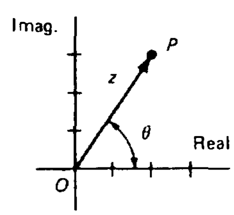
APPENDIX 2
COMPLEX NUMBERS

The algebra of complex numbers is used to handle problems involving steady-state alternating currents and voltages. More advanced techniques are required to treat transient conditions, but we are not concerned with these in this book.

An imaginary number is the product of a real number and the square root of -1, which we designate by j. For example, $4j$ is an imaginary number. A complex number is the vector sum of a real number and an imaginary number. For example, $z = 2 + 3j$ is a complex number. Complex numbers may be represented graphically by marking off the real part of the number on the horizontal, or real, axis and the imaginary part on the vertical, or imaginary, axis. The magnitude of the complex number z is indicated by $|z|$ and if $z = a + bj$, then $|z| = (a^2 + b^2)^{1/2}$.



The vector OP used to represent z makes an angle $\theta = \tan^{-1}(b/a)$ with the real axis.

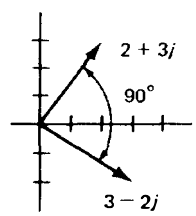
The sum of two complex numbers is found by adding the real parts and adding the imaginary parts, i.e., if $z_1 = 2 + 3j$ and $z_2 = 4 - 5j$, then $z_1 + z_2 = 6 - 2j$.

The product of two complex numbers is found by multiplying the two numbers as binomials and writing $j^2 = -1$. Thus $(2 + 3j)(4 - 5j) = 8 + 12j - 10j - 15j^2 = 23 + 2j$.

Division by complex numbers is not defined. A fraction having a complex denominator is rationalized by multiplying the numerator and denominator by the complex conjugate of the denominator and then simplifying. For example,

$$\frac{2+3j}{3+4j} = \frac{2+3j}{3+4j}\,\frac{3-4j}{3-4j} = \frac{18+1j}{3^2+4^2} = \frac{18}{25} + \frac{1}{25}j$$

A complex number multiplied by j is another complex number and the vectors representing the two are perpendicular to each other. For example, $j(3 - 2j) = (2 + 3j)$. The effect of multiplication by j is to rotate the vector counterclockwise by 90°. Multiplying twice by j rotates the vector by 180°. This is equivalent to multiplication by -1.



Complex numbers may also be represented in polar form by giving the length (or modulus) of the vector and the angle made with the real axis. Thus $z = 3 + 4j$ may be represented by a vector of length 5, making an angle θ $\tan^{-1}$ (4/3) with the real axis, and written $z = (r, \theta)$, with $r = 5$ and $\theta = 53°8'$.

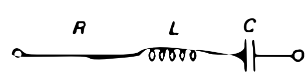
Euler's formula, which can be written

$$z = \exp(j\theta) = \cos\theta + j\sin\theta$$

associates the exponential and trigonometric forms of a complex number. The exponential form is widely used in problems involving alternating currents.

By using complex numbers to represent currents, voltages, and impedances, Kirchhoff's laws can be applied to ac circuits as well as dc circuits. Thus instead of writing $V = I R$, we write $V = I Z$, where it is understood that all three of the quantities are complex. The impedance of a circuit consisting of a resistance R, an inductance L, and a capacitance C, in series, is written as

$$Z = R + j\omega L - j\,\frac{1}{\omega C}$$



where $\omega = 2\pi f$ is called the angular frequency and is measured in radians per second.

Now if an ac voltage $e = E_m \sin(\omega t)$ is impressed on the circuit, the current which flows is given by

$$i = \frac{e}{Z} = E_m \frac{\sin(\omega t)}{R + jX}$$

where X is written for $\omega L - 1/\omega C$.

Rationalizing the denominator as described above we have

$$i = \frac{E_m (R - jX)}{R^2 + X^2} \sin(\omega t)$$

or

$$i = E_m \left(\frac{R}{R^2 + X^2} - j \frac{X}{R^2 + X^2} \right) \sin(\omega t)$$

Now the amplitude of the current is itself a complex number, which means that the vector representing the current is not parallel to the vector representing the voltage. This is equivalent to saying that the current and voltage are not in phase. If we think of the vectors as rotating counterclockwise, and if $(\omega L - 1/\omega C)$ is positive, then the current i lags behind the voltage e by an angle $\theta = \tan^{-1}(X/R)$. If $(\omega L - 1/\omega C)$ is negative, the current i leads the voltage by the same angle; and if $(\omega L - 1/\omega C) = 0$, the current and voltage are in phase with each other.

It is often more convenient mathematically to represent a current or voltage by its exponential form rather than the trigonometric form and to write

$$i = I_m \exp(j\omega t)$$

instead of

$$i = I_m \sin(\omega t) \quad \text{or} \quad I_m \cos(\omega t)$$

We may think of the quantity $\exp(j\omega t)$ as a vector of unit length which makes one complete rotation about the origin in time $2\pi/\omega$. The projection of this vector is equivalent to the projection of a point moving with motion $\sin(\omega t)$.