CHAPTER XII

Piezoelectric Crystals in Oscillator Circuits

By I. E. FAIR

12.1 INTRODUCTION

A STUDY or an explanation of the performance of a piezoelectric crystal in an oscillator circuit involves a study or explanation of oscillator circuits in general and a study of the crystal as a circuit element. Nicolson[1] appears to have been the first to discover that a piezoelectric crystal had sufficient coupling between electrical electrodes and mechanical vibratory movement so that when the electrodes were suitably connected to a vacuum tube circuit, sustained oscillations were produced. In such an oscillator the mechanical oscillatory movement of the crystal functions as does the electrical oscillatory circuit of the usual vacuum tube oscillator. His circuit is shown in Fig. 12.1. Cady[2] independently though later made the same discovery, but he utilized it somewhat differently and expressed it differently. He found that when the electrodes of a quartz crystal are connected in certain ways to an electric oscillator circuit, the frequency is held very constant at a value which coincides with the period of the vibrating crystal. He made the further discovery that due to the very sharp resonance properties of the quartz crystal, the constancy in frequency to be secured was far greater than could be obtained by any purely electric oscillator.

The development of analytical explanations of the crystal controlled oscillator came along rather slowly. Cady explained the control in terms of operation upon the electrical oscillator to which the crystal was attached. He said that the "capacity" of the crystal changes rapidly with frequency in the neighborhood of mechanical resonance, even becoming negative. This "capacity" connected across the oscillator tuned circuit or in other places prevented the frequency from changing to any extent, as any frequency change caused such a "capacity" change in the crystal as to tend to tune the circuit in the other direction. Cady, however, devised one circuit, Fig. 12.2, in which no tuned electrical circuit was used, but he confined his explanation to "a mechanically tuned feedback path from the plate to the grid of the amplifier". Pierce[3] came along later with a two-electrode crystal connected between plate and grid, and no tuned circuit, and also with a

[1] See bibliography end of chapter.

two-electrode crystal connected between grid and cathode and no tuned circuit, where the operation would not be satisfactorily explained by Cady's

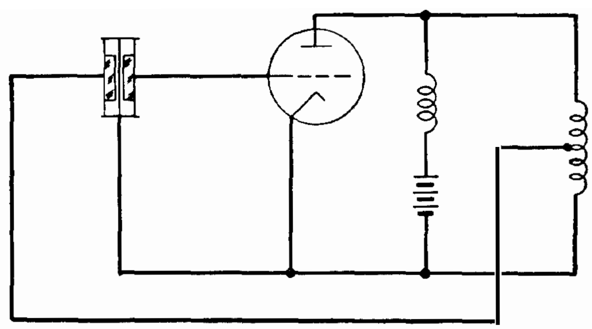

Fig. 12.1—Nicolson's crystal oscillator circuit

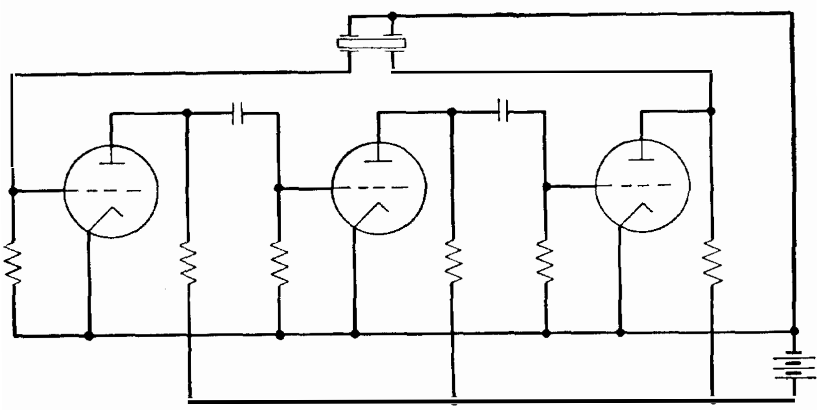

Fig. 12.2—Cady's oscillator circuit using a crystal as a "mechanically tuned feedback path"

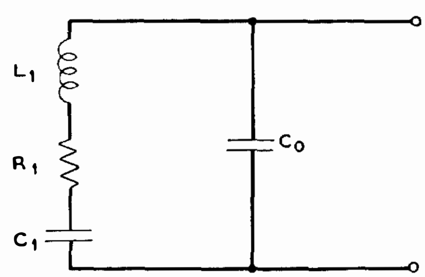

Fig. 12.3—Equivalent electrical circuit of a piezoelectric crystal near its resonant frequency

method. His circuits would require the crystal to exhibit inductive reactance, rather than the capacitance Cady spoke of. Miller[4] also produced a circuit with a two-electrode crystal connected between grid and cathode but with a tuned circuit in the plate lead, which circuit required the crystal to provide inductive reactance.

It was not until after Van Dyke[5] showed that the crystal could be represented by the circuit network of Fig. 12.3 that it was possible to explain these various phenomena. With this view of the crystal, and using the differential equation method of circuit analysis, Terry[6] pointed out that, as with electrical oscillators, the frequency is not completely governed by the resonant element, in this case the crystal, but is influenced somewhat by the circuit elements. The circuit as a whole is quite complex and the equations are difficult to use. Wright[7] and Vigoureux[8] also made analyses of the Pierce type oscillator. Because of the complexity of the equations, the frequency, amplitude, or activity are not computed directly, but the effects of the circuit variables are analyzed in a qualitative manner and the results compared with experimental data.

Oscillators employing crystals may be classified in a number of ways. One classification is based upon whether or not the circuit without the crystal is in itself an oscillator. If it is, the oscillator is called a "crystal controlled" oscillator. If it is not, it is called a "crystal" oscillator. All of Cady's oscillator circuits, except the one shown in Fig. 12.2, are of the first named class. This type of circuit will oscillate at a frequency determined by the tuned circuit if the crystal becomes broken or disconnected, or if high resistance develops in the crystal, or if the electric tuned circuit should become tuned too far from the resonant frequency of the crystal. This property at times is an advantage and at other times a disadvantage. This type of circuit will oscillate under control of the crystal with much less active crystals than most of the other types.

Nicolson's, Pierce's, Cady's of Fig. 12.2 and Miller's oscillators belong to the second named class. They will cease oscillating if the crystal breaks, develops high resistance or is disconnected. Failure of the oscillator to function at all then serves as a warning that something has happened to the crystal.

This second named class of crystal oscillators has been used much more than the first named. The crystal is the principal frequency determining element in the circuit. Often there are required only resistances, or resistances and an inductance, as the other elements to embody along with the vacuum tube and crystal. The simplicity, low costs, and usually no tuning, have made this class attractive. Most analytical studies of oscillator circuits have been made upon this class. For that reason the discussion in this chapter will be limited to this class.

An analytic study of the crystal oscillator can readily start by looking upon the oscillator as consisting only of inductances, capacitances, and resistances, along with the vacuum tube. The crystal is replaced by the proper circuital elements arranged as in Fig. 12.3. This circuit or equivalent of the crystal is that of a series resonant circuit having capacitance paralleling it. The circuit will show both phenomena of series resonance and parallel resonance, the two frequencies being very close together. By making suitable measurements on a crystal, the magnitudes of the inductance, resistance, and the two capacitances can be determined. It is usually found that the series inductance is computed as hundreds or thousands of henries, and the series capacitance is a small fraction of a micromicrofarad. The magnitudes of the inductance and capacitance are beyond what it is possible to construct in the usual forms of building inductances and capacitances. This accounts for its superior frequency control properties.

Although reducing the crystal to an equivalent electrical circuit provides one notable step in understanding the performance of the crystal oscillator, it does not readily lead to a full understanding. The electric oscillator in itself is not fully and completely analyzed in all its ramifications, although it has been under study for over 25 years. These studies have been mathematical and experimental in character, but in all cases it appears there have been approximations of some kind, made because the variable impedance characteristics both of the plate circuit and the grid circuit of the tubes did not lend themselves readily to a rigorous analysis. The earlier investigations assumed a linear relation between grid voltage and plate current and assumed constant plate impedance. Later investigations brought in further elements and further variables, the different investigators attacking the problem in different ways and attempting to prove different points. By this means a large number of factors in oscillators have been ascertained to a first degree of approximation so that a qualitative review of the performance of the electric oscillator is very well known. It is the quantitative view upon the first order magnitude which is still difficult or uncertain. This is particularly true of the crystal oscillator because of the slightly different circuit.

It is proposed, therefore, in this paper to cover briefly a number of the studies on crystal oscillators so as to point out the different modes of attack and the different behavior points in the oscillators which the various investigators have studied. After covering these points, there will be discussed the frequency control properties of the crystal and the frequency stability of crystal oscillators. The performance of the crystal in the oscillator with respect to activity is then treated. There will be introduced two new yardsticks for measuring or indicating crystal quality, one called "figure of merit" and the other called "performance index." These are related to the crystal constants and paralleling capacitances which are usually involved.

They will be defined and their method of use and application in oscillators will be pointed out.

12.2 Solution by Differential Equations

The most direct method of determining the oscillating conditions in a circuit is to analyze the differential equation for the current in some particular branch of the circuit. The relations existing between the coefficients determine whether the current builds up, dies out, or is maintained at a constant value and frequency. Unfortunately the equations resulting from the application of this method to the crystal oscillator circuit are quite complicated. However, lower order differential equations result from the

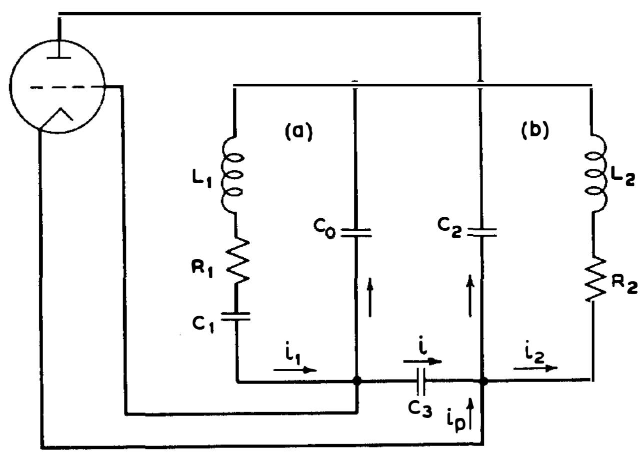

Fig. 12.4—Equivalent circuit of oscillator with crystal connected between grid and plate

application of this method to similar electric oscillator circuits, and certain qualitative information obtained from the latter is applicable to crystal oscillators. Thus Heising's[9] analysis of the Colpitts and Hartley circuits gives much information directly applicable to the Pierce and Miller types of crystal oscillators. From this the circuit conditions necessary for oscillations to exist and the effect of certain circuit variables upon the frequency are ascertained. The more complex qualitative view is given by Terry[6] who shows the relations of the coefficients of linear differential equations of the 2nd, 3rd, and 4th orders, and applies them to the analysis of three common types of crystal oscillator circuits. The resulting equations, together with certain qualitative information regarding their interpretation, are repeated here. In making this analysis the grid current is disregarded and the static tube characteristic is considered linear.

The equation is the same for the three types of circuits considered and is derived for the current i_1, in Figs. 12.4 and 12.5, although it may be set up in terms of any of the currents or voltages existing in the circuit. It is of the form

$$\frac{d^4 i_1}{dt^4} + P_1 \frac{d^3 i_1}{dt^3} + P_2 \frac{d^2 i_1}{dt^2} + P_3 \frac{di_1}{dt} + P_4 i_1 = 0 \tag{12.1}$$

The P coefficients are functions of the circuit elements and are defined for each type of circuit in the following sections.

The solution of (12.1) normally represents a doubly periodic function arising from the two coupled antiresonant meshes (a) and (b). The normal

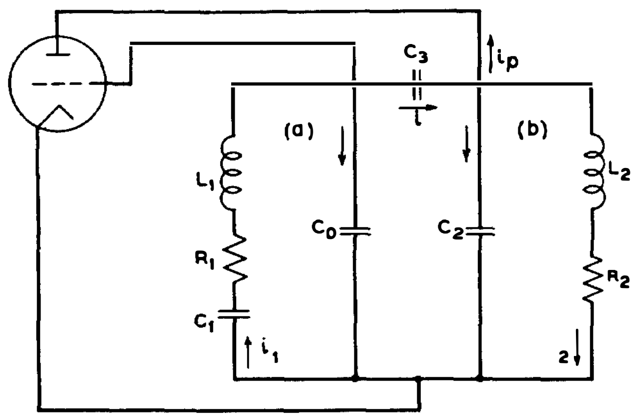

Fig. 12.5—Equivalent circuit of oscillator with crystal connected between grid and cathode

modes of oscillation consist of two currents in each mesh with frequency and damping factors β_1 and α_1, β_2 and α_2 respectively.

The conditions for undamped oscillation as derived from the general equation (12.1) are expressed in terms of the coefficients by

$$\frac{P_3}{P_1} = \frac{P_2 \pm \sqrt{P_2^2 - 4P_4}}{2} \tag{12.2}$$

and the angular frequencies are

$$\beta^2 = \frac{P_2 \pm \sqrt{P_2^2 - 4P_4}}{2} \tag{12.3}$$

where the plus sign gives the condition for one damping factor to be zero and the minus sign that for the other to be zero.

The frequency at which oscillations are maintained is determined by the required phase relation of voltages applied to the tube. With crystal from grid to plate, as in Fig. 12.4, the phase difference of grid and plate voltages is such that the circuit oscillates at only one of the normal modes, and with crystal connected between grid and cathode, as in Fig. 12.5, it oscillates at the other only.

12.21 Crystal Between Grid and Plate

With the crystal connected between the grid and plate of the tube, as in Fig. 12.4, the coefficients of the general equation (12.1) are

$$\left.\begin{aligned}
P_1 &= \frac{R_1}{L_1} + \frac{R_2}{L_2} + \frac{1}{R_p C_b'} \\
P_2 &= \frac{1}{L_1 C_a} + \frac{R_1 R_2}{L_1 L_2} + \frac{1}{L_2 C_b} + \frac{1}{R_p}\left(\frac{R_1}{L_1} + \frac{R_2}{L_2}\right)\frac{1}{C_b'} \\
P_3 &= \frac{R_2}{L_1 L_2 C_a} + \frac{R_1}{L_1 L_2 C_b} + \frac{1}{R_p}\left(\frac{1}{L_1 C_a C_b'} + \frac{R_1 R_2}{L_1 L_2 C_b'} - \frac{1}{L_1 C_m C_m'}\right) \\
P_4 &= \frac{1}{L_1 L_2 C_a C_b} - \frac{1}{L_1 L_2 C_m^2} + \frac{R_2}{R_p}\left(\frac{1}{L_1 L_2 C_a C_b'} - \frac{1}{L_1 L_2 C_m C_m'}\right)
\end{aligned}\right\} \tag{12.4}$$

where

$$\frac{1}{C_a} = \frac{1}{C_1} + \frac{1}{C_0} - \frac{C_x}{C_0^2} \qquad \frac{1}{C_m} = \frac{C_x}{C_2 C_0}$$

$$\frac{1}{C_b} = \frac{1}{C_2} - \frac{C_x}{C_2^2} \qquad \frac{1}{C_m'} = \frac{1}{C_m} - \frac{\mu C_x}{C_0 C_3}$$

$$\frac{1}{C_b'} = \frac{1}{C_b} + \frac{\mu C_x}{C_2 C_3} \qquad \frac{1}{C_x} = \frac{1}{C_0} + \cancel{\frac{1}{C_2}} + \frac{1}{C_3}$$

μ = the amplification factor of the tube.

$$R_p = \frac{\partial e_p}{\partial i_p} \quad (e_g \text{ constant})$$

The uncoupled damping factors, α_a and α_b, the uncoupled undamped angular frequencies, β_a and β_b, and the coupling coefficient τ may be introduced as follows:

$$\alpha_a = \frac{R_1}{2L_1}, \qquad \beta_a^2 = \frac{1}{L_1 C_a} \qquad \tau^2 = \frac{C_a C_b}{C_m^2}$$

$$\alpha_b = \frac{R_2}{2L_2}, \qquad \beta_b^2 = \frac{1}{L_2 C_b}$$

Note that C_a is the total capacitance across L_1 and R_1, and C_b is the total capacitance across L_2 and R_2.

The coefficients of (12.4) become

$$\left.\begin{aligned}
P_1 &= 2(\alpha_a + \alpha_b) + \frac{1}{R_p C_b'} \\
P_2 &= \beta_a^2 + 4\alpha_a\alpha_b + \beta_b^2 + \frac{1}{R_p}(\alpha_a + \alpha_b)\frac{2}{C_b'} \\
P_3 &= 2(\alpha_b\beta_a^2 + \alpha_a\beta_b^2) + \frac{1}{R_p}\left[(\beta_a^2 + 4\alpha_a\alpha_b)\frac{1}{C_b'} - \frac{1}{L_1 C_m C_m'}\right] \\
P_4 &= \beta_a^2\beta_b^2\left[1 - \tau^2 + \frac{R_2}{R_p}\left(\frac{C_b}{C_b'} - \frac{C_m}{C_m'}\tau^2\right)\right]
\end{aligned}\right] \qquad (12.5)$$

The coefficients as given by (12.5) satisfy (12.2) and (12.3) only when the plus sign is used.

The equations are simplified by dividing through by β_a^2 thus

$$\frac{P_3}{\beta_a^2 P_1} = \frac{\dfrac{P_2}{\beta_a^2} + \sqrt{\left(\dfrac{P_2}{\beta_a^2}\right)^2 - \dfrac{4P_4}{\beta_a^4}}}{2} \qquad (12.6)$$

$$\frac{\beta^2}{\beta_a^2} = \frac{\dfrac{P^2}{\beta_a^2} + \sqrt{\left(\dfrac{P_2}{\beta_a^2}\right)^2 - \dfrac{4P_4}{\beta_a^4}}}{2} \qquad (12.7)$$

which gives the ratio of driven frequency of the crystal to its undriven value. The common variable R_p must satisfy both (12.6) and (12.7). The method of computing the frequency would be to solve for R_p in (12.6) and substitute in (12.7). However, the equations are too complicated a function of R_p for this to be practical. Terry solved them graphically by plotting (12.6) and (12.7) as functions of R_p for assigned values of the circuit, and the intersection of these curves gave the frequency for the different circuit conditions. The results are shown in Fig. 12.6. The G-P curves show the frequency change as a function of plate circuit tuning for the grid to plate connection of the crystal.

12.22 Crystal Between Grid and Cathode

With the crystal connected between the grid and cathode of the tube, the circuit is as shown in Fig. 12.5. The coefficients of equation (12.1) are as follows:

$$\left.\begin{aligned}
P_1 &= \frac{R_1}{L_1} + \frac{R_2}{L_2} + \frac{1}{R_p C_b''} \\
P_2 &= \frac{1}{L_1 C_a} + \frac{R_1 R_2}{L_1 L_2} + \frac{1}{L_2 C_b} + \frac{1}{R_p}\left(\frac{R_1}{L_1} + \frac{R_2}{L_2}\right)\frac{1}{C_b''} \\
P_3 &= \frac{R_2}{L_1 L_2 C_a} + \frac{R_1}{L_1 L_2 C_b} + \frac{1}{R_p}\left(\frac{1}{L_1 C_a C_b''} + \frac{R_1 R_2}{L_1 L_2 C_b''} - \frac{1}{L_1 C_m C_m''}\right) \\
P_4 &= \frac{1}{L_1 L_2 C_a C_b} - \frac{1}{L_1 L_2 C_m^2} + \frac{R_2}{R_p}\left(\frac{1}{L_1 L_2 C_a C_b''} - \frac{1}{L_1 L_2 C_m C_m''}\right)
\end{aligned}\right] \tag{12.8}$$

With the substitution of uncoupled frequencies, damping factors and coupling coefficient as described in the previous section, they become

$$\left.\begin{aligned}
P_1 &= 2(\alpha_a + \alpha_b) + \frac{1}{R_p C_b''} \\
P_2 &= \beta_a^2 + 4\alpha_a\alpha_b + \beta_b^2 + \frac{1}{R_p}(\alpha_a + \alpha_b)\frac{2}{C_b''} \\
P_3 &= 2(\alpha_b\beta_a^2 + \alpha_a\beta_b^2) + \frac{1}{R_p}\left[(\beta_a^2 + 4\alpha_a\alpha_b)\frac{1}{C_b''} - \frac{1}{L_1 C_m C_m''}\right] \\
P_4 &= \beta_a^2\beta_b^2\left[1 - \tau^2 + \frac{R_2}{R_p}\left(\frac{C_b}{C_b''} - \frac{C_m}{C_m''}\tau^2\right)\right]
\end{aligned}\right] \tag{12.9}$$

Where

$$\frac{1}{C_m''} = \frac{1}{C_m} + \frac{\mu}{C_d}$$

$$\frac{1}{C_b''} = \frac{1}{C_b} + \frac{\mu}{C_m} \qquad \mu = \text{amplification factor of tube}$$

$$\frac{1}{C_m} = \frac{C_x}{C_2 C_0} \qquad R_p = \frac{\partial e_p}{\partial i_p} \quad (e_g \text{ constant})$$

$$C_b = C_2 + \frac{C_0 C_3}{C_0 + C_3} \qquad \frac{1}{C_x} = \frac{1}{C_0} + \frac{1}{C_2} + \frac{1}{C_3}$$

$$C_d = C_0 + \frac{C_2 C_3}{C_2 + C_3}$$

These equations of conditions for oscillation in this case satisfy (12.4) and (12.5) only when the minus sign is used. That is

$$\frac{P_3}{P_1} = \frac{P_2 - \sqrt{P_2^2 - 4P_4}}{2} \tag{12.10}$$

$$\beta^2 = \frac{P_2 - \sqrt{P_2^2 - 4P_4}}{2} \tag{12.11}$$

Again dividing by β_a^2 to obtain the frequency as a ratio of driven to undriven crystal frequency, we have

$$\frac{P_3}{\beta_a^2 P_1} = \frac{\dfrac{P_2}{\beta_a^2} - \sqrt{\left(\dfrac{P_2}{\beta_a^2}\right)^2 - \dfrac{4P_4}{\beta_a^4}}}{2} \tag{12.12}$$

$$\frac{\beta^2}{\beta_2^2} = \frac{\dfrac{P_2}{\beta_2^2} - \sqrt{\left(\dfrac{P_2}{\beta_a^2}\right)^2 - \dfrac{4P_4}{\beta_a^4}}}{2} \tag{12.13}$$

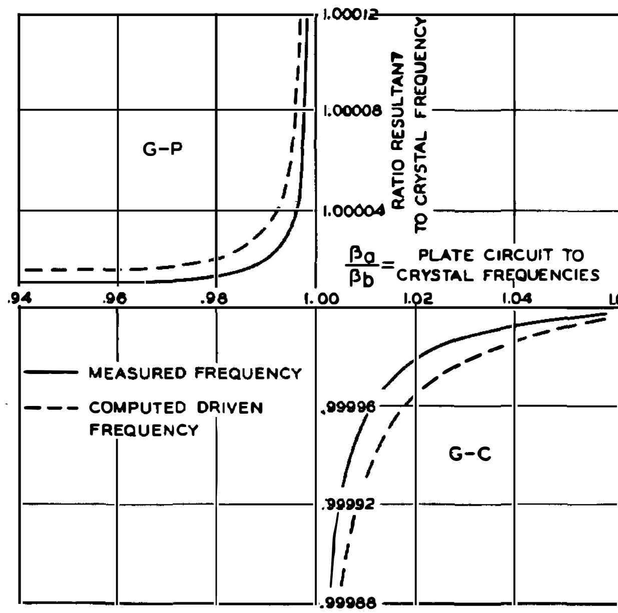

Fig. 12.6—The oscillating frequency as a function of the plate circuit frequency for the crystal connected grid to plate (G-P) and grid to cathode (G-C)

The frequency change as a function of plate circuit tuning was determined graphically in the manner described in section (12.21) and the curves are shown in Fig. 12.6 as the G-C curves.

12.23 Resistance Load Circuit

This is a special case of Plate-Grid connection of the crystal described in section (12.21) in which the plate circuit consists of a capacitance and resistance in parallel. This is a very common Pierce type of oscillator circuit and has the advantage that no tuning adjustment is necessary when using crystals of different frequencies.

Since this circuit is singly periodic, the differential equation for i_1 is of the third order and is derived from (12.1) by setting the plate inductance L_2 of the P coefficients equal to zero. The general equation then becomes

$$\frac{d^3 i_1}{dt^3} + P_1\frac{d^2 i_1}{dt^2} + P_2\frac{di_1}{dt} + P_3 i_1 = 0 \tag{12.14}$$

where

$$\left.\begin{aligned} P_1 &= \frac{R_1}{L_1} + \frac{1}{R_2 C_b} + \frac{1}{R_p C_b'} \\ P_2 &= \frac{1}{L_1 C_a} + \frac{R_1}{R_2 L_1 C_b} + \frac{R_1}{R_p L_1 C_b'} \\ P_3 &= \frac{1}{R_2 L_1 C_a C_b} - \frac{1}{R_2 L_1 C_m^2} + \frac{1}{R_p}\left(\frac{1}{L_1 C_a C_b'} - \frac{1}{L_1 C_m C_m'}\right) \end{aligned}\right] \tag{12.15}$$

With the substitution of the uncoupled damping factors and frequencies, (12.15) becomes

$$\left.\begin{aligned} P_1 &= 2\alpha_a + \frac{1}{R_2 C_b} + \frac{1}{R_p C_b'} \\ P_2 &= \beta_a^2 + \frac{2\alpha_a}{R_2 C_b} + \frac{2\alpha_a}{R_p C_b'} \\ P_3 &= \frac{\beta_a^2}{R_2 C_b} - \frac{1}{R_2 L_1 C_m^2} + \frac{1}{R_p}\left(\frac{\beta_a^2}{C_b'} - \frac{1}{L_1 C_m C_m'}\right) \end{aligned}\right] \tag{12.16}$$

The frequency as obtained from (12.14) is

$$\beta^2 = P_2 \tag{12.17}$$

with the conditions for oscillation

$$P_2 = \frac{P_3}{P_1} \tag{12.18}$$

obtained by setting the damping factor α equal to zero. The ratio of driven to undriven frequency is obtained by dividing (12.17) and (12.18) by β_a^2. That is

$$\frac{\beta^2}{\beta_a^2} = \frac{P_2}{\beta_a^2} = \frac{P_3}{\beta_a^2 P_1} \tag{12.19}$$

12.24 Interpretation of the Equations

It is learned from this analysis that the frequency of oscillation while governed principally by the frequency of the crystal also depends upon all the constants of the circuit. The effect of the plate circuit impedance is

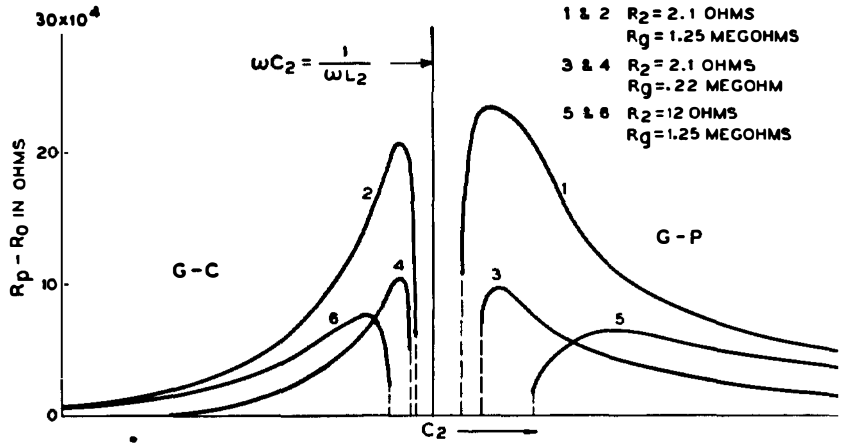

Fig. 12.7—Calculated increase in mean plate resistance against capacitance of the oscillatory circuit

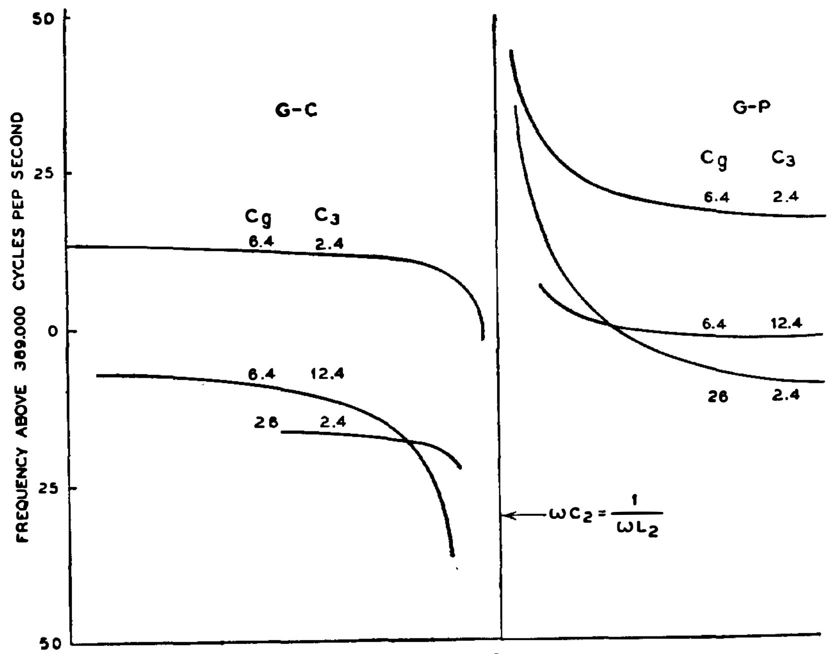

Fig. 12.8—Experimental curves, showing the influence of interelectrode capacitances on the frequency

shown in Fig. 12.6. It is pointed out that the effect of the crystal resistance R_1 is to decrease the frequency for the G-C connection and increase the frequency for the G-P connection. The discrepancy between the measured and experimental values shown on the curves is attributed to the difference between chosen and actual value of R_1. The effect of the input loss of the tube is not shown because the grid current was disregarded; however, this loss may be reduced to an equivalent R_1. The resistance of the plate circuit R_2 affects the frequency in a similar manner. The effects of these resistances on frequency are less for low values of plate circuit impedances.

The required value of R_p gives a measure of amplitude of oscillation because it is necessary for oscillations to build up until the internal plate resistance is equal to the calculated value. It is found that R_p increases

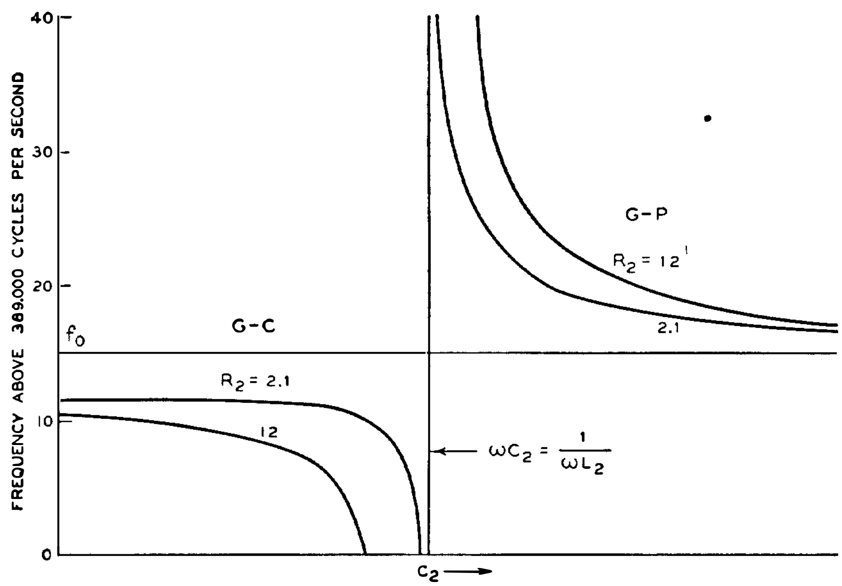

Fig. 12.9—Experimental curves, showing the relation between the frequency and the resistance of the oscillatory circuit

gradually to a maximum as the common frequency for the two types of circuits is approached then abruptly drops.

Vigoureux[8] analyzes the crystal oscillator in a manner similar to Terry and correlates his interpretations of the equations with considerable experimental data, some of which are shown in Figs. 12.7, 12.8, 12.9 and 12.10. He points out that there is an optimum value of grid capacitance with the crystal connected between grid and plate and a certain amount of grid-plate capacitance is required when the crystal is connected between grid and cathode.

Wheeler[10] does not assume a linear static tube characteristic but represents it by a three-term nonlinear expression. The results are more complex and it is necessary in the end to disregard certain resistance terms.

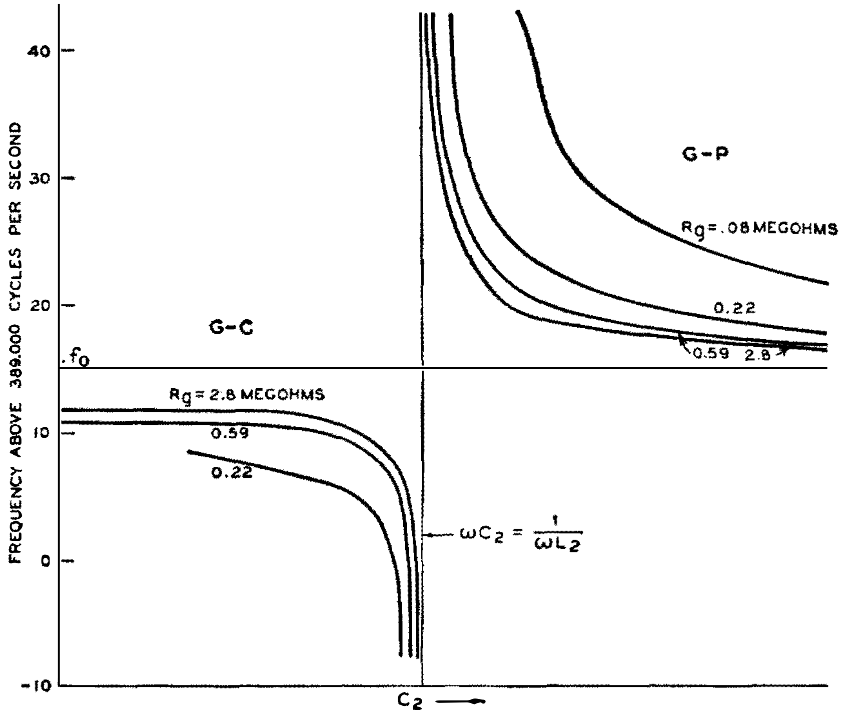

Fig. 12.10—Experimental curves, showing the relation between the frequency of a quartz oscillator and the capacitance of the oscillatory circuit for various values of the grid leak

12.3 Solution by Complex Functions

The analysis of oscillator circuits may be simplified when only steady state conditions are of interest, all circuit elements are considered linear, and certain requirements which define the conditions necessary for oscillations are known. Under these conditions the common circuit equations of complex numbers give the information desired. In this method the voltage induced in the plate circuit is considered the driving voltage which produces a current in the grid circuit (see Fig. 12.11). The network be-

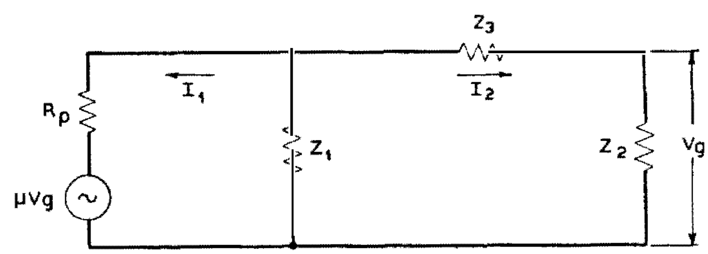

Fig. 12.11—Equivalent circuit of Pierce and Miller types of oscillators shown in Fig. 12.12

tween plate and grid may be of any type and oscillations are maintained when the total gain through the circuit is unity (gain of tubes = attenuation through circuit) and the phase relation between the induced plate voltage (μV_g) and the grid voltage (V_g) is 180° (the phase shift is zero when μ is considered negative). The expression $\mu\beta = 1$ defines these requirements. Llewellyn[11] applies this method to oscillator circuits in general and Koga[12] uses it to study the crystal oscillator in particular.

The equations are developed on the assumption that the grid-voltage vs. plate-current characteristic of the tube is linear. The fundamental equation of $\mu\beta$ is given by the ratio of the voltage developed across the grid

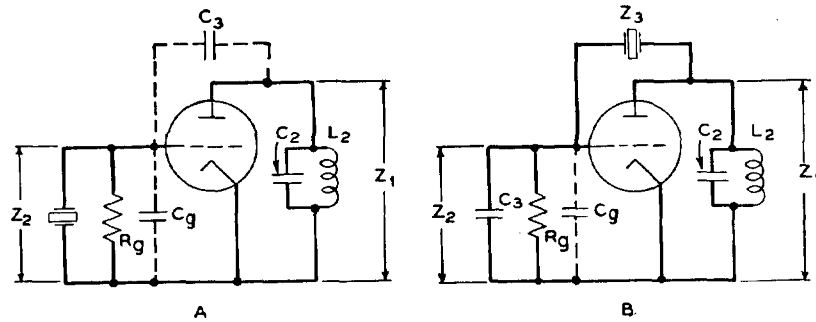

Fig. 12.12—Circuit diagrams of crystal oscillators with crystal connected from grid to cathode (A) and grid to plate (B)

circuit by the fictitious driving voltage μV_g to the voltage V_g. For the general circuit, Fig. 12.11, it is

$$\mu\beta = \frac{I_2 Z_2}{V_g} = \frac{-\mu Z_1 Z_2}{R_p Z_s + Z_1(Z_2 + Z_3)} \tag{12.20}$$

where

$$Z_s = Z_1 + Z_2 + Z_3$$

It is more convenient to write this in the reciprocal form

$$\frac{1}{\mu\beta} = \frac{R_p Z_s + Z_1(Z_2 + Z_3)}{-\mu Z_1 Z_2} = 1 \tag{12.21}$$

In applying this to the crystal oscillator, the additional assumptions made are that the grid current is negligible and the resistance in the plate impedance Z_1 is zero.

12.31 Crystal Grid to Cathode

With the assumptions made above and the crystal connected from grid to cathode of the tube according to Fig. 12.12A, the impedances are

$$Z_1 = jX_1 \qquad Z_2 = R_{cg} + jX_{cg} \qquad Z_3 = jX_3$$

where R_{cg} is the effective resistance and X_{cg} the effective reactance of the crystal, the grid resistance R_g and the circuit capacitance C_g in parallel at the oscillating frequency. Upon substitution of these in (12.21)

$$\frac{1}{\mu\beta} = \frac{[R_{cg}R_p - X_1(X_{cg} + X_3)] + j(X_1R_{cg} + R_pX_s)}{\mu X_1X_{cg} - j\mu X_1R_{cg}} = 1 \tag{12.22}$$

where

$$X_s = X_1 + X_{cg} + X_3$$

Thus $\frac{1}{\mu\beta}$ is of the form

$$\frac{1}{\mu\beta} = P + jQ$$

which means that $P = 1$ and $Q = 0$.
This results in the following two equations obtained from the real and imaginary parts of (12.22) both of which must be satisfied for oscillations to be maintained.

The real part of (12.22) gives

$$-R_p = \frac{X_3(\mu + 1)(R_{cg}^2 + X_{cg}^2) + X_{cg}X_3}{R_{cg}(X_1 + X_3)} \tag{12.23}$$

and from the imaginary part is obtained

$$X_s = \frac{X_1X_3 - R_{cg}R_p}{R_p\phi_{cg}} \tag{12.24}$$

where $\phi_{cg} = \frac{X_{cg}}{R_{cg}}$ (This ratio of reactance to resistance of the crystal circuit will appear in various equations later.)

Equation (12.24) may be said to define the oscillating frequency and is in a convenient form to examine the effect of the various circuit variables upon the frequency. The impedances X_1, R_{cg}, X_{cg} and X_3 may be thought of as forming an oscillating loop (See Fig. 12.11). For oscillations to be maintained in such a loop the sum of the reactances must equal zero and the sum of the resistances must equal zero. But the sum of the resistances cannot equal zero since R_{cg} is the only resistance in the loop and it is positive. It is therefore necessary for the driving voltage μV_g to act upon the circuit and supply the energy dissipated by the resistance R_{cg} (and also R_p through which the energy is supplied). This alters the frequency somewhat and it is no longer determined by setting the three reactances equal to zero as may be seen by equation (12.24). Nevertheless, the right side of this equation is small and approaches zero when R_{cg} approaches zero. It also becomes very small when the reactance X_1 becomes small and R_{cg} is not too great. This is the same condition as found by the differential equation method and illustrated in Fig. 12.6 by the *G-C* curves. As the plate reactance X_1 is made small the frequency increases and approaches a limiting value but does not quite reach it. This limiting value is the frequency at which $X_s = 0$. The dotted *G-C* curve shows that R_{cg} tends to lower the frequency and determines how close the limiting frequency is approached. The plate circuit resistance R_2 (component of Z_1), if considered, would have a similar effect as shown by the experimental curves 12.9. The grid resistance R_g (component of Z_2) has an opposite effect as shown in Fig. 12.10 because increasing R_g is equivalent to decreasing the effective resistance R_{cg}.

The effect of the various constants of the crystal and circuit upon the oscillating frequency may be obtained from (12.24) upon substitution of these constants for the reactances and resistance R_{cg}. The equation is put in a more convenient form for this purpose by Koga.[12] Equation (12.21) is written,

$$\frac{1}{Z_2} + \frac{1}{Z_3} + \frac{\mu}{Z_3\left(1 + R_p/Z_1 + \dfrac{R_p}{Z_2 + Z_3}\right)} = 0 \qquad (12.25)$$

It is assumed that the current in the grid branch is small compared to the plate current. This reduces the equation to

$$\frac{1}{Z_2} + \frac{1}{Z_3} + \frac{\mu}{Z_3(1 + R_p/Z_1)} = 0 \qquad (12.26)$$

The admittance expression for the crystal is

$$\frac{1}{Z_c} = \frac{R_1 - j\left[\omega L_1 - \dfrac{1}{\omega C_1} - \dfrac{1}{\omega(C_0 + C_4)}\right]}{R_1^2 + \left[\omega L_1 - \dfrac{1}{\omega C_1} - \dfrac{1}{\omega(C_0 + C_4)}\right]^2}\left(\frac{C_4}{C_0 + C_4}\right)^2 + j\omega\frac{C_0 C_4}{C_0 + C_4} \qquad (12.27)$$

Note that Koga considers the air gap capacitance C_4 as a separate factor but it may be included in the other constants of the crystal in which case the equivalent circuit is as shown in Fig. 12.3. With the crystal connected between grid and cathode the various circuit admittances are:

$$\frac{1}{Z_1} = \frac{1}{j\omega L_2} + j\omega C_2$$

$$\frac{1}{Z_2} = \frac{1}{Z_c} + \frac{1}{R_g} + j\omega C_g$$

$$\frac{1}{Z_3} = j\omega C_3$$

After substitution of these values of the admittances in (12.26) and setting the real and imaginary parts equal to zero, the following two equations are obtained:

$$\frac{R_1}{R_1^2 + \left[\omega L_1 - \frac{1}{\omega C_1} - \frac{1}{\omega(C_0 + C_4)}\right]^2}\left(\frac{C_4}{C_0 + C_4}\right)^2 + \frac{1}{R_g} - \mu\omega C_3 \frac{R_p\left(\frac{1}{\omega L_2} - \omega C_2\right)}{1 + R_p^2\left(\frac{1}{\omega L_2} - \omega C_2\right)^2} = 0 \tag{12.28}$$

and

$$\frac{\omega L_1 - \frac{1}{\omega C_1} - \frac{1}{\omega(C_0 + C_4)}}{R_1^2 + \left[\omega L_1 - \frac{1}{\omega C_1} - \frac{1}{\omega(C_0 + C_4)}\right]^2} = \omega\left(\frac{C_0 + C_4}{C_4}\right)^2 \left[C_g + \frac{C_0 C_4}{C_0 + C_4} + C_3 + \frac{\mu C_3}{1 + R_p^2\left(\frac{1}{\omega L_2} - \omega C_2\right)^2}\right] \tag{12.29}$$

Equation (12.28) gives the conditions necessary for oscillations and (12.29) gives the oscillating frequency as explained below:

12.32 Frequency of Oscillations for G-C Connection of Crystal

Equation (12.29) for frequency is simplified by the fact that over the narrow frequency range considered, the reactances of L_2 and C_2 do not change appreciably. Also at the oscillating frequency,

$$R_1^2 \ll \left[\omega L_1 - \frac{1}{\omega C_1} - \frac{1}{\omega(C_0 + C_4)}\right]^2$$

With these approximations (12.29) may be written

$$\omega^2 = \frac{1}{L_1 C_1} + \frac{1}{L_1 C_0}\left[1 - \frac{1}{\frac{C_0 + C_t}{C_t} + \frac{C_0}{C_4}}\right] \tag{12.30}$$

where

$$C_t = C_g + C_3 + \frac{\mu C_3}{1 + R_p^2\left(\frac{1}{\omega_0 L_2} - \omega_0 C_2\right)^2}$$

and ω_0 is a constant approximating the oscillating frequency.

Since the frequency is a function of the internal plate resistance of the tube R_p and this is in turn a function of the other circuit variables, the frequency equation (12.30) is not sufficient to calculate the frequency. However, qualitative effects of the various circuit components upon frequency are obtained by assuming R_p an independent variable. It is readily seen that an increase in R_p increases the frequency. The effect of the air gap between crystal and electrodes, which is represented by the capacitance C_4, and the effect of the capacitance across the crystal C_g are illustrated in Fig. 12.13.* To determine the frequency change caused by tuning of

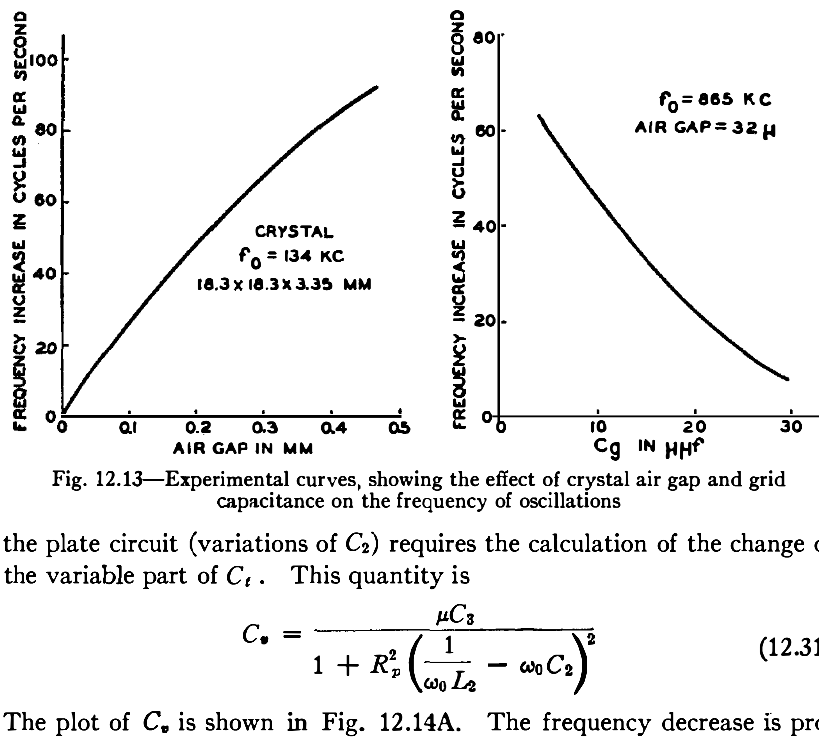

Fig. 12.13—Experimental curves, showing the effect of crystal air gap and grid capacitance on the frequency of oscillations

the plate circuit (variations of C_2) requires the calculation of the change of the variable part of C_t. This quantity is

$$C_v = \frac{\mu C_3}{1 + R_p^2\left(\dfrac{1}{\omega_0 L_2} - \omega_0 C_2\right)^2} \tag{12.31}$$

The plot of C_v is shown in Fig. 12.14A. The frequency decrease is proportional to the increase in C_v. This is indicated in Fig. 12.14B. Oscillations stop before the point $\omega_0 C_2 = \dfrac{1}{\omega_0 L_2}$ is reached. The frequency thus varies in the same manner as shown in Fig. 12.6 but the curve is reversed because of the fact that the independent variable is taken as C_2 instead of the frequency function of C_2.

The frequency change resulting from variations in the grid-plate capacitance C_3 depends also upon the value of C_v as seen from (12.31). It is also

* See also: "The Piezoelectric Resonator and the Effect of Electrode Spacing upon Frequency," Walter G. Cady, *Physics*, Vol. 7, July, 1936.

seen that the smaller the value of C_2 (lower the plate reactance) the less effect will the tube constants μ, R_p and C_3 have upon the frequency. The circuit is therefore more stable. For this reason it has become customary to measure the frequency of crystals with the capacitance C_2 reduced to a value below that which gives maximum amplitude of oscillations.

12.33 Amplitude of Oscillations for G-C Connection of Crystal

A measure of the amplitude of oscillations is obtained from (12.28) which expresses the necessary conditions for oscillations to be maintained. In order for oscillations to start the expression must be negative, and, as the amplitude builds up, R_p increases which reduces the negative terms

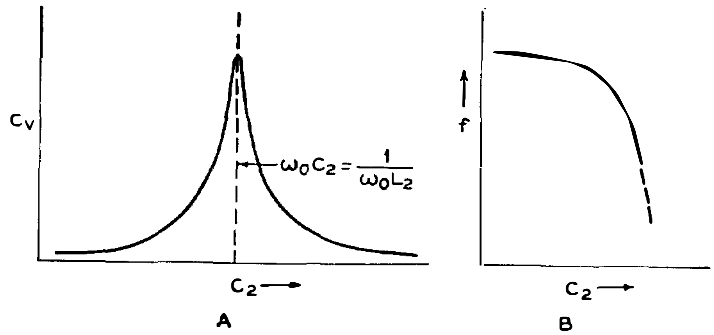

Fig. 12.14—The variation of grid to cathode capacitance (A) and oscillator frequency (B) with change in plate circuit capacitance. Crystal connected grid to cathode

until the equality is satisfied. The difference between the positive and negative terms is therefore a measure of the amplitude of oscillations.

Equation (12.28) may be written

$$\psi - \left[\Phi_0 + \frac{1}{R_g}\right] = A \tag{12.32}$$

where A is a measure of the amplitude,

$$\psi = \mu C_3 \omega_0 \frac{R_p\left(\frac{1}{\omega_0 L_2} - \omega_0 C_2\right)}{1 + R_p^2\left(\frac{1}{\omega_0 L_2} - \omega_0 C_2\right)^2} \tag{12.33}$$

and

$$\Phi_0 = R_1 \omega_0^2 \left(\frac{C_0 + C_4}{C_4}\right)^2 \left[\frac{C_0 C_4}{C_0 + C_4} + C_g + C_3 + \frac{\mu C_3}{1 + R_p^2\left(\frac{1}{\omega_0 L_2} - \omega_0 C_2\right)^2}\right]^2 \tag{12.34}$$

where again R_1^2 is assumed small compared to

$$\left[\omega_0 L_1 - \frac{1}{\omega_0 C_1} - \frac{1}{\omega_0(C_0 + C_4)}\right]^2$$

and ω_0 is considered a constant.

Equation (12.32) shows that in order to obtain a large amplitude ψ should be large and Φ_0 should be small. With this in mind equations (12.33)

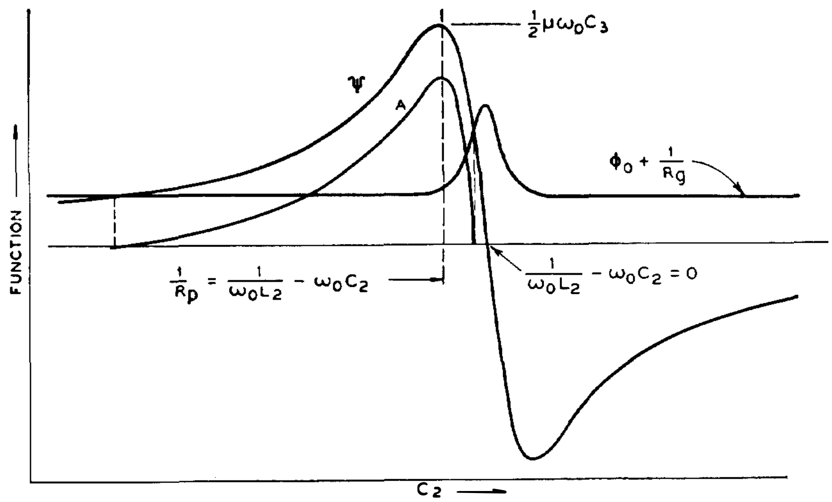

Fig. 12.15—Functions from which the activity variations (A) are determined as the plate circuit capacitance is varied. Crystal connected grid to cathode

and (12.34) may be analyzed to determine the relation between the circuit components and amplitude. It is found that for maximum amplitude

C_g and R_1 should be small,
C_4 should be large,
C_3 has an optimum value, and
R_g should be large.

As to the plate circuit, the amplitude is maximum when $\frac{1}{R_p} = \frac{1}{\omega_0 L_2} - \omega_0 C_2$. A plot of ψ and $\Phi_0 + \frac{1}{R_g}$ is shown in Fig. 12.15. The difference between these two curves is a measure of the amplitude and is shown by curve A. Oscillations can exist only where ψ lies over $\Phi_0 + \frac{1}{R_g}$. The sharpness of ψ varies considerably with the value of R_p and the resistance of the $L_2 - C_2$ circuit. The latter is disregarded for simplicity. Here again the results can only be considered a first approximation, but agree with actual conditions sufficiently to be of considerable interest.

12.34 Crystal Grid to Plate

The equation (12.20) is general and for the condition of crystal connected between grid and plate of the tube (See Figure 12.12B) Z_3 represents the crystal impedance which will be called $Z_c = R_c + jX_c$, also: $Z_1 = jX_1$, $Z_2 = jX_2$ and $X_s = X_1 + X_2 + X_c$.

Note that R_g and C_3 are disregarded in this case because their effects are similar to those determined for the foregoing case of crystal connected grid to cathode.

After substitution of these values in (12.20) the real part is found to be

$$R_p = \frac{(\mu + 1)X_1X_2 + X_1X_c}{R_c} \tag{12.35}$$

and the imaginary part is

$$X_s = -\frac{R_cX_1}{R_p} \tag{12.36}$$

which shows the effect of the various variables on the frequency. The right hand side of equation (12.36) is comparatively small and the frequency is therefore close to a value f_0 which makes $X_s = 0$. In this case the frequency is above the limiting frequency f_0 because the right hand side is positive since X_1 is negative, whereas it was found that the frequency was below f_0 for the crystal connected between grid and cathode. As R_c and X_1 are increased the frequency will increase and as R_p increases the frequency decreases. These interpretations are verified by the *G-P* curves of Figures 12.6, 12.9 and 12.10.

The effects of the various circuit and crystal constants are determined by Koga[12] by writing the general $\mu\beta$ equation as

$$Z_3 + Z_2 + \frac{\mu Z_2}{1 + R_p/Z_1} = 0 \tag{12.37}$$

After substitution for the Z's, the real and imaginary parts are respectively,

$$\left(\frac{1}{\omega C_0}\right)^2 \frac{R_1}{R_1^2 + \left[\omega L_1 - \frac{1}{\omega C_1} - \frac{1}{\omega C_0}\right]^2} + \frac{\mu}{\omega C_g}$$

$$\frac{R_p\left(\frac{1}{\omega L_2} - \omega C_2\right)}{1 + R_p^2\left(\frac{1}{\omega L_2} - \omega C_2\right)^2} = 0 \tag{12.38}$$

and

$$\frac{\dfrac{1}{\omega C_1} + \dfrac{1}{\omega C_0} - \omega L_1}{R_1^2 + \left[\omega L_1 - \dfrac{1}{\omega C_1} - \dfrac{1}{\omega C_0}\right]^2} = \omega C_0^2\left[\frac{1}{C_0} + \frac{1}{C_4} + \frac{1}{C_g} + \frac{\mu}{C_g}\cdot\frac{1}{1 + R_p^2\left(\dfrac{1}{\omega L_2} - \omega C_2\right)^2}\right] \tag{12.39}$$

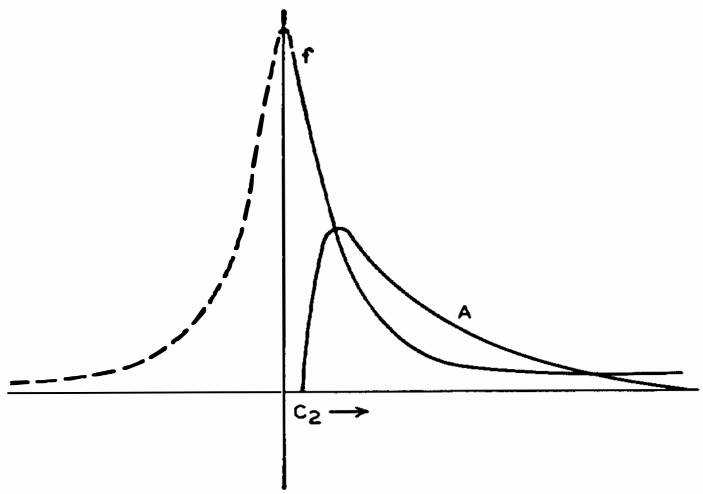

Fig. 12.16—Frequency and activity change for variations in the plate circuit capacitance. Crystal connected grid to plate

There are two values of ω which satisfy (12.39) but only one of these ω_n will satisfy (12.38). At this value of ω_n

$$R_1^2 \ll \left[\omega L_1 - \frac{1}{\omega C_1} - \frac{1}{\omega C_0}\right]^2 \tag{12.40}$$

By introduction of this and the assumption that ω_0 is essentially constant, (12.38) may be written

$$R_1 C_0^2\left[\frac{1}{C_0} + \frac{1}{C_4} + \frac{1}{C_g} + \frac{}{C_g}\cdot\frac{1}{1 + R_p^2\left(\dfrac{1}{\omega_0 L_2} - \omega_0 C_2\right)^2}\right]^2 + \frac{\mu}{\omega_0 C_g}\cdot\frac{R_p\left(\dfrac{1}{\omega_0 L_2} - \omega_0 C_2\right)}{1 + R_p^2\left(\dfrac{1}{\omega_0 L_2} - \omega_0 C_2\right)^2} = 0 \tag{12.41}$$

This is an approximation for the conditions for oscillation and relative amplitude.

The frequency equation (12.39) becomes

$$\omega_n^2 = \frac{1}{L_1}\left[\frac{1}{C_1} + \frac{1}{C_0} - \frac{1}{G}\right] \tag{12.42}$$

where

$$G = C_0^2\left[\frac{1}{C_0} + \frac{1}{C_4} + \frac{1}{C_g} + \frac{\mu}{C_g}\cdot\frac{1}{1 + R_p^2\left(\frac{1}{\omega_0 L_2} - \omega_0 C_2\right)^2}\right]$$

and ω_0 is a fixed value written in place of ω_n. Figure 12.16 shows the frequency and amplitude changes as a function of C_2 for the crystal connected between grid and plate.

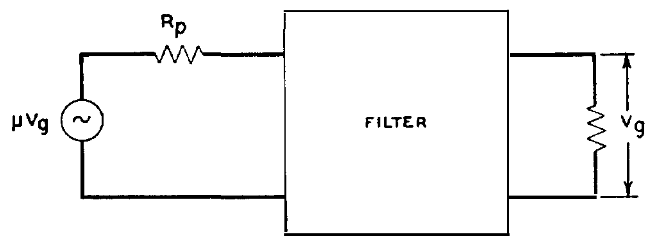

Fig. 12.17—Generalized oscillator circuit in the form of a filter network

12.35 Condition $\mu\beta = 1$ for Circuits in General

It is convenient to apply the rule $\mu\beta = 1$ as the condition for sustained oscillations to more complex oscillator circuits. The circuits may be drawn as shown in Figure 12.17 and the characteristics of the filter network between transmitting and receiving end may be analyzed by conventional filter theory to determine the conditions which fulfill the oscillation requirements. An example of this is the oscillator shown in Fig. 12.18A. The equivalent configuration, Fig. 12.18B, indicates that the crystal is part of a low pass filter and the frequency of operation is that at which the total phase shift is 180°.

Oscillators involving more than one tube may also be inspected in this manner. Fig. 12.19 is a two tube oscillator designed to operate at a frequency close to the resonant frequency of the crystal. The proper phase shift is obtained by a two-stage amplifier and, therefore, no phase shift is required through the crystal network. The crystal thus must operate as a resistance which it can only do at its resonant or antiresonant frequency. Since the transmission through the crystal branch is very low at the antiresonant frequency of the crystal, it will oscillate only at the resonant frequency. Heegner[13] explains a number of crystal oscillator circuits by the method briefly outlined above.

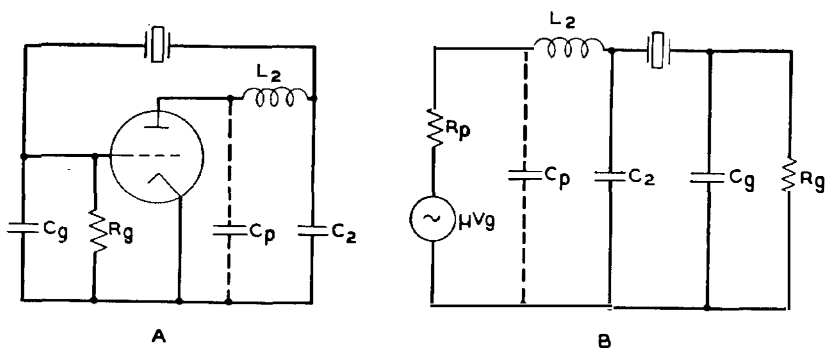

Fig. 12.18—The oscillator circuit (A) is equivalent to the filter circuit (B)

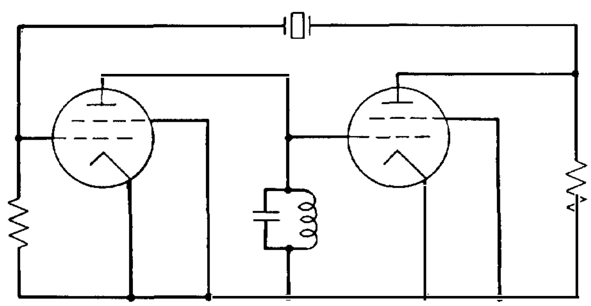

Fig. 12.19—Oscillator circuit in which the crystal operates at its series resonant frequency

12.4 Vector Method of Oscillator Analysis

A convenient method of examining the effect of certain circuit variables on frequency and the necessary conditions for oscillation is by the vector representation of the voltages and currents in the circuit. Much of Heising's[14] early work on the analysis of electric oscillators by vector methods is directly applicable to crystal oscillators. Boella[15] analyzed the crystal oscillator circuit by this method and treated in detail the effect of the decrement of the crystal on the oscillating frequency. Since some engineers prefer this method of qualitative analysis to approximate equations it will be briefly explained.

The vector diagrams for the two conditions, crystal between grid and plate and between grid and cathode, are shown in Fig. 12.20A and B as applied to the circuit diagrams, Fig. 12.12A and B, respectively when in the simplified form of Fig. 12.11. The necessary conditions for oscillations are that V_g is in phase with and equal to μV_g (note that μ is considered negative). Like Koga, Boella assumes the current I_2 small compared to I_1, hence the voltage drop across Z_1 is approximately $Z_1 I_1$. The angle this makes with V_g is determined by the value of Z_1 and the internal plate impedance R_p. Any change in either of these requires a change in the angles ψ and ψ' in order that V_g shall be in phase with μV_g. This means that the frequency must vary to produce this change in ψ and ψ'. Because of the rapid change in the reactance and resistance of the crystal with frequency, these requirements are met with very little change in frequency, which accounts for the high degree of frequency stability obtained with crystals. This is described more in detail in a later section.

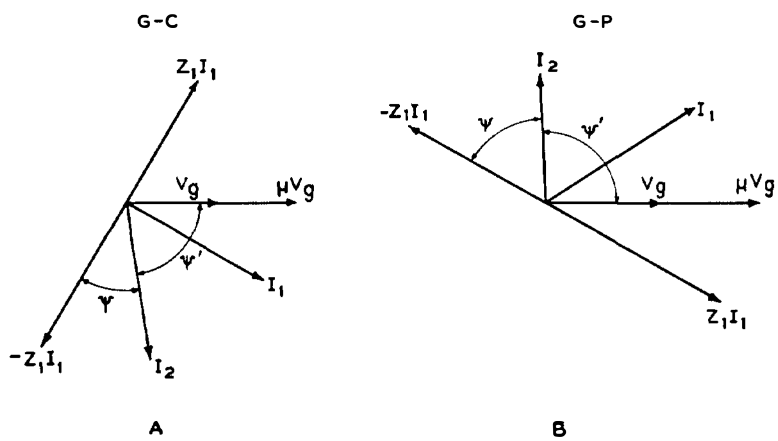

Fig. 12.20—Vector diagrams of currents and voltages in the oscillator circuit Fig. 12.11 with crystal connected grid to cathode (A) and grid to plate (B)

12.41 Change in Frequency with Decrement of Crystal

It has been found that for the crystal connected from grid to cathode there is a maximum theoretical frequency at which the circuit can be made to oscillate by reducing the plate circuit impedance. This also corresponds to the minimum frequency which can be obtained with the crystal connected between grid and plate. This was called the limiting frequency f_0. It is interesting to note that f_0 is determined by the intersection of the reactance curve of the crystal plotted as a function of frequency and the reactance curve of the capacitance in series with the crystal. This series capacitance is the grid-plate capacitance for one case and the grid-cathode capacitance for the other. As illustrated in the curves Fig. 12.21, the limiting frequency f_0 increases as the decrement of the crystal increases.

The difference between the true frequency of oscillations and f_0 increases as the plate impedance is increased and as the losses in any of the circuit elements increase. This is necessary for the proper angle of $\psi + \psi'$ in the vector diagram. With the *G-P* connections, the departure from f_0 and change in f_0 as the decrement of the quartz varies are in the same direction, while for the grid-cathode connection they vary in opposite directions, and the net result will depend upon the value of the internal plate resistance

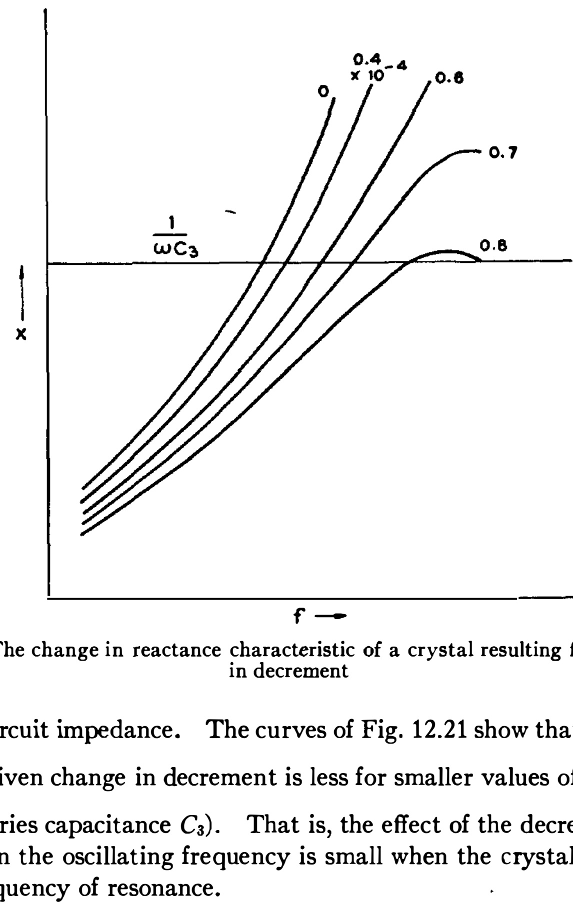

Fig. 12.21—The change in reactance characteristic of a crystal resulting from a change in decrement

and plate circuit impedance. The curves of Fig. 12.21 show that the change in f_0 for a given change in decrement is less for smaller values of $\frac{1}{\omega C_3}$ (larger values of series capacitance C_3). That is, the effect of the decrement of the crystal upon the oscillating frequency is small when the crystal is operated near its frequency of resonance.

12.5 Negative Resistance Method of Analysis

The methods of analyzing oscillator circuits described in the previous sections define the operation in terms of the individual circuit elements and the crystal is treated as one of the circuit elements. Certain advantages result, however, by grouping all the circuit elements, except the crystal, into a single impedance as shown in Fig. 12.22A. Here Z_t represents the impedance looking into the oscillator from the crystal terminals.

The requirements for sustained oscillations are that the sum of the reactances around the loop equal zero and the sum of the resistances equal zero as previously stated in section 12.31. These conditions are obtained when Z_t is a negative resistance ρ in parallel with (or in series with) a capacitance C_t as shown in Fig. 12.22B. The crystal is considered to be operating as an inductance L_c and resistance R_c as determined in the pre-

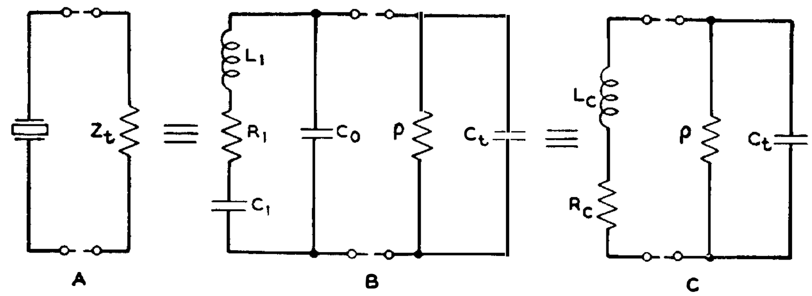

Fig. 12.22—Equivalent representations of crystal and oscillator circuit

vious sections. The frequency equation has been derived by Reich[16] from the differential equation for the current in the loop. It is

$$\omega = \sqrt{\frac{\rho + R_c}{\rho} \cdot \frac{1}{L_c C_t}} \tag{12.43}$$

and the condition for oscillation is shown to be

$$|C_t \rho| \leqq \left|\frac{L_c}{R_c}\right| \tag{12.44}$$

We shall consider the crystal connected between the grid and cathode of the tube, in which case Z_t is the input impedance of the vacuum tube. The expression for $\frac{1}{Z_t}$ was developed by Chaffee[17] from which it is possible to determine the circuit conditions necessary for the input resistance and reactance to be negative. The effect of the circuit variables upon the absolute values of ρ and C_t determines their effect upon the frequency and activity according to equations (12.43) and (12.44).

12.51 Input Admittance of the Vacuum Tube

With the assumption that the grid current is negligible and the static tube capacitances C_p and C_g are part of the external circuit, Chaffee's equation for input conductance becomes

$$g = \frac{C_3^2\omega(K + G_1) + C_3\omega\mu K(C_3\omega - B_1)}{(K + G_1)^2 + (C_3\omega - B_1)^2} \tag{12.45}$$

and for the input susceptance

$$b = -C_3\omega \frac{C_3\omega\mu K(K + G_1) - C_3^2\omega^2(C_3\omega - B_1)}{(K + G_1)^2 + (C_3\omega - B_1)^2} \tag{12.46}$$

where K and μ are defined as follows:

$$K = \left(\frac{\partial i_p}{\partial e_p}\right) \quad (e_g \text{ constant})$$

$$\mu = -\left(\frac{\partial e_p}{\partial e_g}\right) \quad (i_p \text{ constant})$$

and G_1 and B_1 are the conductance and susceptance of the plate circuit. If we let

$$h = \frac{G_1}{|B_1|}$$

$$A = \frac{\omega C_3}{K} \quad \text{and} \quad B = \frac{B_1}{K}$$

(12.45) becomes

$$g = C_3\omega A \frac{(1 + hB) + \mu\left(1 + \frac{B}{A}\right)}{(1 + hB)^2 + A^2\left(1 - \frac{B}{A}\right)^2} \tag{12.47}$$

and (12.46) becomes

$$b = -C_3\omega\left[1 + \frac{\mu(1 + hB) - A^2\left(1 - \frac{B}{A}\right)}{(1 + hB)^2 + A^2\left(1 - \frac{B}{A}\right)^2}\right] \tag{12.48}$$

When the resistance of the plate circuit is neglected (i.e. $h = 0$), and $\mu \gg 1$ we may write

$$\frac{g}{K} = A\,\frac{\mu(A - B)}{1 + (A - B)^2} \tag{12.49}$$

and

$$\frac{b}{K} = -A\left[\frac{\mu - B(A - B)}{1 + (A - B)^2}\right] \tag{12.50}$$

These equations are in a convenient form to determine the effect of the plate tuning $f(B)$ and grid-plate capacitance $f(A)$ upon the resistance ρ and capacitance C_t with the assumptions of no grid current, low plate circuit resistance, and $\mu \gg 1$.

From (12.49) it is seen that in order for g to be negative, B must be positive and greater than A, since A is normally positive. That is, the plate circuit reactance must be positive and less than the grid-plate reactance when the latter is a capacitance. Under these conditions b/K and hence the input reactance will be negative according to (12.50).

Curves of b/K are shown in Fig. 12.23 with B as independent variable and A as parameter. These curves indicate frequency change. On the

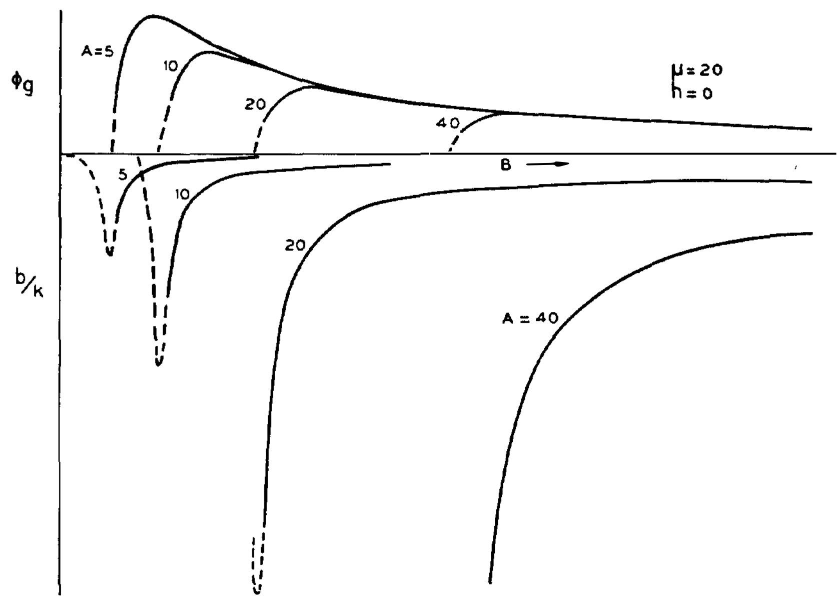

Fig. 12.23—Variations in the input impedance functions of an oscillator circuit for changes in plate circuit tuning

same figure is plotted b/g called ϕ_g. This may be considered the sensitivity of the oscillator or, for a given value of $\omega L_c/R_c$ of the crystal, it represents the activity. The similarity between these curves and the actual change in frequency and activity normally experienced is apparent.

It should be pointed out here that the presence of harmonics is effective in changing the input impedance of the vacuum tube and hence the frequency and activity of the oscillator. The presence of harmonics results from the non-linear characteristics of the vacuum tube. Llewellyn[11] explains that a non-linear resistance may be represented by a linear resistance plus a linear reactance. From what has been said concerning the frequency of the oscillating loop, it is apparent that this effective reactance will alter the frequency. However, this reactance is small when the impedance of the circuit is low at the harmonic frequencies and is zero when the external circuit is a pure resistance.

12.6 Efficiency and Power Output of Oscillators

In many applications of crystal oscillators the efficiency and power output are important factors. These are not treated here but reference is made to the work of Heising[14] which covers this aspect for various electric oscillator circuits. Much of the analysis is directly applicable to crystal oscillators.

12.7 Frequency Stability of Crystal Oscillators

The equations for frequency show that the frequency is governed somewhat by the amplification factor, the grid resistance and internal plate resistance of the vacuum tube. Since these factors are functions of voltages applied to the tube and amplitude of oscillation, they cannot be considered fixed. If the frequency change resulting from these variables is great, the frequency stability is said to be low, and if very little frequency change takes place the frequency is determined principally by the circuit constants and the frequency stability is said to be high.

Llewellyn[11] shows how it is possible to compensate for the change in plate resistance by the proper value of circuit elements. This was done by determining the relations necessary for R_p to be eliminated from the frequency equation. It is sometimes helpful in designing very stable oscillators for frequency standards to select circuit elements which will reduce the effect of plate voltage changes on the frequency. It is more the purpose of this section, however, to show Llewellyn's derivation of the equations for frequency stability which have not heretofore been published and from them point out the characteristic of crystals which enable them to stabilize oscillators.

12.71 The Frequency Stability Equation

The steady state oscillating condition is

$$\mu\beta = 1 \tag{12.51}$$

In general β is a function of the frequency, the amplitude of oscillations, and of some independent variable V. This independent variable is the one for which it is desired to stabilize the frequency. It may be the potential applied to the tube, or it may be a capacitance located somewhere in the circuit. β depends upon these three variables thus:

$$\mu\beta = f(p\,, a\,, V) \tag{12.52}$$

Instead of the frequency, a more general symbol p is used and may be thought of as the differential operator d/dt which occurs in the fundamental linear differential equations taken as describing the oscillatory system. That is

$$p = \frac{d}{dt} = \alpha + i\omega \tag{12.53}$$

The function $\mu\beta$ may have the form

$$\mu\beta = Ae^{i\theta} \tag{12.54}$$

The result of taking a general variation δ of (12.54) is then

$$\frac{\delta A}{A} + i\delta\theta = 0 \tag{12.55}$$

Since (12.54) is a function of the three variables p, a, and V the variational equation (12.55) may be expressed in terms of partial derivatives with respect to these three variables. That is

$$\frac{1}{A}\left[\frac{\partial A}{\partial p}\delta p + \frac{\partial A}{\partial a}\delta a + \frac{\partial A}{\partial V}\delta V\right] + i\left[\frac{\partial \theta}{\partial p}\delta p + \frac{\partial \theta}{\partial a}\delta a + \frac{\partial \theta}{\partial V}\delta V\right] = 0 \tag{12.56}$$

The solution of (12.56) for the variation in p is

$$\delta p = -\frac{\frac{1}{A}\left(\frac{\partial A}{\partial V}\delta V + \frac{\partial A}{\partial a}\delta a\right) + i\left(\frac{\partial \theta}{\partial V}\delta V + \frac{\partial \theta}{\partial a}\delta a\right)}{\frac{1}{A}\frac{\partial A}{\partial p} + i\frac{\partial \theta}{\partial p}} \tag{12.57}$$

It is a property of functions of complex variables that, provided they possess derivatives at all, then the value of the derivative is the same regardless of the direction in which the limiting point is approached. This fact is expressed by

$$\left.\begin{aligned} \frac{\partial A}{\partial p} &= \frac{\partial A}{\partial \alpha} = i\frac{\partial A}{\partial \omega} \\ \frac{\partial \theta}{\partial p} &= \frac{\partial \theta}{\partial \alpha} = i\frac{\partial \theta}{\partial \omega} \\ \text{and } \delta p &= \delta\alpha + i\delta\omega \end{aligned}\right] \tag{12.58}$$

and provides means by which the real and imaginary parts of (12.57) may be separated to yield the two equations

$$\delta\alpha = \frac{\left[\frac{1}{A}\frac{\partial A}{\partial \omega}\left(\frac{\partial \theta}{\partial V}\delta V + \frac{\partial \theta}{\partial a}\delta a\right) - \frac{\partial \theta}{\partial \omega}\left(\frac{1}{A}\frac{\partial A}{\partial V}\delta V + \frac{1}{A}\frac{\partial A}{\partial a}\delta a\right)\right]}{\left(\frac{1}{A}\frac{\partial A}{\partial \omega}\right)^2 + \left(\frac{\partial \theta}{\partial \omega}\right)^2} \tag{12.59}$$

and

$$\delta\omega = \frac{-\left[\frac{1}{A}\frac{\partial A}{\partial \omega}\left(\frac{1}{A}\frac{\partial A}{\partial V}\delta V + \frac{1}{A}\frac{\partial A}{\partial a}\delta a\right) + \frac{\partial \theta}{\partial \omega}\left(\frac{\partial \theta}{\partial V}\delta V + \frac{\partial \theta}{\partial a}\delta a\right)\right]}{\left(\frac{1}{A}\frac{\partial A}{\partial \omega}\right)^2 + \left(\frac{\partial \theta}{\partial \omega}\right)^2} \tag{12.60}$$

The variable p in general may be written as the sum of α and $i\omega$. With the remembrance that p is the differential operator d/dt and that a set of linear equations expresses the transient condition, it is evident that the current will have the form Ie^{pt} which is equivalent to $Ie^{(\alpha+i\omega)t}$. Inspection of this shows that the real part of p, namely α, determines whether the currents in the system are going to build up with time, or die away with time, or remain constant, depending respectively upon whether α is greater than zero, is less than zero, or is actually equal to zero. With this in mind we see that (12.59) and (12.60) state the change in α and ω respectively which would result from some change in the circuit condition. Initially the circuit was oscillating in a steady manner so that α was zero and ω had some particular value. A change in V then occurred. This produced a change in the amplitude accompanied by a change in the frequency as expressed by (12.60) and a change in the transient term α. Suppose now that the change in V were very small. Then in order for oscillations again to assume a steady value it is necessary for the amplitude "a" to change a sufficient amount to cause α to become zero. Thus in (12.59) we put $\delta\alpha$ equal to zero and solve for the required amplitude change. This may then be eliminated from (12.60) resulting in the final expression

$$\delta\omega = \frac{\frac{1}{A}\frac{\partial A}{\partial V}\frac{\partial \theta}{\partial a} - \frac{1}{A}\frac{\partial A}{\partial a}\frac{\partial \theta}{\partial V}}{\frac{1}{A}\frac{\partial A}{\partial \omega}\frac{\partial \theta}{\partial a} - \frac{1}{A}\frac{\partial A}{\partial a}\frac{\partial \theta}{\partial \omega}}\delta V \tag{12.61}$$

which gives the frequency change $\delta\omega$ in terms of the change of the independent variable δV.

12.72 Frequency Stability of Conventional Oscillator

In applying this equation to the oscillator circuit, Fig. 12.24, we must first set up the conditions for oscillations. The $\mu\beta$ equation is

$$\mu\beta = \frac{\mu X_1 X_2 R_g}{i[X_s R_p R_g - X_1 X_2 X_3] - [R_p X_2(X_1 + X_3) + R_g X_1(X_2 + X_3)]} \tag{12.62}$$

The oscillating conditions $\mu\beta = 1$ requires

$$X_s R_p R_g = X_1 X_2 X_3$$

and

$$\mu X_1 X_2 R_g + R_p X_2(X_1 + X_3) + R_g X_1(X_2 + X_3) = 0 \qquad (12.63)$$

It will be assumed that the following relations exist:

$$\mu = f_1(V),\ R_g = f_2(a),\ X_s = X_1 + X_2 + X_3 = f_3(\omega),\ R_p = \text{a constant}$$

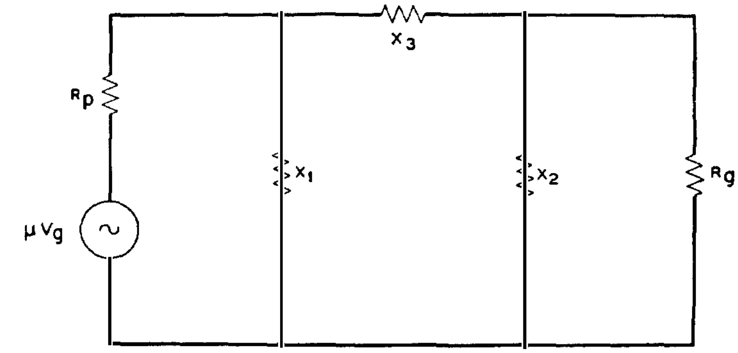

Fig. 12.24—Equivalent oscillator circuit analyzed for frequency stability

Then we obtain from (12.63)

$$\frac{1}{A}\frac{\partial A}{\partial V} = \frac{1}{\mu}\frac{\partial \mu}{\partial V} \qquad \frac{\partial \theta}{\partial V} = 0$$

$$\frac{1}{A}\frac{\partial A}{\partial a} = \frac{1}{R_g}\frac{\partial R_g}{\partial a}\left[1 + \frac{X_2 + X_3}{\mu X_2}\right] \qquad \frac{\partial \theta}{\partial a} = -\frac{1}{R_g}\frac{\partial R_g}{\partial a}\frac{X_0 R_p}{\mu X_1 X_2}$$

$$\begin{aligned}\frac{1}{A}\frac{\partial A}{\partial \omega} = \frac{1}{X_1}\frac{\partial X_1}{\partial \omega}&\left[1 + \frac{R_p X_2 + R_g(X_2 + X_3)}{\mu R_p X_2}\right] \\ &+ \frac{1}{X_2}\frac{\partial X_2}{\partial \omega}\left[1 + \frac{R_p(X_2 + X_3) + R_g X_1}{\mu R_p X_1}\right] \qquad (12.64) \\ &+ \frac{1}{X_3}\frac{\partial X_3}{\partial \omega}\left[\frac{X_3(R_p X_2 + R_g X_1)}{\mu R_p X_1 X_2}\right]\end{aligned}$$

$$\begin{aligned}\frac{\partial \theta}{\partial \omega} = -\frac{R_p}{\mu X_1 X_2}&\left[(X_1 - X_s)\frac{1}{X_1}\frac{\partial X_1}{\partial \omega} + (X_2 - X_s)\frac{1}{X_2}\frac{\partial X_2}{\partial \omega}\right. \\ &\left. + (X_3 - X_s)\frac{1}{X_3}\frac{\partial X_3}{\partial \omega}\right]\end{aligned}$$

By substitution of these values in (12.61) and disregard of X_s in comparison with all other X's the equation for frequency stability is obtained as

$$\frac{d\omega}{dV} = \frac{\frac{1}{\mu}\frac{\partial\mu}{\partial V} X_1 X_2 X_3}{\left(1 - \frac{X_1}{\mu X_2}\right)\left(\frac{\partial X_1}{\partial\omega} + \frac{\partial X_2}{\partial\omega} + \frac{\partial X_3}{\partial\omega}\right) R_p R_g} \tag{12.65}$$

From this we learn that the values of the reactances X_1, X_2, and X_3 should be small and the values of R_p and R_g large to give small changes in ω when V is varied. These variables are more or less limited, however, by the conditions necessary for sustained oscillations according to equation (12.63). It is important to notice that the denominator of (12.65) contains functions which do not appear in equation (12.63) and hence may be of any value. These factors are the rates of change of the various reactances with frequency. For given values of circuit constants, the equation shows that the *frequency stability increases as these rates of change increase.*

12.73 Frequency Stability Coefficient of Crystals

The rate of change of the reactance of an element is referred to as the "frequency stability coefficient"* of the element. Expressed in per cent, we have for the frequency stability coefficient of a reactance

$$F(X) = \frac{dX}{d\omega}\cdot\frac{\omega}{X} \tag{12.66}$$

Let us now examine the frequency stability coefficient of a crystal which is used as the reactance X_2 when connected between grid and cathode of the tube and as X_3 when connected between grid and plate (See Fig. 12.24). The resistance of the crystal will be assumed to equal zero due to the negligible effect of the resistance variations upon the reactance for crystals with average Q and operated at a frequency not too near the anti-resonant frequency. (This may be observed in Fig. 12.21.)

The reactance of the crystal then is

$$X_c = -\frac{j}{\omega C_0}\frac{\omega^2 - \omega_1^2}{\omega^2 - \omega_2^2} \tag{12.67}$$

where

$\omega = 2\pi \times$ frequency
$\omega_1 = 2\pi \times$ resonant frequency
$\omega_2 = 2\pi \times$ anti-resonant frequency

* First suggested by N. E. Sowers.

By substitution of the relations

$$\frac{C_1}{C_0} = \frac{\omega_2^2 - \omega_1^2}{\omega_1^2} \quad \text{and} \quad -\frac{j}{\omega C_0} = X_0$$

into (12.67) we obtained

$$X_c = X_0\left[1 - \frac{C_1}{C_0}\frac{\omega_1^2}{\omega_2^2 - \omega^2}\right] \tag{12.68}$$

and by differentiation

$$\frac{dX_c}{d\omega} = -\frac{X_0}{\omega}\left[1 - \frac{C_1}{C_0}\frac{\omega_1^2}{\omega_2^2 - \omega^2}\right] - X_0\frac{C_1}{C_0}\left[\frac{\omega_1^2\, 2\omega}{(\omega_2^2 - \omega^2)^2}\right] \tag{12.69}$$

Multiply by $\frac{\omega}{X_c}$ to obtain the stability coefficient

$$F(X_c) = \frac{\omega}{X_c}\frac{dX_c}{d\omega} = -\frac{X_0}{X_c}\left[1 - \frac{C_1}{C_0}\cdot\frac{\omega^2}{\omega_2^2 - \omega^2}\right] - \frac{X_0}{X_c}\cdot\frac{C_1}{C_0}\cdot\frac{\omega^2}{\omega_1^2}\cdot\frac{2\omega_1^4}{(\omega_2^2 - \omega^2)^2} \tag{12.70}$$

and eliminate ω by substituting in (12.70) the relations obtained from equation (12.68). These are

$$\left.\begin{aligned} \frac{\omega_1^2}{\omega_2^2 - \omega^2} &= \left(1 - \frac{X_c}{X_0}\right)\frac{C_0}{C_1} \\ \text{and} \qquad & \\ \frac{\omega^2}{\omega_1^2} &= \frac{C_1}{C_0} + 1 - \frac{C_1}{C_0}\frac{1}{1 - \frac{X_c}{X_0}} \end{aligned}\right] \tag{12.71}$$

Thus

$$F(X_c) = -1 - 2\frac{X_0}{X_c}\left(1 - \frac{X_c}{X_0}\right)^2\left[1 + \frac{C_0}{C_1} - \frac{1}{1 - \frac{X_c}{X_0}}\right] \tag{12.72}$$

which may be written

$$F(X_c) = -\frac{X_c}{X_0}\left[1 + \left(1 - \frac{X_0}{X_c}\right) + 2\frac{C_0}{C_1}\left(1 - \frac{X_0}{X_c}\right)^2\right] \tag{12.73}$$

The stability coefficient of a coil and condenser $F(X)'$ may be obtained from (12.73) by letting $C_1 = \infty$. Then

$$F(X)' = -\frac{X_c}{X_0}\left[1 + \left(1 - \frac{X_0}{X_c}\right)\right] \tag{12.74}$$

The comparative stability of the crystal and tuned circuit is given by the ratio

$$\frac{F(X_c)}{F(X)'} = 1 + 2\frac{C_0}{C_1}\cdot\frac{\left(1 - \frac{X_0}{X_c}\right)^2}{1 + \left(1 - \frac{X_0}{X_c}\right)} \tag{12.75}$$

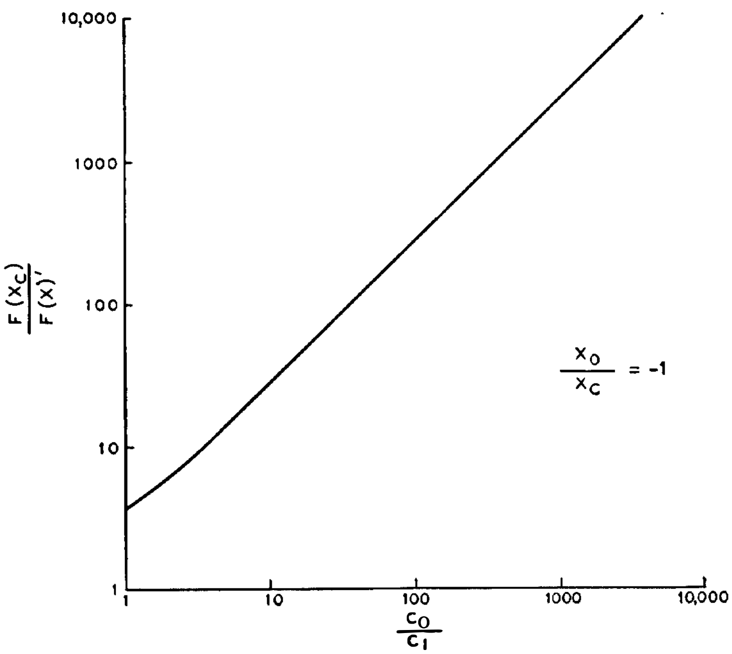

Fig. 12.25—The stability coefficient of a crystal as compared to a coil and condenser for variations of the ratio of capacitances

This ratio is plotted in Fig. 12.25 for $\frac{X_0}{X_c} = -1$ and with $\frac{C_0}{C_1}$ the independent function. It is apparent that the value of $\frac{C_0}{C_1}$ of a crystal is the factor which determines its frequency stability for given operating conditions. For an AT crystal in an air gap holder, the ratio of capacitances is of the order of 10^3 and its stability coefficient is therefore 2.6×10^3 greater than for a simple anti-resonant circuit. Since this is so much greater than the stability coefficients for the other reactances which appear in the denominator of equation (12.65) it represents the order of magnitude of improvement of the frequency stability of an oscillator obtained by the use of a crystal.

The fact that the frequency stability of a crystal oscillator is a function of $\frac{C_0}{C_1}$ explains why a *BT* cut crystal is in general more stable than an *AT* cut. The two may be made equal, however, by adding capacitance across the *AT* cut crystal.

Actually we have compared the frequency stability obtained by the use of one type of circuit (the equivalent crystal circuit) with one of a different configuration obtained by making $C_1 = \infty$. In practice this is usually the case since C_1 must be large to obtain oscillations when using coils and condensers. The limiting factor is therefore the value of $\frac{C_0}{C_1}$ at which oscillations stop and this is determined by the Q of the circuit elements as shown in the next section which deals with activity. It will be shown that the Q required is proportional to $\frac{C_0}{C_1}$ and therefore the maximum frequency stability that can be obtained is directly related to Q.

12.8 Relation Between Crystal Quality and Amplitude of Oscillations

The activity of a crystal is usually thought of as the relative amount of grid current produced in an oscillator circuit. This method of defining activity affords a means of comparing the quality of one crystal with another for a particular set of conditions. The disadvantages are first; it is only a relative measure, and second; it is not possible to compute the activity as thus defined by any method of oscillator analysis so far presented. Curves have been shown of amplitude of oscillations as a function of certain circuit variables, but these represent only qualitative changes associated with plate resistance variations. The first objection has been somewhat rectified by the use of reference oscillators* in which all the circuit elements including the tubes have been carefully matched. There is still the difficulty, however, of comparing crystals of different frequencies for it cannot be assumed that the measurements are independent of this variable. It would be more desirable to have some absolute measure of activity and particularly one which would lend itself to convenient computation from readily measurable constants of the crystal.

12.81 Definition of Crystal Quality for Oscillator Purposes

In deriving an expression for the quality of a crystal, it is convenient to use the negative resistance concept of the oscillator as described in section 12.5. The equations are general and in a form which admit of separating

* Developed by G. M. Thurston.

the crystal from the oscillator circuit. Equation (12.44) which gives the conditions necessary for oscillations to exist, may be written in the form

$$\omega C_t \rho \leqq \frac{\omega L_e}{R_c} \tag{12.76}$$

In order for oscillations to start, the right side of this equation must be equal to or greater than the left side. If it is greater, oscillations build up causing ρ to increase until the equality is satisfied. The difference between these two terms before oscillations start is therefore a relative measure of the final amplitude for a particular oscillator. The absolute value of amplitude cannot be obtained from equation (12.76) since we do not know the relation between ρ and amplitude. However the greater the magnitude of $\frac{\omega L_c}{R_c}$ the greater will be the amplitude of oscillations for a given set of oscillator conditions. This term may therefore be considered a measure of crystal quality. It is the effective Q of the crystal unit as measured at its two terminals and at the operating frequency. To distinguish this from the Q of the crystal as usually spoken of, it will be called φ_c.

In the same respect the left side of equation (12.76) may be thought of as a measure of quality of the oscillator circuit, that is, $\rho\omega C_t = \frac{1}{\varphi_o}$, then (12.76) becomes

$$\varphi_c \varphi_o \geqq 1 \tag{12.77}$$

12.82 Figure of Merit of Crystals — M

It is very inconvenient to use φ_c as a figure of merit of the crystal because it is a complex function of the constants of the crystal circuits, Fig. 12.3, and also the frequency. The computation of φ_c from such measurable characteristics as frequency of resonance f_1, frequency of anti-resonance f_2, resonant resistance R_1, and static capacity C_0, requires considerable time and effort.

It is highly desirable that a simple, easily determined expression for a figure of merit be found. The steps to indicate a suitable one are as follows:

The equation for φ_c in terms of measurable quantities for computing it is derived from equation (12.27) and given by the formula

$$\varphi_c = \frac{\omega L_c}{R_c} = \frac{-1 - \dfrac{\omega L_1^2}{R_1^2} \dfrac{(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)}{\omega^4}}{\dfrac{\omega L_1}{R_1} \dfrac{\omega_2^2 - \omega_1^2}{\omega^2}} \tag{12.78}$$

By letting

$$M = \frac{\omega L_1}{R_1} \frac{\omega_2^2 - \omega_1^2}{\omega^2} \tag{12.79}$$

and

$$n = \frac{\omega_2^2 - \omega^2}{\omega_2^2 - \omega_1^2} \tag{12.80}$$

and with the assumption that over the narrow frequency range that the crystal will operate

$$\frac{\omega_2^2 - \omega_1^2}{\omega_1} \cong \frac{\omega_2^2 - \omega_1^2}{\omega_1 \omega} \tag{12.81}$$

equation (12.78) is reduced to

$$\varphi_c = -\frac{1 + nM^2(n - 1)}{M} \tag{12.82}$$

In this equation it will be observed are only two variables, namely, n which varies widely as the frequency is varied between f_1 and f_2 and M which is substantially constant over the same range.

A set of curves is plotted in Fig. 12.26 for a hypothetical set of crystals having values of M of 1, 2, 5, and 10 with n varied over a range that falls between measured frequencies f_1 and f_2. Studies will show that whenever M increases φ_c will increase. M is readily calculated from measured constants as seen from the following: From (12.79)

$$M = \frac{\omega L_1}{R_1} \frac{\omega_2^2 - \omega_1^2}{\omega^2} = \frac{\omega_1 L_1}{R_1} \frac{\omega_2^2 - \omega_1^2}{\omega_1 \omega} \tag{12.83}$$

With the assumption in (12.81)

$$M = \frac{\omega_1 L_1}{R_1} \frac{C_1}{C_0} = \frac{1}{\omega_1 C_0 R_1} \tag{12.84}$$

which is a simple expression containing three of the four measured quantities mentioned above, and which bears a direct relation to activity for a given value of the frequency variable n. M is the new figure of merit of the crystal.

Fig. 12.26 contains a further indication which is useful on occasions. Here φ_c is not positive at any frequency unless M is greater than 2. But φ_c must be positive for the crystal to oscillate in the two general types of circuits considered here.* It provides a measurable index to separate completely non-useful crystals from those that can oscillate in a given circuit.

Equation (12.84) may be written

$$M = \frac{Q}{r} \tag{12.85}$$

* For a description of oscillator circuits which do not require the crystal to exhibit a positive reactance, see: "A New Direct Crystal-Controlled Oscillator for Ultra-Short-Wave Frequencies" W. P. Mason and I. E. Fair, *Proc. I.R.E.*, Vol. 30, Oct. 1943, p. 464.

where Q is the Q of the crystal and r its ratio of capacitances. Thus the figure of merit involves the dissipation in the crystal determined by Q and the piezo-electric effect determined by r.[18]

It was pointed out in the preceding section that the frequency stability increases as r is increased. The above equation shows that Q must increase

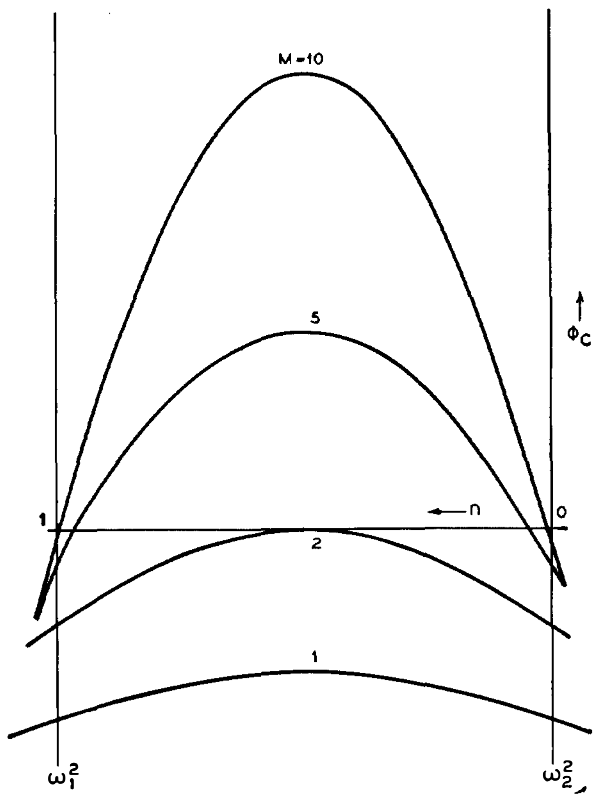

Fig. 12.26—The dependence of the quality function ϕ_c of a crystal upon frequency and figure of merit M

proportionately if the same figure of merit is to be maintained. The frequency stability obtainable in a particular oscillator is therefore limited by the Q of the crystal and the frequency stability coefficients should be compared on this basis.

12.9 Activity of Crystals

In deriving a figure of merit for crystals as oscillators, it was found that the amplitude of oscillations in a given circuit not only depends upon M but also it is a function of frequency relative to the resonant frequency of the crystal. This may be explained by referring to Fig. 12.27 which shows curves of the reactance X_c of the crystal plotted as a function of frequency. The frequency at which oscillations occur depends principally upon the value of circuit capacitance C_t. Equation (12.43) shows that the frequency must adjust itself to a value at which C_t resonates with the reactance of the crystal. This value of reactance is represented on the curve as X_{co} and the corresponding frequency of oscillations as f_o. The

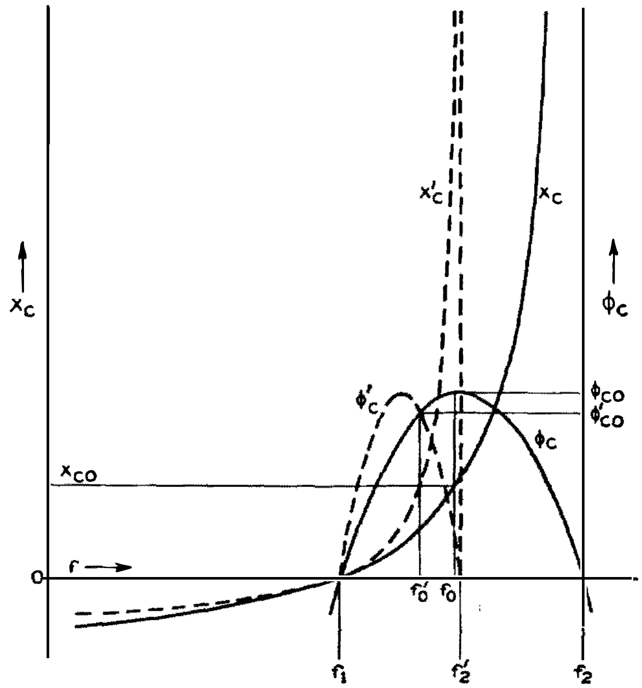

Fig. 12.27—Two crystals having the same figure of merit but with different reactance characteristics X_c and X_c' will operate with different amplitudes according to the relative values of φ_{co} and φ'_o respectively

circuit capacitance (or X_{co}) has been so chosen in the illustration that f_o lies equidistant between the resonant frequency f_1 and the anti-resonant frequency f_2. Then at this value of f_o, φ_c is a maximum, as shown by the φ_c curve, and for a given value of φ_g this will result in the greatest activity. Now let the capacitance C_0 of the crystal be increased. The frequencies f_1 and f_2 will then become closer and at the same time the height of the φ_c curve is reduced. Assume also that the Q is increased in order to maintain the same value of M and hence the same maximum value for φ_c. The reactance-frequency curve for the modified crystal and the corresponding curve for φ_c' are represented by the dotted curves. Note that the oscillating frequency f_o' for the new curve is closer to f_2' than it is for f_1, therefore φ_{co}' is less than φ_{co} and the amplitude of oscillations will be less. Thus two crystals may have the same value of M but will not give the same output unless operated at the same relative frequency with respect to their resonant and anti-resonant frequencies. It would not have been possible to increase the oscillator output by increasing C_t so as to lower the frequency f_o' because by doing so φ_g is decreased more rapidly than φ_c' is increased, and the result would be a further decrease of activity. It is therefore necessary in deriving an expression for activity to include the variable of relative frequency or n which we have shown to be a function of the reactance of the circuit and the crystal constants.

12.91 Derivation of Performance Index of Crystals—PI

It will be assumed in the first derivation that the negative resistance ρ of the circuit is much greater than the effective resistance of the crystal R_c under stable oscillating conditions. Equation (12.43) which expresses the frequency then becomes

$$\omega \cong \sqrt{\frac{1}{L_c C_t}} \tag{12.86}$$

This leads to a very simple solution from which a more exact expression is later obtained.

Equation (12.44), which expresses conditions necessary for oscillations, may be written

$$|\rho| \leqq \left|\frac{\omega L_c}{\omega C_t R_c}\right| \tag{12.87}$$

As before, the numerical difference between ρ and the right side of the equation is a measure of activity. In fact, the right side of the equation may be considered to be an expression of the activity performance provided the terms are themselves independent of ρ. This is not quite true, since previous sections show that the capacitance C_t is not entirely independent of the activity. (See equations (12.30) and (12.46).) However, this effect may be considered negligible for most practical purposes and the value of the right side of (12.87) called the Performance Index PI of the crystal. From equations (12.86) and (12.87) the performance index is found to be

$$PI = \frac{1}{R_c \omega^2 C_t^2} \tag{12.88}$$

This equation may be greatly in error under operating condition which makes R_c large compared to ρ. Also R_c varies rapidly with frequency and is difficult to evaluate. R_c is most readily eliminated from the equation by revising the picture slightly. With reference to the simplified oscillator circuit, Fig. 12.22B, it is apparent that the static crystal capacitance C_0 and the circuit capacitance C_t may be combined. This leaves for the crystal branch the inductance L_e' (different from L_c) which is a function of frequency and the resonant resistance of the crystal R_1 which is not a function of frequency. Now R_1 may be assumed small compared to ρ with considerable accuracy. It is only necessary, then, to replace C_t in equation (12.88) with $(C_0 + C_t)$ and R_e by R_1. This equation then becomes

$$PI = \frac{1}{R_1 \omega^2 (C_0 + C_t)^2} \tag{12.89}$$

An exact equation for PI is derived in section 12.93 and it is shown that the error in the simple expression above will in most cases be very small.

An approximate equation for the relation between R_1 and R_e is obtained by dividing (12.88) by (12.89). We thus find

$$1 = \frac{R_1 (C_0 + C_t)^2}{R_e C_t^2} \tag{12.90}$$

or the effective resistance of the crystal at the operating frequency is

$$R_e = R_1 \left(\frac{C_0}{C_t} + 1\right)^2 \tag{12.91}$$

Because of the approximation in equation (12.88) the equation for R_c above is accurate only when $\left(\frac{C_0}{C_t} + 1\right)^2 \ll M^2$ as will be shown in section 12.93.

The expression for PI as given by (12.89) may be written

$$PI = \frac{1}{\omega^2 C_0^2 R_1 \left(1 + \frac{C_t}{C_0}\right)^2} \tag{12.92}$$

which is the most convenient form for calculating PI from the constants of the crystal and the oscillator circuit.

12.92 Relation Between M and PI

It was found that

$$M = \frac{1}{\omega_1 C_0 R_1} \tag{12.93}$$

and this is essentially equal to $\frac{1}{\omega C_0 R_1}$ over the narrow frequency range considered. Therefore,

$$PI = \frac{M}{\omega C_0 \left(1 + \frac{C_t}{C_0}\right)^2} \tag{12.94}$$

This gives a relation between the performance index and the figure of merit of the crystal.

Another useful relation between M and PI is obtained from (12.89) and (12.93). Equation (12.93) may be written

$$M = \frac{X_0}{R_1} \tag{12.95}$$

This is of the same form as the Q of a coil when X_0 is considered to be the reactance of the coil and R_1 its resistance. Like the Q of a coil M is essentially constant over a wide frequency range. Now if we let C_t approach zero in (12.89) it becomes

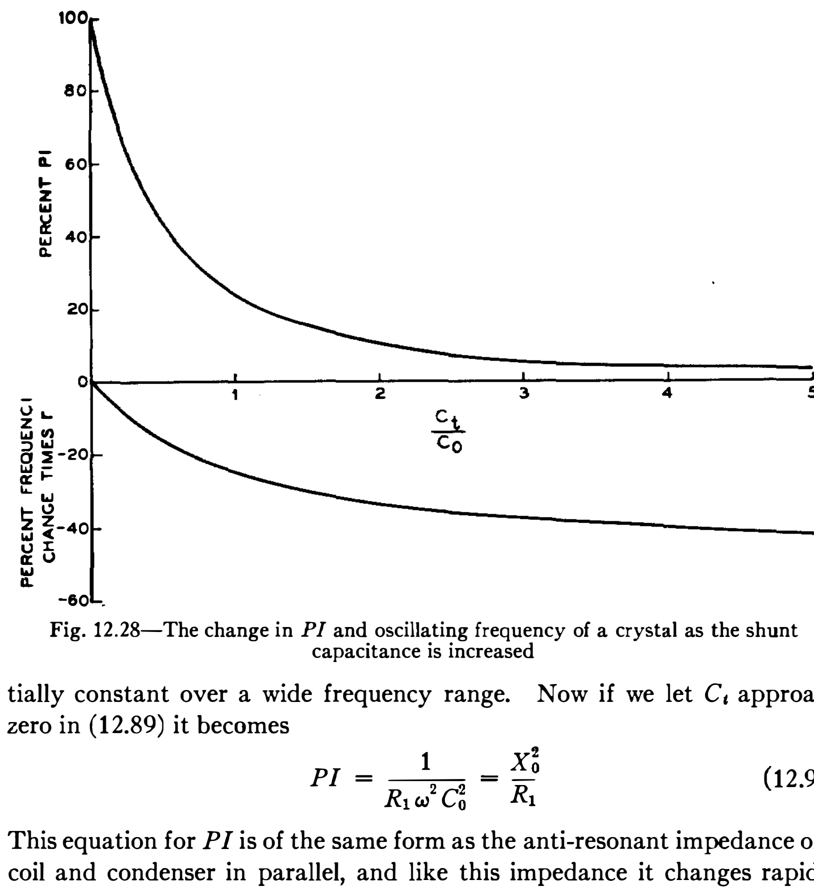

Fig. 12.28—The change in PI and oscillating frequency of a crystal as the shunt capacitance is increased

$$PI = \frac{1}{R_1 \omega^2 C_0^2} = \frac{X_0^2}{R_1} \tag{12.96}$$

This equation for PI is of the same form as the anti-resonant impedance of a coil and condenser in parallel, and like this impedance it changes rapidly with frequency. The maximum value of PI is therefore X_0 times M and is obtained when $C_t = 0$. Fig. 12.28 shows a curve of % PI plotted as a function of $\frac{C_t}{C_0}$. This curve represents the change in activity as capacitance is added across the crystals (increase in C_t).

12.93 Exact Expressions for PI and R_c

The error in PI caused by the assumption that the frequency is independent of the crystal resistance R_1, that is, by use of approximate equation (12.86) for the frequency, may be investigated as follows:

The impedance of the crystal and C_t in parallel is given by

$$Z = \frac{1}{\omega(C_0 + C_t)(1 + m^2 P^2)} [P - j(1 + mP^2(m - 1))] \quad (12.97)$$

where

$$P = \frac{MC_0}{C_0 + C_t} \qquad m = \frac{\omega_3^2 - \omega^2}{\omega_3^2 - \omega_1^2}$$

$\omega_3 = 2\pi$ times frequency of anti-resonance of the crystal and C_t combination when $R_1 = 0$

$\omega_1 = 2\pi$ times frequency of resonance of the crystal and C_t combination when $R_1 = 0$

$\omega \ = 2\pi$ times operating frequency

(Note that P is the figure of merit of the crystal and C_t in parallel.) The imaginary part of Z is

$$X = -\frac{1 + mP^2(m - 1)}{\omega(C_0 + C_t)(1 + m^2 P^2)} \quad (12.98)$$

The condition for stable oscillations requires $X = 0$. For this condition

$$m = \tfrac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{P^2}} \quad (12.99)$$

which defines the exact frequency of oscillation. The negative sign before the radical is used since the effective resistance is greater at this frequency, thus requiring less negative conductance for oscillation.

With P large $(m \to 0)$ the frequency of oscillations coincides with ω_3 which is the same as given by the approximate frequency equation (12.86). The real part of (12.97) is

$$R = \frac{P}{\omega(C_0 + C_t)(1 + m^2 P^2)} \quad (12.100)$$

and when $m = 0$

$$R = \frac{P}{\omega(C_0 + C_t)} = \frac{M}{\omega C_0\left(1 + \frac{C_t}{C_0}\right)^2} \quad (12.101)$$

This is identical to the expression for PI of equation (12.94). *PI is therefore the anti-resonant resistance of the crystal and capacitance C_t in parallel.* Substitution of the value of m as given by (12.99) into (12.100) gives the anti-resonant resistance at the oscillating frequency. Thus

$$R_o = \frac{P}{\omega(C_0 + C_t)}\left[1 - \frac{1 - \sqrt{1 - \frac{4}{P^2}}}{2}\right] \tag{12.102}$$

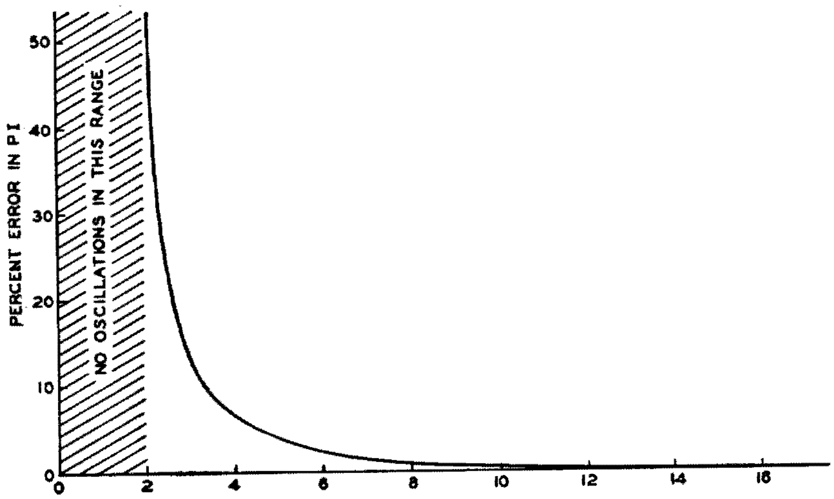

Fig. 12.29—The error in PI resulting from the use of approximate equation (12.94)

which is the exact expression for PI. The differential error resulting from the use of approximate equation (12.94) is then

$$\frac{PI - R_o}{R_o} = \frac{1 - \sqrt{1 - \frac{4}{P^2}}}{1 + \sqrt{1 - \frac{4}{P^2}}} \tag{12.103}$$

The per cent error as a function of P is shown in Fig. (12.29). The error diminishes rapidly with increase in P and is negligible for crystals that are of sufficient quality for most oscillator purposes.

Equation (12.91) for R_e is also approximate because of the assumption that the frequency is independent of R_e. A more exact expression will be derived.

The impedance of the crystal alone is

$$Z_e = \frac{1}{\omega C_0(1 + n^2 M^2)}\,[M - j(1 + nM^2(n - 1))] \tag{12.104}$$

where

$M =$ the figure of merit of the crystal

$$n = \frac{\omega_2^2 - \omega^2}{\omega_2^2 - \omega_1^2}$$

$\omega_1 = 2\pi$ times frequency of resonance of the crystal
$\omega_2 = 2\pi$ times frequency of anti-resonance of the crystal
$\omega =$ the independent variable, 2π times frequency
$C_0 =$ the static capacitance of the crystal

The resistive component of Z_c is

$$R_c = \frac{M}{\omega C_0(1 + n^2 M^2)} = \frac{R_1}{\frac{1}{M^2} + n^2} \tag{12.105}$$

In order to express R_c in terms of C_0 and C_t, the quantity n must be expressed in terms of these variables which define the oscillating frequency. This is accomplished as follows:

The equation of ratio of capacitances of a crystal is

$$\frac{C_1}{C_0} = \frac{\omega_2^2 - \omega_1^2}{\omega_1^2} \tag{12.106}$$

Similarly, when the capacit[illegible] $_t$ [illegible]oss the crystal

$$\frac{C_1}{C_0 + C_t} = \frac{\omega_3^2 - \omega_1^2}{\omega_1^2} \tag{12.107}$$

where ω_3 is 2π times the anti-resonant frequency of the crystal and C_t in parallel. The ratio of (12.107) and (12.106) is

$$\frac{C_0}{C_0 + C_t} = \frac{\omega_3^2 - \omega_1^2}{\omega_2^2 - \omega_1^2} \tag{12.108}$$

The oscillating frequency is given by (12.99) in which m is as defined under (12.97). The oscillating frequency ω is therefore given by

$$\frac{\omega_3^2 - \omega^2}{\omega_3^2 - \omega_1^2} = \frac{1 - \sqrt{1 - \frac{4}{P^2}}}{2} \tag{12.109}$$

or

$$-\frac{\omega_1^2 - \omega^2}{\omega_3^2 - \omega_1^2} = \frac{1 + \sqrt{1 - \frac{4}{P^2}}}{2} \tag{12.110}$$

The angular frequency ω_3 is eliminated by multiplying this by (12.108). Thus

$$\frac{\omega_1^2 - \omega^2}{\omega_2^2 - \omega_1^2} = \frac{-C_0}{C_0 + C_t}\left[\frac{1 + \sqrt{1 - \frac{4}{P^2}}}{2}\right] \tag{12.111}$$

r

$$\frac{\omega_2^2 - \omega^2}{\omega_2^2 - \omega_1^2} = \frac{-C_0}{C_0 + C_t}\left[\frac{1 + \sqrt{1 - \frac{4}{P^2}}}{2}\right] + 1 \tag{12.112}$$

This is the value for n at the oscillating frequency and may be reduced to the form,

$$n = \left[\frac{1 - \sqrt{1 - \frac{4}{P^2}}}{2} + \frac{1 + \sqrt{1 - \frac{4}{P^2}}}{2\left(\frac{C_0}{C_t} + 1\right)}\right] \tag{12.113}$$

When this value of n is substituted in (12.105), the value of R_c is found to be

$$R_c = \frac{R_1}{\frac{1}{M^2} + \left[\frac{1 - \sqrt{1 - \frac{4}{P^2}}}{2} + \frac{1 + \sqrt{1 - \frac{4}{P^2}}}{2\left(\frac{C_0}{C_t} + 1\right)}\right]^2} \tag{12.114}$$

For crystals of usable quality $\frac{4}{P^2} << 1$ and by this assumption the equation reduces to

$$R_c = \frac{R_1}{\frac{1}{M^2} + \frac{1}{\left(\frac{C_0}{C_t} + 1\right)^2}} \tag{12.115}$$

This again reduces to (12.91) when $M^2 >> \left(\frac{C_0}{C_t} + 1\right)^2$.

12.94 Frequency Change Resulting from Paralleling Capacitance

It is often desirable to know how much the frequency of an oscillator may be changed by varying the capacitance C_t across the crystal. This is determined from (12.112) which gives the oscillating frequency as a function of C_t. For practical considerations we may assume $\frac{4}{P^2} \ll 1$ which reduces the equation to

$$\frac{\omega_2^2 - \omega^2}{\omega_2^2 - \omega_1^2} = \frac{C_t}{C_0 + C_t} \tag{12.116}$$

From this and (12.106) we obtain

$$\frac{\omega_2^2 - \omega^2}{\omega_1^2} = \frac{C_1 C_t}{C_0 (C_0 + C_t)} \tag{12.117}$$

Since

$$\frac{\omega_2^2 - \omega^2}{\omega_1^2} = \frac{(\omega_2 - \omega)(\omega_2 + \omega)}{\omega_1^2} \cong \frac{2(\omega_2 - \omega)}{\omega_1}$$

then

$$\frac{\omega - \omega_2}{\omega_1} \cong \frac{-1}{2r\left(\frac{C_0}{C_t} + 1\right)} \tag{12.118}$$

where r is the ratio of the capacitances of the crystal.

A curve of per cent frequency change multiplied by r as a function of $\frac{C_t}{C_0}$ is shown on Fig. 12.28 for comparison with the associated PI change.

12.95 Relation Between PI and Oscillator Activity

The relation between PI and activity obtained in a particular oscillator will now be examined. Let the curves of Fig. 12.30 represent the variations of ρ with amplitude for two oscillators A and B, or they might be for the same oscillator at widely different frequencies. These are characteristics of the oscillator circuits and may be of any shape. However, for oscillators with grid leak bias, the curves normally have no negative slopes. The rate of change of ρ depends upon the rate of change of μ and plate resistance of the vacuum tube as shown by (12.45) for input conductance.* Since ρ builds up to a value equal to PI we may plot PI for ρ. The grid current I_g is usually taken as a measure of amplitude. Therefore, Fig. 12.30 may be plotted as shown in Fig. 12.31 where PI is the independent variable. These curves are the characteristics of the oscillator circuits A and B with PI defining the quality of the crystal when used with a particular value of C_t. It is characteristic of oscillators to "saturate" as shown by the curves.

* It is also a function of grid resistance but this does not appear in the approximate equation (12.45) because of the assumption of no grid current. See Chaffee's[17] complete equation for input admittance.

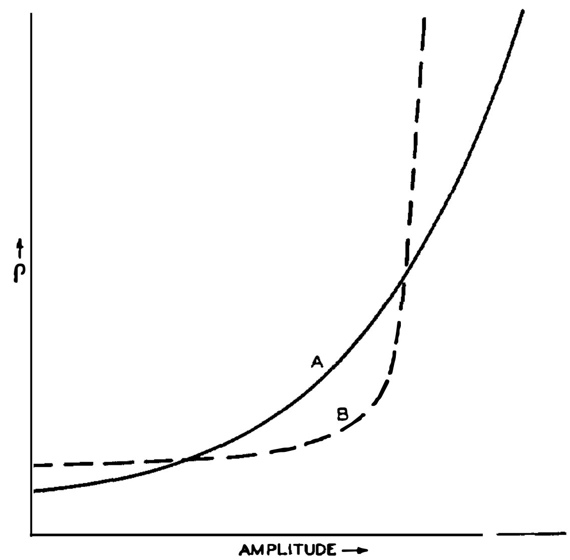

Fig. 12.30—Hypothetical curves illustrating the normal relation between the negative resistance of oscillator circuits and the amplitude of oscillations

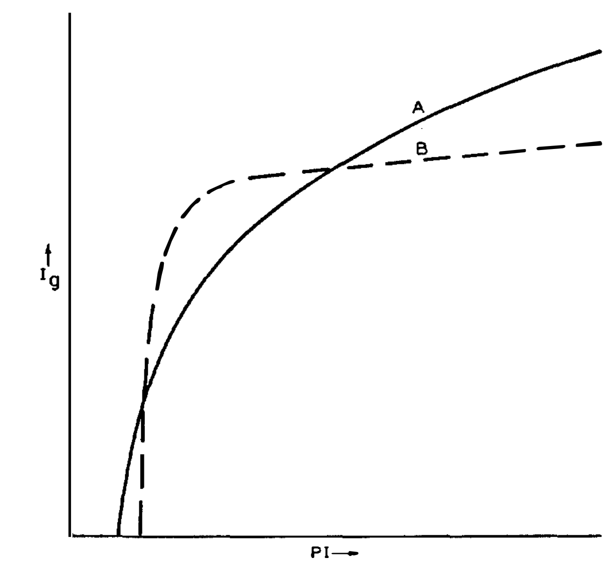

Fig. 12.31—By interchanging the coordinates of Figure 12.30 the curves will represent the relation between *PI* and oscillator grid current

Some oscillators saturate very rapidly and completely according to curve B and no further output is obtained regardless of the improvement in the crystal quality. For this reason it has not been possible in the past to separate the performance of the oscillator and the crystal since both were based upon the grid current as a measure of quality. By defining crystal activities and oscillator sensitivity in the manner outlined, the crystal and circuit can be studied separately. The per cent of crystals obtainable with PI above a certain value will be known and the design and improvement of oscillator circuits will be facilitated.

12.96 Use of PI in Crystal Design

The expression of PI in terms of the crystal constants and C_t as given by equations (12.89) or (12.92) assists in the design of crystals. As an example, the effect of changing the area of the crystal electrodes will be computed. The Q of a crystal is defined as

$$Q = \frac{1}{\omega_1 C_1 R_1} \tag{12.119}$$

By introduction of the ratio of capacitances of the crystal $r = \frac{C_0}{C_1}$ equation (12.119) becomes

$$Q = \frac{r}{\omega_1 C_0 R_1} \tag{12.120}$$

or

$$\frac{Q}{r} = \frac{1}{\omega_1 C_0 R_1} = M \tag{12.121}$$

Assuming Q and r do not vary, that is, disregarding effects such as secondary modes, change in damping produced by the mounting etc., and substituting (12.121) in (12.94) we obtain

$$PI = \frac{Q}{r} \cdot \frac{C_0}{\omega(C_0 + C_t)^2} \tag{12.122}$$

where $\frac{Q}{r}$ is considered constant. Differentiating (12.122) with respect to C_0 we find that PI is a maximum when $C_0 = C_t$. Since C_0 is proportional to the area of the electrodes this establishes the optimum area for a particular value of circuit capacitance.

The capacitance of BT-cut plates is 1.68 mmf per square centimeter per megacycle.* Substitution of this for C_0 in (12.122) gives

$$PI = \frac{.268 \times 10^6 MA}{(1.68\ Af + C_t)^2} \tag{12.123}$$

* All frequencies are referred to the time interval of one second throughout this paper, i.e. *megacycles per second* is called simply *megacycles* (mc) as is customary in the radio field.

where A = the area of the crystal in square centimeters.
f = the frequency in megacycles
C_t = circuit capacitance in mmf.
M = figure of merit of the crystal (assumed constant)

Thus for crystals of a given area, the performance index should decrease as the frequency increases. Fig. (12.32) shows the theoretical variations of PI as the function of the diameter of the electrodes of three frequencies and for

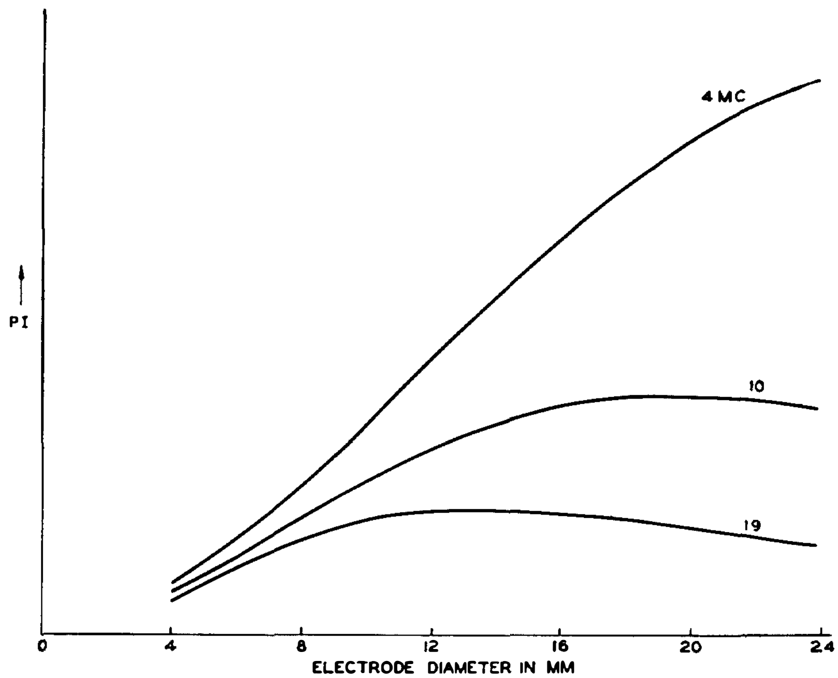

Fig. 12.32—Theoretical curves showing the relations of PI, electrode diameter, and crystal frequency for BT crystals and a circuit capacitance of 50 $\mu\mu$f

a circuit capacitance of 50 mmf. The activity of a 4-mc crystal with 11-mm diameter electrodes is about the same as a 10-mc crystal with 18-mm electrodes. It must be remembered in making this comparison that it is assumed that the damping introduced by the mounting is the same in both cases. Actually the damping is much greater for low-frequency crystals of this type than for high-frequency ones and maximum PI occurs at some intermediate frequency as shown by the curves of Fig. 12.33. These curves show that the damping caused by the particular mounting used was small for frequencies above 6-mc but increases rapidly below this value.

12.97 Measurement of PI and M

In all the discussions so far regarding the performance of crystals in oscillator circuits, the crystal has been represented by the equivalent circuit of Fig. 12.3 in which all the elements were considered constant. It is possible to obtain crystals in which this is essentially the case, but in general there are three secondary effects which complicate the picture. These are, first, the effect of other modes of vibration of the crystal, second, variations in the crystal constants resulting from variations in the amplitude of vibration, and third, the leakage or dielectric loss in the crystal holder. These factors will be considered in the order named.

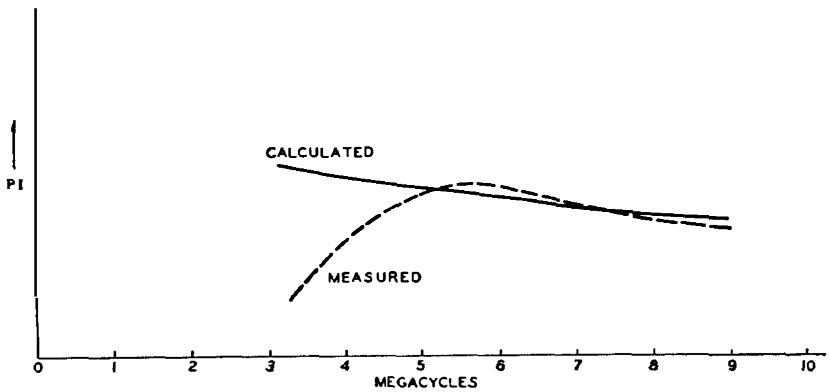

Fig. 12.33—Calculated and measured values of *PI* for *BT* crystals. The discrepancy is a measure of mounting loss

Secondary modes of vibration affect the crystal for oscillator purposes only when the frequencies of these modes are sufficiently close to the principal one to alter its impedance characteristic in the frequency range of oscillation; that is, to alter the reactance as shown in Fig. 12.27 between the frequency f_1 and f_2 and the corresponding effective resistance between these two frequencies. With interfering modes present, the equivalent crystal circuit is so complicated as to make it impractical to compute *PI* or *M* from such measurable quantities as resonant resistance R_1, series resonant frequency f_1, anti-resonant frequency f_2, etc. For this reason it is necessary to measure the reactance and effective resistance of the crystal at the operating frequency in order to obtain a measure of crystal quality which will correlate with the crystal performance. For the same reason it is important when comparing oscillator circuits that the crystal should be operated at the same frequency in each case.

It is believed that the non-linear effect noticed in crystals when used as oscillators is produced by the changes in the mounting as the amplitude of vibration varies. The PI of some clamped or pressure-mounted crystals has been found to vary as much as 50% with change in drive. Noticeable frequency change also occurs. A change in the nature of secondary modes as the amplitude is varied has also been observed. Some secondary modes which interfere with large amplitude of vibrations practically disappear when the amplitude is reduced. This may be explained by the fact that certain modes are damped out by the pressure of the mounting and with large amplitude of vibration the effect of the pressure is reduced.

The dielectric loss of the holder was considered negligible in the theory but it is found that certain phenolic holders have equivalent high-frequency leakage resistances less than 100,000 ohms. This resistance is in parallel with the crystal and will therefore reduce the PI according to the equation

$$PI = \frac{PI_c R_L}{PI_c + R_L} \tag{12.124}$$

where

PI = resulting PI
PI_c = calculated PI
R_L = equivalent high-frequency leakage resistance

Because of these secondary effects which are not negligible it is essential in measuring crystal activity that the frequency and voltage across the crystal be known. Standard test circuits should simulate operating conditions in this respect. With these considerations, a crystal PI meter has been developed in which the frequency and amplitude may be adjusted to correlate with various oscillators. The principle of operation and performance of this meter is described by C. W. Harrison in Chapter XV.

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