APPENDIX 1
BASIC DIELECTRIC THEORY

Consider a parallel-plate capacitor of area A with a quartz plate of thickness e as a dielectric. When the terminals of the capacitor are connected to a battery, the charging current which flows is composed of three parts:

1. The Maxwell displacement current
2. The dielectric polarization current
3. The piezoelectric polarization current

If the dielectric were vacuum instead of quartz, only 1 would flow. If the dielectric were glass, 1 and 2 would be present. But if the dielectric is a piezoelectric material such as quartz, then all three are present.

The first current

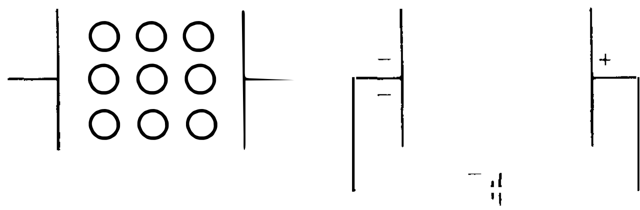
$$i_1 = A \frac{dD_1}{dt}$$

is called the Maxwell displacement current. D_1 is called the Maxwell displacement. It is the charge per unit area which appears on the plates of the capacitor. The displacement is related to the electric field between the plates by the equation

$$D_1 = \epsilon_0 E \tag{a}$$

where D_1 and E have the same direction and ϵ_0 is called the permittivity of vacuum.

The second current is the dielectric polarization current. We may think of a dielectric as being composed of + and - charges with limited freedom of motion relative to each other. Each molecule (or other unit) is said to have an induced dipole moment p when the field is applied. The dipole moment of each mole-



Dielectric without field

cule is the product of the magnitude of the charge and the distance between the two charges. If we have n dipoles per unit volume, the dipole moment per unit volume, or the polarization, is

$$P = n\,p$$

If we let Q be the charge per unit area induced on the electrodes by the polarization of the dielectric, then QA is the charge on one electrode. The other electrode has charge $-QA$. The dipole moment of the dielectric is therefore QAe. But Ae is the volume of the dielectric, so that Q is the dipole moment per unit volume or the polarization. Hence we have the important result that the polarization of a dielectric is equal to the induced surface charge per unit area. This is true whether the polarization is due to dielectric or piezoelectic effects or both.

When the capacitor is charged the dielectric is said to be polarized. It should be noted that the polarization P which is taken to point from the negative charge toward the positive charge of the dipole is opposite to the direction of the electric field which points from the positive toward the negative electrode.

In many dielectric materials the polarization and hence the electric displacement is proportional to the applied field, so we may write

$$-P_2 = D_2 = \eta E \qquad (b)$$

where η is called the dielectric susceptibility.

The third current is the piezoelectric displacement current resulting from the polarization due to the strain in the crystal. It is found that the polarization in quartz is proportional to the strain, so that we may write

$$P_3 = \epsilon\, x$$

But the strain is related to the electi the equation

$$x = d\,E$$

Therefore the polarization and electric displacement are given by

$$-P_3 = D_3 = d\epsilon E \qquad (c)$$

Combining Eqs. (*a*), (*b*), and (*c*) we have

$$D = D_1 + D_2 + D_3 = (\epsilon_0 + \eta + d\,\epsilon)\,E = k'E$$

or

$$D = \epsilon_0 \left(1 + \frac{\eta}{\epsilon_0} + \frac{d\epsilon}{\epsilon_0}\right) E$$

The quantity $1 + \eta/\epsilon_0$ is called the dielectric constant in the case of an ordinary dielectric, i.e., one which is not piezoelectric. It is usually designated K_e. The quantity $1 + \eta/\epsilon_0 + d\,\epsilon/\epsilon_0$ is analogous to the dielectric constant in an ordinary dielectric.

The force acting on a charge q in an electric field is

$$F = -qE$$

where the minus sign signifies that the force on a positive charge is opposite in direction to E.

The work done in moving a charge q a distance dy parallel to E is

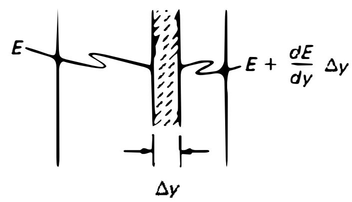
$$F\,dy = -q\,E\,dy$$

The work per unit charge, which is $-E\,dy$, is called the potential difference dV. Therefore $dV = -E\,dy$ and

$$E = -\frac{dV}{dy} \qquad (d)$$

Let us now suppose that the charge density in the dielectric is not constant but varies from point to point. This is the case if the dielectric is also piezoelectric and the strain is not uniform. We suppose the strain and therefore the charge density, q_v, to vary in the y direction. Let $q_v\,(y)$ represent the charge density expressed as a function of y. Consider an element in the form of a thin slice Δy. Let E be the field intensity at the left side of the element. Then the field intensity at the right side is

$$E + \frac{dE}{dy}\,\Delta y$$



By Gauss' law

$$k\,\epsilon_0 \int_s E \cdot dA = \int_v q_v(y)\,A\,dy$$

On the left side of the element the surface integral yields $k\epsilon_0 A E$ directed toward the left. On the right side it yields

$$k\,\epsilon_0 A\left(E + \frac{dE}{dy}\,\Delta y\right)$$

directed toward the right. The integrals over the other sides of the element are zero, since the field is assumed parallel to the y direction. Therefore the integral over the entire surface of the element is

$$k\,\epsilon_0\,A\,\frac{dE}{dy}\,\Delta y$$

But the charge enclosed by the element is simply $q_v A\,\Delta y$ and therefore by Gauss' law

$$k\,\epsilon_0\,\frac{dE}{dy} = q_v(y) \tag{e}$$

We may note, in passing, that if $q_v = 0$, then $dE/dy = 0$ and therefore E is constant. But if q_v is not zero, then E is a function of y. This is the case when the strain varies in the y direction due to vibration in a piezoelectric material.

The electric displacement $D = k\,\epsilon_0\,E$, and therefore

$$\frac{dD}{dy} = k\,\epsilon_0\,\frac{dE}{dy}$$

From (e),

$$\frac{dE}{dy} = \frac{q_v}{k\,\epsilon_0} \tag{f}$$

Therefore,

$$\frac{dD}{dy} = q_v(y)$$

Since P is oppositely directed to D and E

$$\frac{dP}{dy} = -q_v(y) \tag{g}$$

From (d),

$$\frac{dE}{dy} = -\frac{d^2 V}{dy^2}$$

By using (f) we have Poisson's equation:

$$\frac{d^2 V}{dy^2} = -\frac{q_v}{k\epsilon_0} \tag{h}$$

If $q_v = 0$ we have LaPlace's equation

$$\frac{d^2 V}{dy^2} = 0 \tag{i}$$

Equations (g), (h), and (i) are required in deriving the equivalent circuit of the quartz resonator in Chap. 6.