10
INHARMONIC OVERTONE MODES AND FILTER CRYSTALS

INTRODUCTION

The use of thickness-shear resonators as components in HF and VHF wave filters was delayed for many years because of the existence of the inharmonic overtone modes which are present in all such piezoids. As we have seen in Chap. 5, the frequencies of these modes lie in a relatively narrow range immediately above that of the main mode where they cause severe problems in the design of wave filters. These modes came to be called *spurious modes* and later *spurs.*

It is the purpose of this chapter to review the nature of these modes and to discuss the methods which are available for dealing with them in the fabrication of crystal units for use in HF and VHF filters.

Before continuing with this chapter the reader may find it useful to review the material presented in Chap. 5.

NATURE OF THE INHARMONIC OVERTONE MODES

The inharmonic overtone modes, or spurs, are not an expression of the perversity of nature or of demon possession. On the contrary they are natural phenomena just as are the overtones of an organ pipe or piano string. They are more closely analogous to the overtones of a bell which, in general, are all higher in pitch than the fundamental and have frequencies which are not harmonically related to the fundamental or to each other.

The overtones of any vibrating system are determined primarily by the nature of the medium and the boundary conditions and very little by the method of excitation. For example, a violin string tuned to the note G produces this note whether the string is plucked or bowed or even if the violin is dropped on the floor. It is most important to note this point. It is especially applicable to quartz resonators in which the damping losses are so small that any mode which is physically possible and permitted by the boundary conditions may be (and probably will be) excited.

It should be recalled that inharmonic modes do not occur in one-dimensional vibrating systems in which the only possible modes are the harmonic overtone modes. In two- and three-dimensional systems inharmonic as well as harmonic modes are possible. An infinitely large AT-cut plate with electrodes covering the entire surface would behave as a one-dimensional resonator and would have only harmonic modes with $n = 1, 3, 5, \ldots$. In a finite plate different sets of inharmonic overtones are produced depending on whether the edges are clamped or free. If the edges are clamped, nodes of motion must occur at the edges while a free edge acts as an antinode of motion.

THE GEOMETRY OF THE PIEZOID

Since we are here concerned with the overtone modes of the resonators used in HF and VHF wave filters, we shall consider a typical unit consisting of a circular AT-cut plate. The electrodes are deposited on the quartz in the form of a circular area concentric with the boundary of the plate. The plate is assumed to have plane, parallel surfaces although in most cases the surfaces are slightly convex due to the nature of the lapping process used in fabricating the blanks.

Except for a small amount of fringing at the edge of the electrodes, the electric field is confined to the region between the electrodes and the piezoelectric strain is therefore confined to this portion of the plate. The zone of quartz surrounding the electrodes is not excited piezoelectrically, so we may think of it as a device for supporting the vibrating area much as the frame of a drum supports the stretched membrane which forms the drumhead. We will see later that the surrounding zone plays an important role in determining the characteristics of the resonator.

THE FUNDAMENTAL, OR *n*01, MODE

The primary excitation of the AT-cut plate consists of the $X_y{}'$ shear strain. The resulting vibration at the resonant frequency consists of a transverse wave traveling in the thickness direction of the plate. The particle motion is in the X-direction. The plane parallel to and midway between the surfaces is a nodal plane and consequently the displacement on opposite sides of it are in opposite directions.

At the surface of a vibrating plate the mechanical displacement and the electric polarization have similar spacial distributions. This may be a bit confusing, since the polarization depends on the strain and not the displacement. It must be remembered, however, that the polarization depends on the strain *in* the plate rather than the strain at the surface (which is zero) and that the internal strain is greatest at the center of the plate. The displacement, strain, and polarization all become zero near the edge of the electrode.

Numerous experiments show that the amplitude of the motion decreases substantially to zero at a distance less than 20 times the plate thickness from the edge of the electrode. It is likely that a part of the excitation in the zone immediately surrounding the electrodes is due to fringing of the field. Figure 10.1 gives some idea of

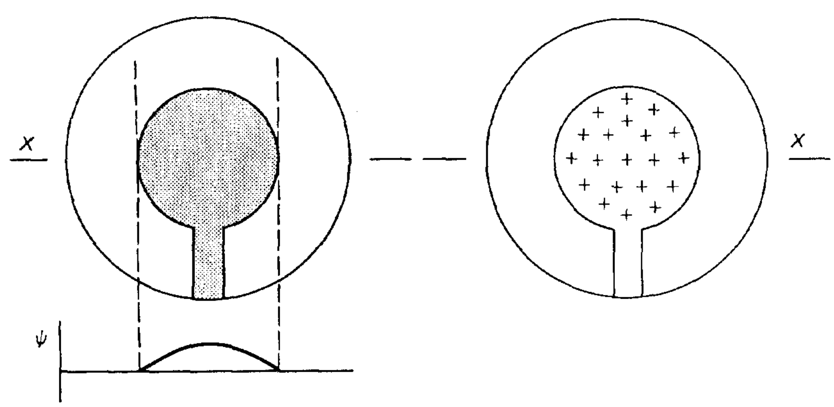

Fig. 10.1. The displacement and charge distribution which would result if an unelectroded plate were excited in the *n*01 mode.

the distribution of the displacement across the diameter of the plate in the direction of the X-axis.

If the plate were infinitely large and the driving field were uniform over the entire surface, the distribution of the strain and the surface charge density would be constant over the entire surface. The motion would be that of a plane wave traveling in the thickness direction of the plate. In an actual resonator the strain must decrease to zero at some point not far from the edge of the electrode area. Furthermore, the conducting electrodes make the surface charge density approximately constant over the surface. This results in an apparent increase in the elastic constant and tends to raise the frequency of the resonator. Since this effect is small, we will ignore it along with the effects of anisotropy.

The distribution of motion over the vibrating area may be considered to result from a wave traveling radially on the surface and being reflected at the boundary between the vibrating and nonvibrating regions. The result is a standing wave with a nodal line around the circumference and an antinode in the center. The standing wave is the result of interference between waves incident on, and reflected from, the edge in the same way that standing waves are set up on a string by interference between the waves traveling in opposite directions on the string. A closer analogy is that of the vibration of a circular drumhead in which the center of the drumhead moves in a direction perpendicular to its plane while the edge remains stationary. (It must be emphasized that this is only an analogy. The motions are quite different in the two cases.)

THE FIRST-INHARMONIC OVERTONE, OR $n11$, MODE

When the plate is excited in the $n11$ mode, waves travel around the plate in opposite directions, setting up a standing wave with a nodal line along a diameter of the vibrating area. The displacement and strain on opposite sides of the nodal line are opposite in direction and the charges developed have opposite signs. Figure 10.2 shows the distribution of the strain along a diameter perpendicular to the nodal line. The nodal line around the edge is, of course, still present.

The piezoelectric current to the resonator is the algebraic sum of the currents to the two regions and the fact that these two currents are 180° out of phase suggests that if the strains, their distribution,

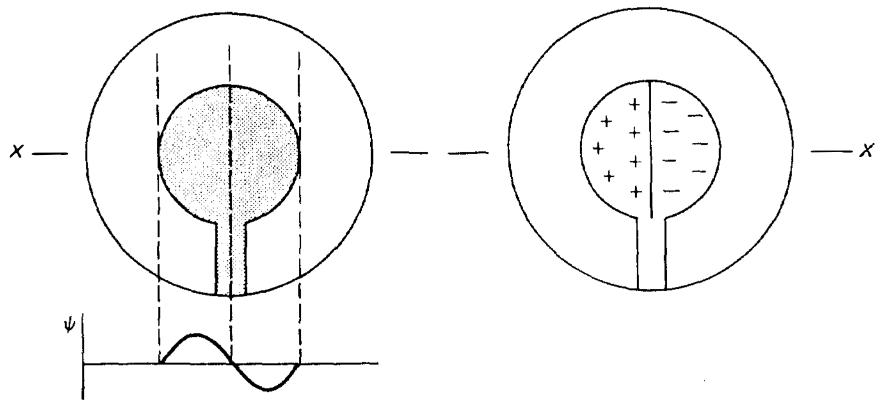

Fig. 10.2. The displacement and charge distribution on the surface of an unelectroded AT-cut plate vibrating in the 111 mode.

and the areas of the two regions were identical, then the piezoelectric current should be zero and the mode should not be excited. If excitation of the mode were desired, it could be achieved by dividing the plating at the nodal line and crisscrossing the connections so that the fields would be opposite in the two portions.

In practice the quartz plate should be lapped to have as nearly perfect symmetry as possible and the electrodes should be carefully aligned and located concentric with the blank. This is important because the n11 mode is the first overtone above the fundamental and is therefore the most troublesome one. Although it is possible, in theory, to suppress the nm1 modes by careful control of symmetry, and it is true that the modes are often missing or very weakly excited, it is difficult to eliminate them entirely in this way. The slightest asymmetry is sufficient to cause the plate to be excited in one of these modes.

THE n02 MODE

The second-overtone mode of a circular plate is the n02 mode. When this mode is excited a radial standing wave is set up. The distance from the center of the plate to the edge is one wavelength and consequently a second circular nodal line is formed concentric with the nodal line at the circumference. This divides the surface of the plate into two zones. Naturally the displacement, strain, and polarization are opposite in the two zones. The displacement along any diameter of the plate is shown in Fig. 10.3.

It would be difficult or impossible to suppress the $n02$ mode or any other mode having circular nodal lines by controlling the symmetry of the quartz blank and the plating, since the strains and charges also have circular symmetry.

THE EFFECTS OF CONTOURING

The frequency separation between the main and overtone modes is inversely proportional to the square of the radius, or the area, of the vibrating region, as shown in Eq. (42). Therefore, reducing the size of the vibrating area should cause the troublesome inharmonic modes to move to higher frequencies, which is indeed the case.

One method of reducing the size of the vibrating area is to make the blank thinner at the edges than in the center. This is usually done by lapping one or both sides to have a spherical shape. The process of generating a spherical surface on a quartz plate is called *contouring*. Spherical surfaces are used, not because they are necessarily the best configuration, but because spherical surfaces are relatively easy to generate using the techniques for grinding spherical surfaces on lenses.

As the radius of curvature of the surface of the quartz plate is reduced, the frequencies of the inharmonic overtone modes move to higher frequencies and the overtone spectrum becomes more dispersed.

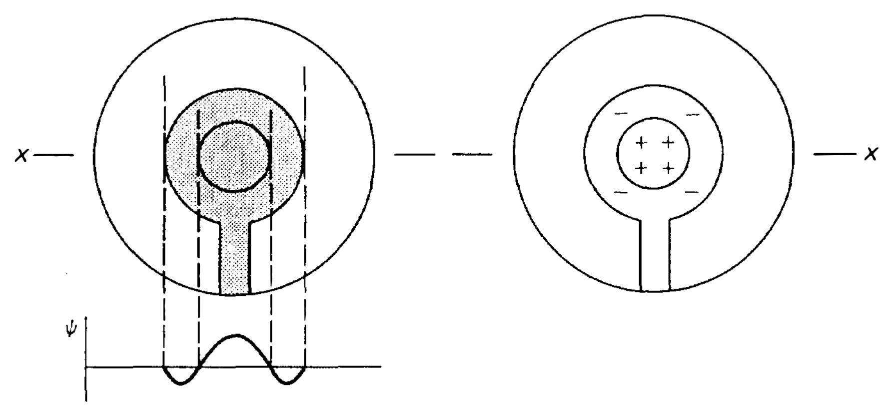

Fig. 10.3. The displacement and charge distribution on the surface of a typical AT-cut plate excited in the $n02$ mode.

It is easy to increase the frequency interval between the main mode and the first inharmonic overtone mode by 2 percent or more of the nominal frequency.

Figure 10.4 shows the effect of contouring on the overtone frequency spectrum of a circular AT-cut plate having a diameter of 0.550 in (1.4 cm). Initially the surfaces of the plate were plane and parallel. One side of the blank was contoured using an optical lap to generate a spherical surface. The entire surface of the blank can be made spherical without changing the frequency of the fundamental mode if care is taken not to reduce the thickness of the plate at the center.

Laps with successively smaller radii of curvature were used. The ordinate of Fig. 10.4 is given in units of $1/2R$, where R is the radius of curvature of the lap measured in meters. (This corresponds approximately to the diopter rating used by opticians.) An ordinate of 1 corresponds to a radius of curvature of 0.5 m, an ordinate of 2 corresponds to a radius of 0.25 m, etc.

The effect of decreasing the radius of curvature of the spherical surface is readily apparent. Initially, when the surface is "flat" the

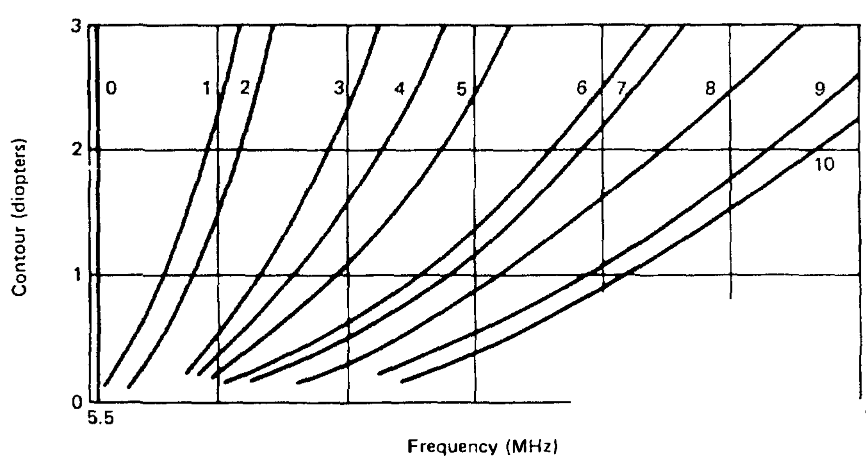

Fig. 10.4. Graph showing the (experimental) changes of the frequencies of the fundamental and first 10 overtones of a 5.5-MHz AT-cut blank as the radius of curvature of one side was changed from infinity (a flat blank) to 0.16 m. The ordinate is in units of $1/2R$, where R is in meters. The numbers are roughly equal to the so-called diopter reading which is marked on optical laps.

frequencies of the overtone modes are very closely grouped just above the frequency of the main mode. The frequencies are so close together that their resonances tend to overlap and it is impossible to determine the resonant frequencies uniquely. As contouring proceeds, the frequency of each overtone mode moves higher in a quite regular way. Dust-pattern studies show that each mode retains its characteristic identity and that the area of the vibrating portion of the blank decreases as contouring proceeds. The vibrating areas may be either smaller or larger than the electrode area and the nodal pattern is independent of both the size and location of the electrode, although the relative amplitudes or equivalent resistances of the modes is thereby affected.

Other effects accompany the changes in the overtone spectrum as the blank is contoured. Slight contouring reduces the value of the equivalent resistance R by reducing damping losses through the mounting supports and by moving the spurious modes away from the resonant frequency. Further contouring causes the value of R to increase because of the decrease in the vibrating area.

The values of L and C are also affected. As the contouring proceeds L increases and C decreases. Hence, contouring provides a method for controlling the values of these two parameters.

The capacitance ratio $r = C_0/C$ is also influenced by contouring. It is shown in App. IV that the theoretical value of r for a large, thin AT-cut plate should be about 154. In arriving at this value it is assumed that the distribution of strain is uniform over the entire electroded area and that no fringing occurs at the electrode edge. Since neither of these conditions is satisfied and because the holder adds some capacitance to C_0, the measured value of r is always greater than 154. By careful attention to the size of the electroded area in relation to the contour, values of r as low as 190 can be achieved.

It is shown in App. IV that the value of r is closely associated with the value of κ, the piezoelectric coupling coefficient, which is a measure of the effectiveness of the electric field in exciting mechanical vibration in the quartz. Hence, contouring affords a method for changing the value of κ.

Although contouring does provide a degree of control over the values of L and C and the frequencies of the overtone modes, its usefulness is limited by the fact that the two effects are not independent.

The contour required to disperse the overtone modes may not yield the values of L and C which are simultaneously required.

Contouring is most useful in the range of frequencies between about 2 and 5 MHz. At higher frequencies, 10 to 100 MHz, which is the range in which many applications occur, contouring is impractical or at least a very difficult production process.

Fortunately a method of eliminating or at least suppressing the undesired overtone modes without unduly influencing the motional inductance and capacitance is available and it is especially applicable at the higher frequencies. It is called *energy trapping*.

ENERGY TRAPPING

Resonance in a vibrating system is always associated with the phenomenon of standing waves. In order for standing waves to occur, certain simple relationships must exist between the dimensions of the resonator and the wavelength of the disturbance being propagated on it. For example, the length of a stretched string, vibrating in a resonant mode, must be an integral number multiplied by half the wavelength of the disturbance on the string. The wave which travels one way on the string is reflected at the end and the interference between the incident and reflected waves produces the standing waves and resonance. It is impossible for resonance to occur in a string of infinite length because the reflected wave would not return to the source in a finite time. In an actual vibrating system some energy is dissipated as the wave is propagated and consequently the wave may be dissipated before it returns. Such a system could not sustain standing waves. If the incident wave could be completely absorbed by some mechanism, then no energy would be reflected and again no standing wave would result.

Some waves travel in a tangential direction in circular and spherical vibrating systems. The condition for standing waves and resonance to occur in these cases is that the length of the path followed by the wave must be an integral number of wavelengths. Those modes of motion (see Fig. 5.8) having nodes along diameters are associated with waves traveling in the tangential direction. The number of nodal diameters is equal to the number of wavelengths divided into the length of the path. These waves are not reflected at any boundary but result from the interference of two waves traveling in opposite direction in the tangential directions.

On the other hand, resonance associated with waves traveling in the radial direction depend on reflection at the circular boundary. In this case the nodal lines are concentric with the center of the plate.

Both types of standing waves occur in circular plates. It is important to note that the modes which produce circular nodes are due to reflection of waves at the edge of the plate, but those modes which result in nodes along the diameters are not due to reflection but rather to the condition that the length of a closed path must be an integral number of wavelengths.

The theory of energy trapping is based on the principle that traveling waves can be propagated and thereby produce standing waves and resonances only for frequencies above a certain minimum value called the *cutoff* frequency. The fundamental mode frequency is the cutoff frequency of a one-dimensional system, but this is of only trivial interest.

As we have seen, in a multidimensional system some modes exist which have frequencies only a little above the fundamental mode. It can be shown that these modes may be trapped and therefore prevented from being reflected, thereby setting up the resonances which are called the inharmonic modes or spurs.

The mathematics of the problem for a system with circular symmetry and including the effects of anisotropy is very complex, but we can see the general features and even make some useful calculations by considering the propagation of waves in a much simpler system. For this purpose we consider the propagation of waves in an isotropic plate of thickness e and width w, oriented with respect to orthogonal axes, as shown in Fig. 10.5.

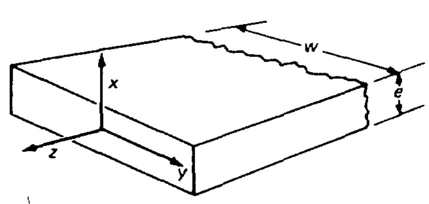

Fig. 10.5. Thin plate with waves propagated in the length or z direction. All surfaces are assumed to be free (not clamped).

The equation for the displacement, expressed in rectangular coordinates, is

$$\frac{\partial^2 \psi}{\partial t^2} = \nu^2 \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right) \tag{93}$$

We are interested in solutions of Eq. (93) which describe traveling waves in the z direction. Such waves, reflected at a boundary, produce standing waves which are associated with the inharmonic overtones.

Such a solution is

$$\psi = [X(x)]\,[Y(y)] \cdot [\exp j\,(\omega t \pm kz)] \tag{94}$$

where $k = 2\pi/\lambda_z$ and λ_z is the wavelength of the wave traveling in the z direction produced by the disturbance of frequency ω.

By computing the various second derivatives of Eq. (94) and substituting in Eq. (93) we have

$$k^2 - \frac{\omega^2}{\nu^2} = \frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 y}{dy^2}$$

Using the usual method of separation of variables

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -\zeta^2 = k^2 - \frac{\omega^2}{\nu^2} - \frac{1}{Y}\frac{d^2 y}{dy^2} \tag{95}$$

where ζ^2 is a constant to be determined.

The solution of the equation involving x is

$$X = A \cos(\zeta x) + B \sin(\zeta x)$$

The boundary conditions require that $\psi = 0$ at $x = 0$ and to be a maximum at $x = \pm \frac{1}{2} e$. These conditions are satisfied if $A = 0$ and $\zeta = (n\,\pi/e)$, where $n = 1, 3, 5, \ldots$. The portion of Eq. (95) involving y can be written

$$\frac{d^2 y}{dy^2} = \gamma^2 Y \tag{96}$$

where

$$\gamma^2 = \zeta^2 + k^2 - \frac{\omega^2}{\nu^2} \tag{97}$$

The solution of Eq. (96) depends on the sign of γ^2. There are three separate cases. If γ^2 is positive, the solution is

$$Y = C \cosh(\gamma y) + D \sinh \gamma y)$$

If $\gamma^2 = 0$, the solution is

$$Y = Ey + F$$

If γ^2 is negative, the solution is

$$Y = G \sin(\gamma y) + H \cos(\gamma y)$$

The last is the only solution which is compatible with a resonant condition. If we assume that $Y = 0$ at the center of the bar where $y = 0$ and is a maximum at the edges where $y = \pm \frac{1}{2} w$, then $H = 0$ and $\gamma = (m\pi/w)$.

Substituting the value for ζ and γ in Eq. (97) and solving for k^2 we obtain

$$k^2 = \frac{\omega^2}{\nu^2} - \pi^2 \left(\frac{n^2}{e^2} + \frac{m^2}{w^2}\right) \tag{98}$$

Now $k^2 = (2\pi/\lambda_g)$,^{2} where λ_g is the length of the wave propagated in the z direction in the plate, and $(\omega/\nu)^2 = (2\pi/\lambda)^2$, where λ is the wavelength in an infinite plate, both waves being excited by the same frequency.

Hence we may write

$$\left(\frac{1}{\lambda_g}\right)^2 = \left(\frac{1}{\lambda}\right)^2 - \frac{1}{4}\left(\frac{n^2}{e^2} + \frac{m^2}{w^2}\right) \tag{99}$$

from which it is easily seen that λ_g is greater than λ unless $m = n = 0$ or unless both e and w are infinite.

Now k^2 in Eq. (98) must be positive or no wave exists. If $k^2 = 0$, the wavelength λ_g is infinite; and if k^2 is negative, the wavelength is imaginary. The wavelength λ for which λ_g is infinite is called the *cutoff wavelength*. It is given by

$$\lambda_c = \frac{2}{(n^2/e^2 + m^2/w^2)^{1/2}} \tag{100}$$

Any actual wave propagated in the rectangular plate must have a wavelength (in an infinite plate) less than λ_c. The frequency $f_c = v/\lambda_c$ is called the *cutoff frequency*. It is the lowest frequency which can be propagated. Waves excited by frequencies lower than f_c cannot be propagated and are attenuated exponentially around the source.

Those familiar with the propagation of electromagnetic waves in wave guides will immediately note the similarity between the behavior of waves traveling in wave guides and waves traveling in solids. The cutoff phenomenon is commonly used in wave guides to eliminate undesired modes and frequencies.

The phenomenon in actual crystalline solids is, of course, much more complicated than that of the hypothetical case developed here. Not only have we ignored the effects of anisotropy but we have also neglected all cross-coupling effects. But the neglect of both effects may not be as serious as might at first be thought. The anisotropy effects in quartz are not large. For example, the three elastic constants involved – c'_{66}, c'_{55}, and c'_{44} – have values 29.3, 69.8, and 41.4×10^9 N/m^2 for the AT-cut orientation. The velocities of the shear waves depending on these constants are in the ratio of the square roots of the numbers or approximately 5:8:6.

The effects of coupling to other modes, through c'_{56}, c'_{12}, etc., have little effect unless the resonant frequencies of the coupled modes are within a few tenths of a percent of each other.

Let us now consider an AT-cut plate of indefinite lateral extent having a thickness e so that its fundamental resonant frequency is given by

$$f_S = \frac{1}{2e}\left(\frac{c'_{66}}{\rho}\right)^{1/2}$$

The area on which the electrodes are plated is designated region E and the surrounding unplated area region S. The mass added by the electrodes makes the effective thickness of region E equal to e'. The frequency of region E is therefore

$$f_E = \frac{1}{2e'} \left(\frac{c'_{66}}{\rho}\right)^{1/2}$$

By means of a signal generator of variable frequency the region under the electrodes can be set into resonance at the frequency f_E. The resulting mode of motion may be described as a plane wave traveling in the thickness direction of the plate. This mode is called the fundamental, or first-harmonic, mode.

As the frequency of the generator is increased other discrete resonances are observed. These are the inharmonic overtone modes, commonly called spurious modes, or simply spurs. They are also associated with standing waves in region E. However, these waves are not propagated in the thickness direction but in directions having components in the plane of the blank. Resonance occurs when these waves are reflected at the discontinuity between regions E and S. The frequencies of these inharmonic modes depend on the dimensions of the plated region and, of course, the thickness of the plate. When the plate is excited at one of the inharmonic overtone frequencies, standing waves may be observed in region E. As stated before, the condition for standing waves to exist is that *reflection must occur at the boundary between regions E and S.*

It is characteristic of any vibrating system that only those modes having frequencies above a certain minimum or cutoff frequency can occur because standing waves of lower frequency cannot exist in the system. The cutoff frequency for region E is f_E and that for region S is f_S. Waves having frequencies higher than f_E and lower than f_S can exist in region E but not in region S, where only waves of frequency equal to or greater than f_S can exist.

Waves in region E having frequencies between f_E and f_S are totally reflected at the boundary between regions E and S. If these waves are excited by the correct frequencies, standing waves may be generated, resulting in spurs. Waves generated by frequencies greater than f_S are not reflected and travel outward toward the edge of the blank.

Suppose that rectangular electrodes having dimensions $w \times 1$ are deposited on region E. The eigenfrequencies of a rectangular (isotropic) plate with perfectly reflecting edges are, from Eq. (37),

$$f_{nmp} = \frac{v}{2}\left[\left(\frac{n}{e}\right)^2 + \left(\frac{m}{w}\right)^2 + \left(\frac{p}{1}\right)^2\right]^{\frac{1}{2}} \qquad \begin{aligned} n &= 1, 3, 5, \ldots \\ m &= 1, 2, 3, \ldots \\ p &= 1, 2, 3, \ldots \end{aligned}$$

If w and 1 are each large compared with e, we may write,

$$f_{nmp} \simeq f_{n11}\left[1 + \frac{1}{2}\left(\frac{me}{nw}\right)^2 + \frac{1}{2}\left(\frac{pe}{nl}\right)^2\right]$$

where $f_{n11} = (nv/2e)$ is the frequency of harmonic mode of order n.

The condition for a mode of frequency f_{nmp} *not* to be excited is that $nf_s < f_{nmp}$, which requires that

$$nf_s < f_{n11}\left[1 + \frac{1}{2}\left(\frac{me}{nw}\right)^2 + \frac{1}{2}\left(\frac{pe}{nl}\right)^2\right]$$

and

$$nf_s - f_{n11} < f_{n11} \cdot \left(\frac{1}{2n^2}\right)\left[\left(\frac{me}{w}\right)^2 + \left(\frac{pe}{l}\right)^2\right] \qquad (101)$$

Equation (101) gives the plate-back required to suppress the *nmp* mode and all inharmonic modes of higher frequency.

As an illustration of the principle we consider a 10.000 MHz AT-cut plate having square electrodes with dimensions $w = l = 0.300$ cm (0.118 in). The thickness $e = 0.0169$ cm (0.00655 in). The unit is operated on its first-harmonic mode $n = 1$. The first harmonic mode is the 112 (or 121) mode and therefore the plate back must be *less than*

$$10.000 \times 10^6 \times \frac{1}{2}\left[\left(2\ \frac{0.0169}{0.300}\right)^2 + \left(1\ \frac{0.0169}{0.300}\right)^2\right] = 79 \text{ kHz}$$

In order to fabricate such a device the plate is first lapped or etched to a frequency somewhat *below* 10.079 MHz. The electrodes are then applied reducing the frequency to 10.000 MHz. The frequencies of all inharmonic overtones are higher than 10.079 MHz and are therefore not excited, since waves of this and higher frequencies are *not* reflected at the boundary between regions E and S.

To review: Spurious modes can exist in the E region only if waves are reflected at the boundary between E and S. If the frequency of every *potential* mode in E is greater than the cutoff frequency in S, then no wave is reflected and no spurious mode can exist.

As another example we consider the same blank to be supplied with circular electrodes having a diameter of 0.300 cm. The eigenfrequencies of a circular plate with a clamped boundary are found, from Eq. (42), to be

$$f_{nmk} = f_{n01}\left[1 + 2\left(\frac{\chi_{mk}\, e}{n\pi\, d}\right)^2\right] \qquad \begin{aligned} n &= 1, 3, 5, \ldots \\ m &= 0, 1, 2, 3, \ldots \\ k &= 1, 2, 3, \ldots \end{aligned}$$

where χ_{mk} is the kth root of Bessel's function of order m. The frequency of the nth harmonic mode is f_{n01}.

The condition that a mode with frequency f_{nmk} *not be excited* is that $nf_s < f_{nmk}$ and therefore that

$$nf_s < f_{n01}\left[1 + \frac{M_{mk}}{n^2}\left(\frac{e}{d}\right)^2\right] \qquad \text{where } M_{mk} = 2\left(\frac{\chi_{mk}}{\pi}\right)^2$$

or that

$$nf_s - f_{n01} < f_{n01}\,\frac{M_{mk}}{n^2}\left(\frac{e}{d}\right)^2 \tag{102}$$

The plate-back required to suppress the first, or $n11$, mode, and all higher modes, can now be computed. The value of $\chi_{11} = 3.832$. Therefore the plate-back must *be less than*

$$10.000 \times 10^6\ \frac{2.98}{n^2}\left(\frac{0.0169}{0.300}\right)^2 = \frac{95}{n^2}\ \text{kHz}$$

Thus, if the resonator is to be used at 10 MHz with all spurs suppressed, the maximum plate-back is 95 kHz. But if the resonator were to be used at its third harmonic at 30 MHz, the maximum plate-back would be only 10 kHz. This might not provide adequate electric conductance.

It is obvious that a tradeoff exists between the thickness/diameter ratio and the plate-back. The allowable plate-back may be increased by reducing the electrode size, but this influences the values of the parameters of the equivalent circuit; in particular, it reduces the value of the motional capacitance C. One solution to this problem is to use the method of "dot" resonators in which several small isolated resonators, all having the same frequency, are operated in parallel.

In order to suppress all inharmonic overtone modes in a resonator with square electrodes we have, from Eq. (101),

$$\text{Plate-back} < f_s \, \frac{2.5}{n^2} \left(\frac{e}{w}\right)^2 \qquad w = 1 \tag{103}$$

Similarly, in order to suppress all inharmonic overtone modes in a resonator with circular electrodes we have, from Eq. (102),

$$\text{Plate-back} < f_s \, \frac{2.98}{n^2} \left(\frac{e}{d}\right)^2 \tag{104}$$

In both cases the maximum permissible plate-back is proportional to the frequency of the resonator, inversely proportional to the square of the harmonic order, and proportional to the square of the thickness/lateral dimension ratio.

It should not be expected that the constants 2.5 and 2.98 are exactly correct, because the effects of anisotropy have been ignored and also because a number of approximations have been made in the derivations. But the general forms of Eqs. (103) and (104) do apply and with the constants determined empirically should provide a good starting point for the designer.

It is possible that, in some cases, the first overtone may be absent because of charge cancelation. In such instances, the next overtone is the first one which must be considered and the constants are larger. It must be noted that any convexity of the surfaces of the blank also influences the overtone frequencies.

Finally, some modes which are not suppressed by control of the plate back may often be suppressed by loading the blank at certain points on the unplated area with dots of some cement. The dots are located empirically and experience (and perhaps intuition) helps in determining the correct size and location.

For those readers who are interested in a nonmathematical explanation of energy trapping, we present the following qualitative explanation.

1. An elastic plate having a given set of dimensions, bounded by reflecting edges, has an overtone spectrum of discrete, inharmonically related frequencies.
2. The boundary between the plated and unplated regions of an AT-cut plate acts as a reflecting edge for all acoustic waves having frequencies less than that of the resonant frequency of the unplated area.
3. If the frequency of the unplated area is *less* than that of the overtone of lowest frequency, then the plated area can be excited into resonance *only* at its own harmonic frequency (provided no other reflecting edges are present).

A more complete discussion of the energy-trapping concept is given by Shockley, Curran, and Koneval.[1] If Eq. (24) of this paper is solved for $f_e - f_s$, the result is the same as Eqs. (103) and (104) except for the values of the constants. Those interested in a more complete discussion of this subject should consult the original article.

[1] *J. Acoust. Soc. Amer.*, **41**:985-993 (1967).