APPENDIX 4
THE PIEZOELECTRIC COUPLING COEFFICIENT

An electromechanical transducer is a device which converts electric energy to elastic (or mechanical) energy, and vice versa. The quartz crystal does so through the piezoelectric effect. When a voltage V is applied to the electrodes of a large quartz plate of area A and thickness e ($A \gg e^2$), the energy stored electrically is given by

$$W_e = \frac{1}{2} C_0 V^2 = \frac{1}{2} k \epsilon_0 \left(\frac{V^2 A}{e} \right)$$

k is the dielectric constant of the unclamped crystal and therefore W_e is the sum of the dielectric and piezoelectric strain energies.

The piezoelectric strain energy per unit volume in the crystal is

$$\frac{1}{2} c x^2 = \frac{1}{2} \left(\frac{\epsilon^2}{c} \right) \left(\frac{V}{e} \right)^2$$

The piezoelectric strain energy in the crystal is therefore

$$W_p = \frac{1}{2} \left(\frac{\epsilon^2}{c} \right) \left(\frac{V^2 A}{e} \right)$$

where ϵ is the applicable piezoelecric constant and c is the applicable stiffness coefficient.

By definition the coupling coefficient κ is given by $\kappa = \sqrt{W_p/W_e}$, so

$$\kappa = \sqrt{\frac{\epsilon^2}{c k \epsilon_0}}$$

The coupling coefficient κ is closely related to the capacitance ratio r. From page 98 we have

$$r = \frac{C_0}{C} = \frac{\pi^2 k c \epsilon_0}{8\, \epsilon^2}$$

and therefore

$$r = \frac{\pi^2}{8}\left(\frac{1}{\kappa^2}\right)$$

Note: In the derivation above the difference between the dielectric constants of the free and unclamped plates has been neglected for simplicity. When the difference is included

$$r = \frac{\pi^2}{8}\left(\frac{1}{\kappa^2} - 1\right)$$

The difference is negligible in quartz although not necessarily so with other materials.

The value of κ can be computed for the AT-cut by using the applicable values. The applicable piezoelectric constant is $\epsilon'_{26} = \epsilon_{14} \sin\theta \cos\theta - \epsilon_{11} \cos^2\theta$. With the values of ϵ_{11} and ϵ_{14} from page 62, and $\theta = 35^\circ$

$$\epsilon'_{26} = 0.097\ C/\mathrm{m}^2$$

The value of the applicable stiffness coefficient $c'_{66} = 29.3 \times 10^9\ \mathrm{N/m^2}$ (page 142). The applicable dielectric constand is $k'_{33} = 4.54$ (page 35). Using these values the piezoelectric coupling coefficient for the AT-cut plate is found to be

$$\kappa = 0.089 \qquad \text{and} \qquad \kappa^2 = 0.0080$$

The capacitance ratio for the AT-cut plate is therefore

$$r = \frac{C_0}{C} = \frac{\pi^2}{8\kappa^2} = 154$$

This is the theoretical value for an infinite plate where it is assumed that the strain is uniform over the entire surface of the plate; that no fringing of the field occurs at the electrode edges; and that no stray holder capacitances exist. None of these conditions is satisfied in an actual resonator, so that typical values measured on a well-designed AT-cut resonator operating on its fundamental mode are $r = 200$ and $\kappa^2 = 0.006$.

The value of r is proportional to the square of the harmonic order. Thus the value of r for an AT-cut resonator operating on its third-harmonic overtone mode is typically 2000 or more. At the fifth harmonic r is greater than 5000.

The antiresonant frequency of a crystal unit is given by

$$f_A = f_R \left(1 + \frac{C}{2C_t}\right)$$ see page 123.

From this equation it is obvious that f_A can be varied by changing C_t with a load capacitance. The degree to which f_A can be varied by changing C_t is called *pullability*. The pullability obviously depends on the value of the motional capacitance C. A crystal unit having a small C is said to be a "stiff" crystal unit.

The frequency of a crystal unit is often "trimmed" to a specified value by varying C_x (and therefore C_t). In temperature-controlled oscillators (TCXOs) C_x is varied by means of a semiconductor diode whose capacitance can be changed by means of a biasing voltage.

For maximum frequency stability a crystal unit should have a small motional capacitance, i.e., be a "stiff" crystal. For maximum pullability the motional capacitance should be large.

It is readily seen that κ^2 should be small for the greatest frequency stability. Physically this means that if the piezoelectric coupling coefficient is small, changes in the parameters of the electric circuit have minimum effect upon the mechanical properties of the piezoid.