proportional to ω. An analysis based on normal mode theory is given by Kino and Wagers [78].