7
THE CRYSTAL UNIT AS A CIRCUIT COMPONENT

INTRODUCTION

In the previous chapter we have shown that the piezoelectric resonator is electrically equivalent to the network of Fig. 6.5. It is impossible, by any set of electrical measurements made at terminals A and B, to distinguish between a piezoelectric resonator and an electric circuit made up of these components. One might ask, if this be true, "What is the advantage of the quartz crystal unit over a conventional circuit?" The answer lies in the values which conventional components would have to provide in order to simulate the behavior of the crystal unit. The simulation of C_0 would present no problem, but the simulation of the other parameters is, in general, quite impossible. For example, the value of C in a typical 5-MHz AT-cut unit may be of the order of 0.01 pF. A capacitor of this size could scarcely be built, since its leads, if any, would probably have more capacitance than this. The value of L is of the order of 0.1 H and R is perhaps 10 Ω. A coil having an inductance of 0.1 H would ordinarily require many turns of wire on a ferromagnetic core. To construct such a coil with a resistance of only 10 Ω in a volume of less than 0.5 cm^3 is probably impossible.

The properties which give the quartz crystal unit its unique ability to control a frequency are its high Q, or quality, factor and its low κ, or electromechanical, coupling.

The Q of any resonant system is given by

$$Q = 2\pi \frac{\text{Energy stored per cycle}}{\text{Energy dissipated per cycle}}$$

The Q of a series-resonant electrical circuit is given by $Q = \omega L/R = 1/\omega RC$ and may have a value of a few hundred. The Q of a well-made tuning fork may be a few thousand, but the Q of a quartz crystal unit is usually not less than 10^5 and may be as high as 10^7.

The value of κ, the electromechanical coupling coefficient, is given by

$$\kappa = \sqrt{\frac{\text{Energy stored elastically}}{\text{Energy stored electrically}}}$$

The value of κ^2 for an AT-cut plate (see App. IV) is less than 1 percent. This means that the electric driving system is very loosely coupled to the mechanical system and that the former, therefore, has very little influence over the latter. But because of the high Q of quartz, even this small effect is sufficient to excite and maintain vibrations in the quartz piezoid, which is what makes it an effective transducer for controlling a frequency.

THE RESONANT FREQUENCIES

The admittance Y_{AB} between the terminals A and B of Fig. 6.5 is given by

$$Y_{AB} = \frac{1}{Z_{AB}} = \frac{1}{R + j(\omega L - 1/\omega C)} + j\omega C_0$$

$$= \frac{1 + j\omega RC_0 - \omega^2 LC_0 + C_0/C}{R + j(\omega L - 1/\omega C)}$$

$$Z_{AB} = \frac{R + j(\omega L - 1/\omega C)}{1 - \omega^2 LC_0 + C_0/C + j\omega RC_0}$$

Multiplying the numerator and denominator by the complex conjugate of the latter gives

$$Z_{AB} = \frac{R + j(\omega L - 1/\omega C - \omega^3 L^2 C_0 + 2\omega L C_0/C - C_0/\omega C^2 - \omega R^2 C_0)}{C_0^2/C^2 + 1 - 2\omega^2 L C_0 + 2C_0/C + \omega^2 R^2 C_0^2 - 2\omega L C_0^2/C + \omega^4 L^2 C_0^2} \tag{62}$$

The condition for the circuit to be resonant is that the imaginary part of Eq. (62) should be equal to zero. Setting the imaginary part of Eq. (62) equal to zero and clearing of fractions we have

$$\omega^2 LC^2 - C - \omega^4 L^2 C^2 C_0 + 2\omega^2 LCC_0 - C_0 - \omega^2 R^2 C^2 C_0 = 0 \tag{63}$$

For most practical resonators and within the accuracy of many measurements, the value of the last term containing the resistance R is negligible compared with the other terms and may be omitted in calculating the resonant frequencies. Omitting this term and rearranging gives

$$\omega^4 L^2 C^2 C_0 - \omega^2 (LC^2 + 2LCC_0) + (C + C_0) = 0 \tag{64}$$

Solving Eq. (64) by means of the quadratic formula, we obtain

$$\omega^2 = \frac{2LCC_0}{2L^2 C^2 C_0} \qquad \text{and} \qquad \omega^2 = \frac{2LC^2 + 2LCC_0}{2L^2 C^2 C_0}$$

The first root gives the series resonant frequency and the second the parallel or antiresonant frequency. Denoting these frequencies by f_R and f_A, respectively, we have

$$\omega_R^2 = \frac{1}{LC} \qquad \text{or} \qquad f_R = \frac{1}{2\pi}\sqrt{\frac{1}{LC}} \tag{65}$$

$$\omega_A^2 = \frac{1}{LC} + \frac{1}{LC_0} \qquad \text{or} \qquad f_A = \frac{1}{2\pi}\sqrt{\frac{1}{LC} + \frac{1}{LC_0}} \tag{66a}$$

If a capacitance C_x is connected in parallel with C_0, the antiresonant frequency is given by

$$f_A = \frac{1}{2\pi}\sqrt{\frac{1}{LC} + \frac{1}{LC_t}} \tag{66b}$$

where $C_t = C_0 + C_x$.

From Eq. (65) it follows that the series-resonant frequency is independent of the capacitance C_0. If the term involving R had been included in Eq. (64), the value of f_R would have been modified, but only by an insignificant amount. The frequency of parallel resonance f_A (sometimes called the antiresonant frequency) does depend on C_0. Therefore, when the unit is operated at the parallel-resonant frequency, the frequency is determined by the sum of C_0 and all capacitances in parallel with the crystal unit. Thus, while f_R is unique, f_A depends not only on the crystal unit but also on the associated circuit. From Eqs. (65) and (66*a*) it is seen that f_A is higher than f_R. In a sense, a crystal unit has an antiresonant frequency when it is not connected to any external circuit or to an external circuit which has zero reactance but the unit is usually not employed in this way.

THE IMPEDANCES AT THE RESONANT FREQUENCIES

The impedance of the crystal unit at the resonant frequency f_R is found by substituting $\omega_R^2 = 1/LC$ for ω in Eq. (62). The imaginary part of the numerator is zero, so the remainder of Eq. (62) reduces to

$$Z_R = \frac{R}{1 - (\omega R C_0)^2} \doteq R \tag{67}$$

The approximation in Eq. (67) is justified since for a typical AT unit operated at a frequency of 10 MHz, R is typically about 10 Ω, while the quantity $(\omega R C_0)^2$ is of the order of 10^{-5}. To a high degree of approximation the impedance of a crystal unit at the series-resonant frequency is thus equal to R. This approximation is valid in most cases.

The impedance of the crystal unit at the parallel or antiresonant frequency is found by substituting $\omega_A^2 = 1/LC + 1/LC_0$ into Eq. (62). In this way the antiresonant impedance is found to be

$$Z_A = \frac{1}{\omega^2 C_0^2 R} \tag{68a}$$

Any capacitance C_x in parallel with the crystal unit must be added to the capacitance C_0 so that

$$Z_A = \frac{1}{\omega^2 C_t^2 R} \qquad \text{where } C_t = C_0 + C_x \tag{68b}$$

The quantity $1/\omega^2 C_t^2 R$ has been called the *performance index* and it has been used as a measure of the quality of the crystal unit. It is not generally used now. If no extraneous coupled mode exists, the impedance of the crystal unit Z_A can be predicted at any frequency within the resonance interval. If coupled modes do exist, the value of Z_A may vary considerably from the calculated value.

At the series-resonant frequency f_R, the impedance of the crystal unit is a pure resistance having a relatively low value. Typical values for AT-cut units in the frequency range from 5 to 10 MHz vary between 10 and 100 Ω.

The impedance at the parallel-resonant frequency is also a pure resistance, but the value is much higher and, of course, depends upon the capacitance across the crystal unit. Typically Z_A is of the order of 10^4 to 10^5 Ω.

THE PARAMETERS OF THE EQUIVALENT CIRCUIT

The values of the reactive components of the equivalent circuit can be determined from measurements of the series- and parallel-resonant frequencies. Subtracting Eq. (65) from Eq. (66*a*) we have

$$\omega_A^2 - \omega_R^2 = (\omega_A - \omega_R)(\omega_A + \omega_R) = \frac{1}{LC_t}$$

where $C_t = C_0 + C_x$ is the total capacitance across the crystal unit.

Since ω_A and ω_R are very nearly equal we may write

$$2\omega\Delta\omega = \frac{1}{LC_t}$$

from which

$$L = (8\pi^2 f\, C_t\, \Delta f)^{-1} \qquad \text{where } \Delta f = f_A - f_R \tag{69}$$

Using this result and Eq. (65) we have

$$C = \frac{2\,\Delta f\, C_t}{f} \tag{70}$$

The quality factor Q, which is defined as $\omega L/R$, is thus given by

$$Q = (4\pi R\; C_t\; \Delta f)^{-1} \tag{71}$$

MEASUREMENT OF THE CIRCUIT PARAMETERS C_0 AND R.

The static capacitance C_0 may be measured by any convenient bridge or substitution method provided that the measurement is made near, but not too near, the resonant frequency. The measurement frequency should be near the operating frequency in order to take into account the distributed capacitance and inductance of the holder, both of which are usually frequency-dependent. A frequency about 1 percent below the resonant frequency is to be preferred. It is essential that the measurement be made at a frequency which does not coincide with one of the many inharmonic overtones of the plate. The range immediately below the resonant frequency is least likely to coincide with one of these modes. It is advisable to make the measurement at two or more different frequencies to ensure the absence of extraneous responses. The Q-meter provides a simple substitution method for measuring C_0. The accuracy is adequate for most purposes.

The value of R is easily determined by a circuit of the type illustrated in Fig. 7.1 (or its solid-state counterpart), which was the predecessor of the CI-meter to be discussed later.

The circuit is a simple self-excited shunt-fed Colpitts oscillator with a socket in series with the inductance. The circuit will oscillate

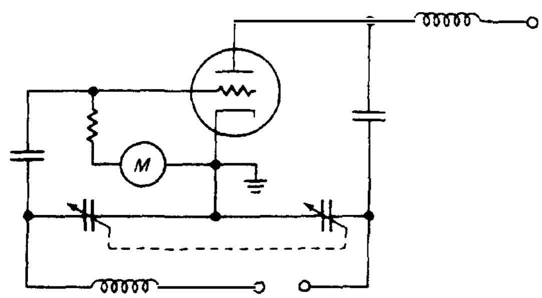


Fig. 7.1. Simple circuit for measuring R in the equivalent circuit.

if a resistor is plugged into the socket, provided the resistance is not too great. In practice a nonreactive resistor is inserted in the socket and the frequency is adjusted to be approximately equal to the resonant frequency of the crystal unit to be measured. The resistor is then replaced by the crystal unit and the frequency and grid current are noted. The crystal unit is then replaced by a resistor which results in the same grid current, and the tuning is adjusted until the same frequency and grid current are obtained with the resistor and the crystal unit. Under this condition, the crystal unit must be presenting the same reactance and resistance to the oscillator as the resistor and it is therefore operating at its series-resonant frequency. Thus both R and f_R are determined.

MEASUREMENT OF L AND C

The values of the motional parameters L and C are determined by measuring the frequencies of resonance and antiresonance and using Eqs. (69) and (70). We have just seen how the resonant frequency can be determined, but the measurement of the antiresonant frequency presents a more difficult problem.

From Eq. (66b) we see that the frequency f_A depends on the capacitance C_x across the crystal unit. Both C_x and f_A must be known to calculate L and C. When the crystal unit is used as an active element in an oscillator such as the Pierce or Miller circuits, the effective capacitance presented to the crystal unit is unknown and not readily measurable, because it depends not only upon the socket and wiring capacitances but also upon the transit time, Miller effect, electrode or junction capacitance, etc.

Since it is scarcely possible to determine C_x when the crystal unit is operated as the active element in an oscillator, it is usually operated as a passive element in *series* with the load capacitance C_x. The possibility of using this method depends on the result that the frequencies of zero reactance are the same in both cases, a fact that has resulted in considerable discussion and no little confusion since the CI-meter was invented in the mid-1940s. Before continuing with the discussion of methods of determining L and C we digress to show that the frequency of antiresonance given by Eq. (66*b*) is the same as the frequency of resonance of the network consisting of the crystal unit in series with the load capacitor C_x.

THE CRYSTAL UNIT IN SERIES WITH A CAPACITOR

In this section we derive the expressions for the frequencies of zero reactance and the corresponding impedances when the crystal unit is operated in series with a capacitor. Figure 7.2 shows the equivalent circuit of the crystal unit with a load capacitor C_x in parallel with the crystal unit (*a*) and in series with it (*b*).

The impedance between terminals A and B of the circuit of Fig. 7.2*b* is given by

$$\begin{aligned} Z_{AB} &= \frac{R + j(\omega L - 1/\omega C)}{1 - \omega^2 L C_0 + C_0/C + j\omega R C_0} + \frac{1}{j\omega C_x} \\ &= \frac{j\omega R C_x - \omega^2 L C_x + C_x/C + 1 - \omega^2 L C_0 + C_0/C + j\omega R C_0}{j\omega C_x - j\,\omega^3 L C_0 C_x + j\,\omega\, C_0 C_x/C - \omega^2 R C_0 C_x} \\ &= \frac{1 - \omega^2 L C_0 + C_0/C + C_x/C - \omega^2 L C_x + j\omega R C_t}{-\,\omega^2 R C_0 C_x + j(\omega C_x - \omega^3 L C_0 C_x + \omega\, C_0 C_x/C)} \end{aligned}$$

The foregoing equation is rationalized by multiplying the denominator and numerator by the complex conjugate of the denominator. The result is

$$Z_{AB} = \frac{N}{\omega^4 R^2 C_0^2 C_x^2 + (\omega C_x - \omega^3 L C_0 C_x + \omega C_0 C_x/C)^2} \quad (72)$$

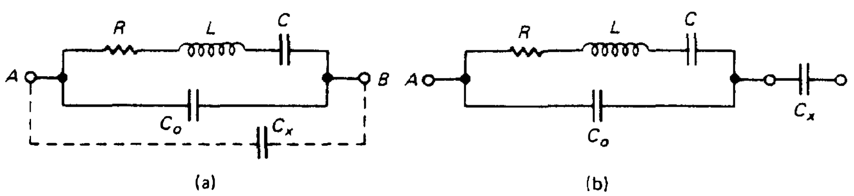


Fig. 7.2. Equivalent circuit of the crystal unit with load capacitance C_x in (*a*) parallel and (*b*) series.

where the numerator N is

$$N = \omega^2 RC_x^2 - j\Bigg(\omega^5 L^2 C_0^2 C_x + \omega^5 L^2 C_0 C_x^2 - \frac{2\,\omega^3 LC_0 C_x^2}{C}$$

$$-\frac{2\,\omega^3 LC_0^2\, C_x}{C} - \omega^3\, LC_x^2 + \omega^3 R^2\, C_0 C_x^2 + \omega^3 R^2 C_0^2\, C_x$$

$$+\frac{2\,\omega\, C_0 C_x}{C} + \omega\, C_x$$

$$-2\omega^3 LC_0\, C_x + \frac{\omega C_x^2}{C} + \frac{\omega C_0^2 C_x}{C^2} + \frac{\omega\, C_0 C_x^2}{C^2}\Bigg)$$

In order to determine the frequencies of zero phase we set the imaginary part of N equal to zero. The terms in the imaginary part of N which contain R are some six orders of magnitude smaller than any of the other terms and therefore may be neglected in calculating the resonant frequencies. This is in agreement with the well-known fact that the frequency is affected very little by damping in a circuit in which Q is large.

One solution is $\omega = 0$, corresponding to the direct-current case. Dividing through by ω after dropping the terms containing R leaves

$$\omega^4 \left(L^2 C_0^2 C_x + L^2 C_0 C_x^2\right) - \omega^2 \left(\frac{2LC_0 C_x^2}{C} + \frac{2LC_0^2 C_x}{C} + 2LC_0 C_x + LC_x^2\right) + \left(\frac{2\, C_0 C_x}{C} + C_x + \frac{C_x^2}{C} + \frac{C_0^2\, C_x}{C^2} + \frac{C_0 C_x^2}{C^2}\right) = 0$$

This expression can be simplified by multiplying through by (C^2/C_x) and letting $C_t = (C_0 + C_x)$ to give (assuming that $C_x \neq 0$)

$$\omega^4 L^2 C^2 C_0 C_t - \omega^2\, [2LCC_0 C_t + LC^2\, (C_0 + C_t)] + [(C_0 + C)(C_t + C)] = 0$$

This equation can be solved for ω^2 by the use of the quadratic formula. The result is

$$\omega^2 = \frac{LC^2(C_0 + C_t) + 2LCC_0C_t \pm LC^2(C_0 - C_t)}{2L^2C^2C_0C_t}$$

Using the + sign,

$$\omega^+ = \sqrt{\frac{1}{LC} + \frac{1}{LC_0}} \tag{73}$$

Using the - sign,

$$\omega^- = \sqrt{\frac{1}{LC} + \frac{1}{LC_t}} \tag{74}$$

The impedances at the frequencies ω^+ and ω^- are found by using Eq. (72). The term involving R in the denominator of Eq. (72) is negligible compared with the other terms, and of course, the imaginary part of N is equal to zero. Substituting ω^- from Eq. (74) for ω in Eq. (72) gives

$$Z_{AB}^- = \frac{R}{(1 - C_0/C_t)^2} \tag{75}$$

The impedance Z_{AB}^+ at frequency ω^+ is found by substituting from Eq. (73) into Eq. (72). In this case, however, all of the denominator except the term involving R is identically zero leaving only

$$Z_{AB}^+ = \frac{1}{\omega^2 RC_0^2} \tag{76}$$

From Eqs. (75) and (76) it is easily shown that Z_{AB}^+ is very large, typically 10^5 to 10^6 Ω; while Z_{AB}^- is of the same order of R as long as C_x is appreciably greater than C_0. It is logical, therefore, to call ω^+ the frequency of antiresonance and ω^- the frequency of resonance. For convenience of comparison, the results are tabulated below for the two cases where C_x is in parallel and in series with the crystal unit.

Parallel (Fig. 7.2*a*)	Series (Fig. 7.2*b*)
$\omega_R = \sqrt{\dfrac{1}{LC}}$	$\omega'_R = \sqrt{\dfrac{1}{LC} + \dfrac{1}{LC_t}}$
$\omega_A = \sqrt{\dfrac{1}{LC} + \dfrac{1}{LC_t}}$	$\omega'_A = \sqrt{\dfrac{1}{LC} + \dfrac{1}{LC_0}}$
$Z_R = R$	$Z'_R = \dfrac{R}{(1 - C_0/C_t)^2}$
$Z_A = \dfrac{1}{\omega^2 C_t^2 R}$	$Z'_A = \dfrac{1}{\omega^2 C_0^2 R}$

Two important results of the foregoing derivation should be noted since they form the basis of the methods most generally used to determine the values of L and C. First, the series-resonant (low-impedance) frequency of the series combination is the same as the parallel-resonant (high-impedance) frequency of the parallel combination. This reduces a difficult measurement of f_A to a simple one. The other important result is that the value of R can also be obtained from the measurement of the impedance of the series combination at the low-impedance resonant frequency. If we let $R_e = Z'_R$ be the impedance of the series combination at the resonant frequency, then

$$R = R_e \left(1 - \frac{C_0}{C_t}\right)^2 = R_e \left(\frac{C_x}{C_t}\right)^2 \tag{77}$$

THE PROBLEM OF OSCILLATOR CORRELATION

When crystal units first began to be used to control the frequencies of oscillators in communication equipments, the crystal units were usually fabricated to give the desired frequency in the actual equipment. Since to transport the actual equipment to the site of the crystal manufacturer was often impractical, common practice was to extract the oscillator from the equipment and supply it to the crystal manufacturer.

Circuits which simulated the oscillator were later supplied. Since these "test sets" were often enclosed in cabinets which were painted black, they became known as *black boxes*. A crystal-unit manufacturer would be supplied with one or more black boxes for each type of crystal unit he produced. It was soon recognized that the frequency developed by a crystal unit when operated in one black box did not equal that developed in another black box, which was supposed to be identical with the first. The problem of correlation became more severe as frequency tolerances were decreased.

Ultimately it became clear that the specification of frequency and activity does not adequately describe a crystal unit. Suppose, for example, that a crystal unit is specified to have a certain frequency when operated in a given oscillator at a parallel-resonant frequency. Due to unavoidable manufacturing tolerances the capacitances presented to the crystal unit will be different in different oscillators of the same type. From Eq. (66*b*) it is seen that the frequency f_A can be obtained in an infinite number of ways by varying L, C, and C_0 even if C_x is constant. Suppose now that another crystal unit, specified to have the same frequency, has different values of L, C, and C_0 but produces the same frequency, in the original oscillator. Now if both crystal units are operated in another oscillator having a slightly different C_x the frequencies of the two crystal units will not only differ from the original frequency but may differ from each other.

The solution is to specify not only the frequency but also C_0, and either L or C (but not both). Specified in this way, crystal units are interchangeable among oscillators of the same type with acceptable frequency differences.

The CI-meter was devised to meet the need for an instrument to measure the motional parameters L, C, and R of the crystal unit. A recent modification of the CI-meter has also included a circuit for measuring C_0.

THE CI-METER

The crystal impedance meter, or CI-meter, is similar in principle to the simple circuit of Fig. 7.1. A simplified diagram is shown in Fig. 7.3. It differs primarily from the simpler circuit in that the inductance is divided and the feedback path is provided by a connection

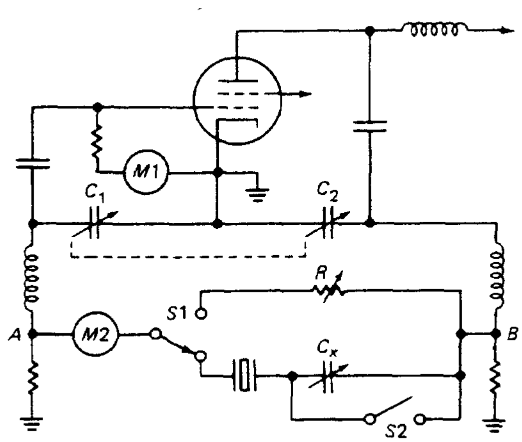


Fig. 7.3. Simplified circuit of the CI-meter.

between points A and B. The circuit is shielded so that oscillation cannot occur except by feedback through the path AB. Switches S_1 and S_2 are provided for convenience and resistor R is a set of calibrated, noninductive decade resistors which may be substituted for the crystal unit.

Meter M_1 reads the grid current of the oscillator and is used in determining the value of R and R_e. When the crystal unit and the resistor result in the same grid current (which is a measure of the amplitude of the oscillation), then the impedance of the crystal unit is equal to that of the resistor. M_2 is a radio-frequency milliammeter which is used to measure the current through the crystal unit. C_x is a calibrated variable capacitor used to supply the load capacitance specified for the particular crystal unit.

The series-resonant frequency and the series resistance are measured with S_2 closed. When the same frequency and grid current are obtained with a resistor and the crystal unit, the frequency is f_R and the resistance R is read from the decade resistor.

To measure the parallel-resonant frequency switch, S_2 is opened and C_x is adjusted to the value specified for the crystal unit. Although C_x is in series with the crystal unit, the frequency of oscillation is the same as the frequency which the crystal unit would produce in an oscillator circuit in which C_x is in parallel with the crystal unit, as we have shown in the previous section. The tuning capacitors C_1 and C_2

and the decade resistor are adjusted until the frequency and grid current are not changed by switching S_1 from the crystal unit to the resistor. When the grid current and the frequency are unchanged, the frequency is f_A and the resistance is R_e. The equivalent resistance R_e may be used as a measure of the quality of the crystal unit. If the value of R is required, it may be calculated using Eq. (77). The impedance which the crystal unit will have in parallel with the load capacitance C_x in the parallel-resonant mode may be found by combining Eqs. (68*b*) and (77). It is

$$Z_A = \frac{1}{\omega^2 C_x^2 R_e} \tag{78}$$

Whereas the CI-meter is simple in theory and convenient to use, requiring only a frequency counter in addition, it does have certain disadvantages. The construction of the instrument is critical because of the necessity of eliminating or compensating all stray reactances. These may cause the readings obtained on different instruments to differ slightly. Nevertheless, since its invention, most crystal units have been specified in terms of measurements made with the CI-meter.

More recently the development of highly stable yet continuously variable sources of radio-frequency signals, called *frequency synthesizers*, and an instrument called the *vector voltmeter*, have made possible measurements on the crystal unit as a passive element. This method has begun to replace the CI-meter, especially for measurements requiring the highest precision and reproducibility.

PASSIVE-NETWORK METHODS

Figure 7.4 shows, in block-diagram form, the system used for measuring the parameters of the equivalent circuit by using the crystal unit as an element in a passive network.

The crystal unit is part of a π network consisting of the low-resistance, noninductive resistors R_1 and R_2. Careful shielding is required to ensure that no current is induced in R_2 by inductive or capacitive coupling with R_1. The inputs to the vector voltmeter are connected to points A and B. The vector voltmeter may be used to

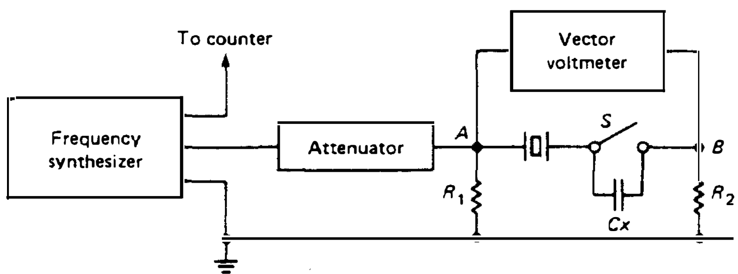


Fig. 7.4. Apparatus used in measuring the parameters of the equivalent circuit by a passive method.

measure the voltage at A or B, but its important function is to measure the phase difference between the voltages at A and B, which is, of course, the phase angle of the impedance between points A and B. It is capable of measuring this phase angle to a fraction of a degree.

We now show that the phase difference between points A and B is an exceedingly sensitive indicator of the resonant frequency of the crystal unit. At frequencies very near f_R the crystal unit appears to be a series-resonant network consisting of R, L, and C in series having impedance

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

This impedance can be represented graphically by the triangle shown in Fig. 7.5 in which $\tan\theta = (\omega L - 1/\omega C)/R$. But since the phase angle θ is very small (not over a few degrees), we may replace the tangent of the angle by the angle and write

$$\theta = \frac{\omega L - 1/\omega C}{R}$$

Fig. 7.5. Impedance of the crystal unit very near the resonant frequency f_R.

Differentiating the equation above with respect to ω we have

$$
\begin{aligned}
d\theta &= \left(L + \frac{1}{\omega^2 C}\right) \frac{d\omega}{R} \\
&= \frac{\omega^2 LC + 1}{\omega^2 RC}\, d\omega = \frac{2}{\omega^2 RC}\, d\omega \\
&= \frac{2\,Q}{\omega}\, d\omega \qquad \text{since } Q = \frac{1}{\omega RC}
\end{aligned}
$$

If $d\theta$ is expressed in degrees we have

$$d\theta = \frac{360\,Q}{\pi\;f}\,df \tag{79}$$

For example, if a 10-MHz crystal unit has a Q of 10^5, a frequency change of 1 Hz causes a phase change of slightly more than one degree, which is well within the sensitivity of the vector voltmeter. If Q is larger, the change in phase angle is proportionally greater also.

In practice the instrument is first set up with a nonreactive resistor in place of the crystal unit. The frequency synthesizer is set to the approximate frequency of the crystal unit. A phase-shifting network in the vector voltmeter is used to compensate for phase shifts in the cables and the π network. When the phase shifter has been set to show zero phase, the crystal unit is placed in the socket and the frequency adjusted to produce zero phase. By definition, this frequency is f_R. The value of R can be determined by substituting for the crystal unit, a resistor which gives the same voltage at point B.

The antiresonant frequency f_A and the equivalent resistance R_e are determined by opening switch S, thereby placing load capacitance C_x in series with the crystal unit. The frequency is adjusted for zero phase, giving the frequency f_A. R_e can be determined by substituting a resistor between points A and B which yields the same voltage at point B as the crystal unit in series with C_x.

Having determined R, f_R, f_A, and C_0, the values of L, C, Q, and Z_A can be determined using the same equations as those used with the CI-meter.

THE EFFECT OF SHUNT CAPACITANCE

When a crystal unit is used as the frequency-controlling element in a Pierce or Miller oscillator circuit it operates at the parallel-resonant frequency f_A, which is given by

$$f_A = \frac{1}{2\pi} \sqrt{\frac{1}{LC} + \frac{1}{LC_t}} \qquad C_t = C_0 + C_x$$

Since the frequency depends on C_t, it follows that any variation in C_t due to a change in C_x must influence the frequency of oscillation. Sometimes the value of C_x is deliberately changed to vary the frequency of the oscillator. The frequency of a typical 5-MHz AT unit can be "warped" or "trimmed" by a few kilohertz in this way.

Usually, however, the variations of frequency which occur as a result of changes in C_t are undesirable and are to be minimized as much as possible.

Differentiating the equation above with respect to C_t gives

$$df_A = -\frac{1}{4\pi LC_t^2 \, (1/LC + 1/LC_t)^{1/2}} \, dC_t$$

But since C_t is very large compared with C, we may write

$$df_A = -\frac{\sqrt{LC}}{4\pi LC_t^2} \, dC_t = -\frac{fC}{2C_t^2} \, dC_t \tag{80}$$

Equation (80) shows that an increase in C_t causes the frequency to decrease. It also shows that the change in frequency is proportional to C, the motional capacitance of the crystal unit. In order to minimize the change of frequency, C should therefore be made small. Such a crystal is often called a *stiff* crystal. The change of frequency with shunt capacitance varies inversely with the square of C_t; and to minimize the frequency change, C_t should be made as large as possible.

Because of variations in components and wiring tolerances, no two oscillator circuits are identical, so a given crystal unit may operate at slightly different frequencies in different oscillators of the same type. Furthermore, aging of components and temperature effects result in changes of the shunt (or load) capacitance with accompanying changes of frequency. These effects are minimized, but not eliminated, by careful selection of components and care in circuit design.

A particularly severe problem is that of changes in the effective value of the shunting capacitance due to dynamic effects. Active elements such as vacuum tubes and transistors are not strictly unilateral devices. Variations in the reactance of the load at the output terminals of tubes and transistors produce variations in the reactance at the input terminals. This phenomenon is often called the Miller effect. The changes of reactance result in changes of C_t and changes in the antiresonant frequency. The effect of dynamic changes of C_t are minimized by keeping the coupling between the crystal unit and the active element as loose as possible. For maximum stability, the crystal unit should be operated at minimum amplitude of oscillation, i.e., at minimum drive level.

LIMITING CONDITION FOR OSCILLATION

The advantage of using a large value of C_t has been discussed in the previous paragraph. However, there is a maximum value of C_t beyond which the crystal unit cannot be used in a circuit which requires that the crystal unit develop an inductive reactance.

We can readily determine the limiting value of C_t by returning to Eq. (63), which may be written (substituting C_t for C_0) as

$$\omega^4 L^2 C^2 C_t - \omega^2 (LC^2 + 2LCC_t - R^2 C^2 C_t) + (C + C_t) = 0$$

The condition for the existence of real roots of the equation is that the discriminant $(b^2 - 4ac) \geqslant 0$. This requires that

$$(4LC^3 R^2 - R^4 C^4) C_t^2 + 2LC^4 R^2 C_t - L^2 C^4 \leqslant 0 \qquad (81)$$

The inequality above has the form $ax^2 + bx - c \leqslant 0$, which is true only if the value of x lies within the closed interval bounded by the roots of the equation $ax^2 + bx - c = 0$, where a, b, and $c > 0$.

The roots of the equation formed by setting Eq. (81) equal to zero are

$$\frac{LCR \pm 2L\sqrt{LC}}{R^3 C - 4LR}$$

But since $LCR \ll 2L\sqrt{LC}$ and $R^3C \ll 4\,LR$ we may write that the roots are

$$\pm\,\frac{\sqrt{LC}}{2R}$$

and therefore the condition that Eq. (63) have real roots (and therefore real frequencies of oscillation) is that

$$C_t \leqslant \frac{\sqrt{LC}}{2R} = \frac{1}{2\,\omega R} = \frac{QC}{2}$$

or, in other words, the maximum value of C_t, which we shall denote C_t^*, is

$$C_t^* = Q\,\frac{C}{2} \tag{82}$$

If $Q = 10^5$ and $C = 1 \times 10^{-14}$ F, the maximum possible load capacitance is therefore 500 pF. It is a necessary condition that the $C_t \leqslant C_t^*$ but not a sufficient condition for oscillation to occur. The amplifier must be capable of providing a loop gain of 1, that is, it must replace the power dissipated. The circuit may fail to oscillate for either reason.

THE IMPORTANCE OF Q

Equation (82) may be written in another form to say

$$Q^* = 2\,\frac{C_t}{C}$$

where Q^* is the minimum value Q may have if the crystal is to oscillate with load capacitance C_t. Equation (80) may be written in a different form as follows:

$$\frac{df_A}{f_A} = -\frac{1}{2}\,\frac{C}{C_t}\,\frac{dC_t}{C_t} \tag{83}$$

which says that the fractional change in the frequency of oscillation is directly proportional to the fractional change in C_t. The highest stability is achieved when C_t is a maximum. But the maximum value of C_t is set by Eq. (82). Hence, the highest possible frequency stability with respect to variations in C_t is

$$\frac{df_A}{f_A} = \frac{1}{Q} \frac{dC_t}{C_t^*} \tag{84}$$

Equation (84) shows the importance of Q in determining the frequency stability of a crystal-controlled oscillator. Since the Q of an ordinary quartz crystal unit is usually between 10^5 and 10^6, it follows that a 1 percent change in C_t^* results in a frequency change of only 0.1 to 0.01 parts per million. This presupposes that C_t is the maximum possible value, but the actual value will likely be smaller than this by a factor less than 10. The frequency change is, of course, proportionally larger.

THE CRITICAL FREQUENCIES OF THE QUARTZ RESONATOR

It has been shown in Chap. 6 that the quartz crystal unit can be represented by the electric network of Fig. 6.5. We now show that this network has six critical frequencies; each is of importance in the piezoelectric resonator. These six frequencies are:

1. The series-resonant frequency: $f_R = 1/2\pi\sqrt{1/LC}$, [Eq. (65)].
2. The parallel-resonant frequency: $f_A = 1/2\pi\sqrt{1/LC + 1/LC_0}$, [Eq. (66)].
3. The frequency of zero phase f_r, at which Z_{AB} is a pure resistance having a low value.
4. The frequency of zero phase f_a, at which Z_{AB} is a pure resistance having a large value.
5. The frequency at which Z_{AB} is a minimum, f_m.
6. The frequency at which Z_{AB} is a maximum, f_n.

THE FREQUENCIES OF ZERO PHASE

We have previously neglected the effect of dissipation of energy in determining the frequencies of resonance [see Eqs. (63) and (64)]. However the frequencies at which Z_{AB} is a pure resistance do depend to a slight degree on the value of R.

These frequencies can most easily be determined by calculating the admittance of the network of Fig. 6.5 and then setting the imaginary part equal to zero and solving for the frequencies.

Let $Y_1 = j\omega C_0$ be the admittance of the capacitative branch of the equivalent network and let $Y_2 = 1/(R + jX)$ where $X = R + j(\omega L - 1/\omega C)$ is the admittance of the series branch. Then

$$Y_{AB} = Y_1 + Y_2 = \underline{\qquad\qquad} \qquad \underline{\qquad\qquad} - \omega C_0 \Big)$$

Setting the imaginary part equal to zero gives $\omega C_0 X^2 - X + \omega C_0 R^2 = 0$. By use of the quadratic formula we find

$$X = \frac{1 \pm \sqrt{1 - 4(\omega C_0 R)^2}}{2\,\omega C_0}$$

Now since $(2\,\omega C_0 R)^2 \ll 1$ we may write

$$X = \frac{1 \pm [1 - 2(\omega C_0 R)^2]}{2\,\omega C_0}$$

Using, first, the minus sign, we have

$$X = \omega C_0 R^2$$

However, $X = \omega L$ -1/ written

$$\frac{\qquad}{\omega C}$$

Using $\omega_R^2 LC = 1$ we may write

$$X = \underline{\qquad\qquad\qquad} = \frac{(\omega - \omega_R)(\omega + \omega_R) L}{\omega}$$

But since $(\omega + \omega_R$

$$X = 2(\omega - \omega_R) L$$

Equating the two values for X, we have $\omega C_0 R^2 = 2(\omega - \omega_R) L$, from which $\omega_R = \omega(1 - C_0 R^2/2L)$. Then, because $C_0 R^2/2L \ll 1$, we may write $\omega \doteq \omega_R\ (1 + C_0 R^2/2L)$. This is the lower frequency at which Z_{AB} is a pure resistance. Calling this frequency f_r we have

$$\boxed{f_r = f_R \left(1 + \frac{r}{2Q^2}\right)}$$

where we have written $r = C_0/C$ and $Q = \omega L/R = 1/\omega RC$.

Taking the plus sign, we have

$$X = \frac{2 - 2(\omega C_0 R)^2}{2\omega C_0} = \frac{1 - (\omega C_0 R)^2}{\omega C_0}$$

Just as in the previous case, we set the two values of X equal and solve for ωR. In this way we obtain

$$\omega_R = \omega \left(1 - \frac{1}{2\omega^2 L C_0} + \frac{C_0 R^2}{2L}\right)$$

Now $1 = \omega_R^2 LC$, so

$$\frac{\omega_R^2 LC}{2\omega^2 L C_0} \doteq \frac{C}{2C_0} = \frac{1}{2r}$$

Since the last two terms are each small compared to 1, we may write

$$\omega_a = \omega_R \left(1 + \frac{1}{2r} - \frac{r}{2Q^2}\right)$$

or

$$\boxed{f_a = f_R \left(1 + \frac{1}{2r} - \frac{r}{2Q^2}\right)}$$

The frequency f_a is the antiresonant frequency (or the frequency of parallel resonance including the effect of damping). Comparing with Eq. (66)[1] in which the effect of damping is neglected, we see that the effect of damping is to make $f_a < f_A$. (We have just seen that the effect of damping is to make $f_r > f_R$.) Hence, the effect of damping is to reduce the frequency difference between f_a and f_r. The effect is very small, since it depends inversely on Q^2. Typically f_A and f_a, or f_R and f_r, differ by a few parts in 10^8 in high-frequency fundamental-mode AT-cut units.

FREQUENCIES OF MINIMUM AND MAXIMUM IMPEDANCE

In order to determine the frequencies at which the impedance Z_{AB} has a minimum and a maximum value, we first compute the modulus of the admittance, which is

$$\left|Y_{AB}\right|^2 = \left(\frac{R}{R^2 + X^2}\right)^2 + \left(\frac{X}{R^2 + X^2} - \omega C_0\right)^2$$
$$= \frac{1 - 2\,\omega C_0 X}{R^2 + X^2} + \omega^2 C_0^2$$

In order to find the minimum and maximum values of $|Y_{AB}|^2$, we differentiate the expression with respect to X (which is a function of ω) and set the result equal to zero. The result is

$$\omega C_0\, X^2 - X - \omega\, C_0 R^2 = 0$$

Using the quadratic formula to find the values of X gives

$$X = \frac{1 \pm \sqrt{1 + (2\,\omega C_0 R)^2}}{2\,\omega C_0} \approx \frac{1 \pm [1 + 2\,(\omega C_0 R)^2]}{2\omega C_0}$$

---

[1] Equation (66) may be written

$$\omega_A^2 = \frac{1}{LC} + \frac{1}{LC_0} = \frac{1}{LC}\left(1 + \frac{1}{r}\right) = \omega_R^2\left(1 + \frac{1}{r}\right)$$

or

$$\omega_A \doteq \omega_R\left(1 + \frac{1}{2r}\right) \quad \text{and} \quad f_A \doteq f_R\left(1 + \frac{1}{2r}\right)$$

Using first the minus sign, we have $X = -\omega C_0 R^2$.
But also $X = 2(\omega - \omega_R)L$, so we have, by setting equal the two expressions for X,

$$\omega = \omega_R \frac{2L}{2L + C_0R^2} \doteq \omega_R \left(1 - \frac{C_0R^2}{2L}\right)$$

This is the frequency of minimum impedance ω_m. Therefore, $f_m = f_R\ (1 - C_0R^2/2L)$, or

$$\boxed{f_m = f_R \left(1 - \frac{r}{2Q^2}\right)}$$

Now using the plus sign we find

$$X = \frac{1 + (\omega C_0 R)^2}{\omega C_0}$$

Again setting this equal to $X = 2(\omega - \omega_R)L$ we have that

$$\omega_R = \omega \left(1 - \frac{1}{2\,\omega^2 LC_0} - \frac{C_0R^2}{2L}\right)$$

Since the last two terms are very small compared with 1, we may write with very small error that

$$\omega \doteq \omega_R \left(1 + \frac{1}{2\,\omega^2 LC_0} + \frac{C_0R^2}{2L}\right)$$

This is the frequency of maximum impedance, so we may write

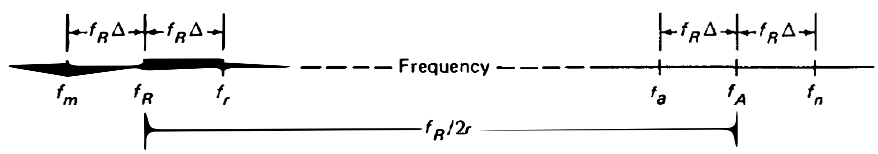
$$\boxed{f_n = f_R \left(1 + \frac{1}{2r} + \frac{r}{2Q^2}\right)}$$

All of these results may be briefly summarized in the following way:

1. The series-resonant frequency: $f_R = 1/2\pi\ \sqrt{1/LC}$.
2. The parallel-resonant frequency: $f_A = 1/2\pi \sqrt{1/LC + 1/LC_0}$ $\doteq f_R\ (1 + 1/2r)$.
3. The lower frequency of zero phase: $f_r = f_R\ (1 + \Delta)$.
4. The upper frequency of zero phase: $f_a = f_R\ (1 + 1/2r - \Delta)$.
5. The frequency of minimum impedance: $f_m = f_R\ (1 - \Delta)$.

6. The frequency of maximum impedance: $f_n = f_R\ (1 + 1/2r + \Delta)$, where $\Delta = r/2Q^2$ and $r = C_0/C$ (or C_t/C, where $C_t = C_0 + C_x$). [See Eq. (66*b*).]

The results may also be shown graphically as follows.



THE IMPEDANCE DIAGRAM

At the angular frequency $\omega = 2\pi f$ the impedance Z of the circuit consisting of an inductance L, a capacitance C, and a resistance R in series is given by

$$Z = R + jX$$

where $X = (\omega L - 1/\omega C)$
If $\omega L > 1/\omega C$ then X is positive and the circuit has an inductive reactance. Conversely, if X is negative the circuit has a capacitive reactance. If $\omega_R^2 LC = 1$ the angular frequency ω_R is called the series resonant frequency.

The impedance Z can be represented by a vector in the complex plane (Fig. 7.6). The real component of OP, which is OA, represents the resistance R and the imaginary component AP represents the re-

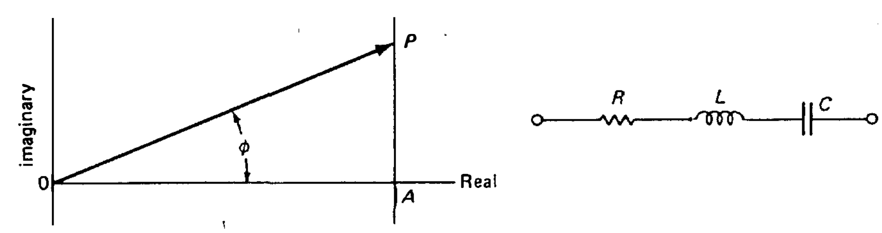


Fig. 7.6. The impedance diagram for the series circuit R, L, C.

actance X. As the frequency is varied from 0 to ∞ the point P moves *up* the line AP from $-\infty$ to $+\infty$.

It is possible to calibrate the line AP for frequency. In the immediate neighborhood of A, corresponding to f_R, the calibration is almost linear. This may be shown in the following way. The reactance

$$
\begin{aligned}
X &= \omega L - 1/\omega C = (\omega^2 LC - 1)/\omega C = (\omega^2 LC - \omega_R^2 LC)/\omega C \\
&= (\omega - \omega_R)(\omega + \omega_R)(L/\omega) \\
&= 2\,\Delta\omega L
\end{aligned}
$$

where $\Delta\omega = \omega - \omega_R$ and we have taken $(\omega + \omega_R) = 2\omega$
Therefore

$$X = 4\,\pi L\,\Delta f$$

and the distance AP which is proportional to X is also proportional to $(f - f_R) = \Delta f$.

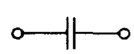
The phase angle ϕ is given by $\phi = \tan^{-1}(AP/OA)$

THE ADMITTANCE DIAGRAM

The admittance Y is the reciprocal of the impedance Z so that

$$
\begin{aligned}
Y &= 1/Z = 1/(R + J\,X) \\
&= R/(R^2 + X^2) - jX/(R^2 + X^2) \\
&= g - j\,b.
\end{aligned}
$$

The quantity $g = R/(R^2 + X^2)$ is called the conductance and the quantity $b = X/(R^2 + X^2)$ is called the susceptance. The admittance $Y = g - j\,b$ can also be represented by a vector in the complex plane by making a suitable choice of scale factors. It must be noted, however, that the signs of the imaginary parts of Z and Y are reversed as shown in the following chart.

		
Reactance	+	–
Susceptance	–	+

To construct an admittance diagram for the series circuit L, C, R we let 0 be the origin of coordinates in the complex plane (Figure 7.7). We draw a circle with diameter OA making $R = s_i \cdot OA$ where s_i is some convenient scale factor (ohms/cm). The circle is called the admittance circle. A line is drawn through point A perpendicular to OA. The triangle OPA is the impedance triangle. We then mark point Q where line OP intersects the circle. As the point P moves along the line AP the point Q moves around the circle. Hence to every point on the line AP there corresponds a point on the admittance circle. We shall now show that the distance OQ represents the admittance Y, the line QM the (negative of the) susceptance b, and the line OM the conductance g. To do so we require a theorem from analytic geometry which says that

$$OP \cdot OQ = (OA)^2$$

The proof of this theorem is relatively simple. We draw the line QA. Then in right triangle OQA

$$(OQ)^2 + (QA)^2 = (OA)^2$$

In right triangle OPA

$$(OP)^2 - (PA)^2 = (OA)^2$$

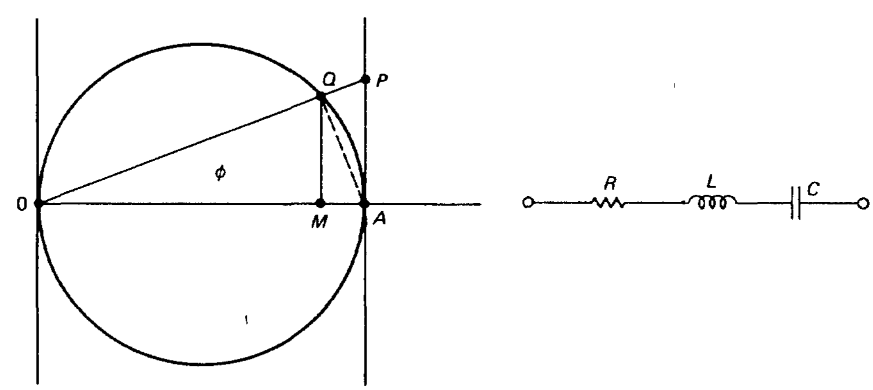


Fig. 7.7. Admittance diagram of the series circuit R, L, C.

and in right triangle QAP

$$(PA)^2 - (QA)^2 - (QP)^2 = 0$$

Obviously

$$(QP)^2 = (OP - OQ)^2 = (OP)^2 + (OQ)^2 - 2\, OP \cdot OQ$$

Adding the first three equations and substituting from the last gives the required proof.

In the impedance triangle OPA we have $R = s_i \cdot OA$, $X = s_i \cdot AP$ and $Z = s_i \cdot OP$ where s_i is the impedance scale factor (ohms/cm).

In the admittance triangle OQM we take $g = s_a \cdot OM$, $-b = s_a \cdot QM$, and $Y = s_a \cdot OQ$ where s_a is the admittance scale factor (mhos/cm). From the above theorem we have

$$(Z/s_i) \cdot (Y/s_a) = (R/s_i)^2$$

from which

$$s_a = s_i/R^2 \qquad \text{(since } Z \cdot Y = 1\text{)}$$

To calibrate the line AP for frequency we take

$$X = 4\pi\, L\, \Delta f = s_f\, \Delta f$$

from which

$$AP = \frac{4\pi L}{s_i}\, \Delta f = s_f \Delta f$$

the quantity $s_f = 4\pi\, L/s_i$ is called the frequency scale factor (cm/Hz).

If care is taken in drawing the diagram to scale the approximate values of the various quantities can be determined by measuring the lengths of the corresponding lines and multiplying by the appropriate scale factors. It must be remembered that the signs of X and b are reversed and that the phase angles of Z and Y are also opposite although they are equal in magnitude.

We may now illustrate the use of the admittance-impedance diagram by considering the series circuit of $R = 20$ ohms, $L = 0.031\, H$, and $C = 33 \times 10^{-14}$ F. The figures are typical values of the motional parameters of a 5-MHz AT-cut quartz crystal unit.

We first choose a convenient impedance scale factor s_i, say 1 ohm/cm, and draw the line OA having a length R/s_i = 20 cm. A circle with diameter OA is then drawn and the line perpendicular to OA through the point A is constructed.

Suppose that we desire to know the values of R, X, Z, g, b, Y and ϕ at the frequency 10 Hz above f_R. We first compute the value of the frequency scale factor $s_f = 4\pi L/s_i$ which is found to be 0.39 cm/Hz and mark the point P making AP = 3.9 cm. Then by direct measurement and use of the scale factors we find

$$\begin{array}{ll} R = s_i \cdot OA = 20 \text{ ohms} & g = s_a \cdot OM = 0.048 \text{ mhos} \\ X = s_i \cdot AP = 3.9 \text{ ohms} & -b = s_a \cdot QM = 0.0098 \text{ mhos} \\ Z = s_i \cdot OP = 20.4 \text{ ohms} & Y = s_a \cdot OQ = 0.049 \text{ mhos} \end{array}$$

$$\phi = \tan^{-1} (AP/OP) = 11°$$

It is considerably easier and quicker to determine the values of the various quantities in this way than by using algebraic methods and for many purposes the results are sufficiently accurate. A resonance curve can easily be made by plotting the admittance against the frequency, both being obtained from the diagram of Fig. 7.7.

The principal advantage of the admittance method is found in dealing with circuits having parallel branches such as the equivalent circuit of the piezoelectric resonator. Sometimes it is a little inconvenient to make the drawings for these circuits to scale because the scale factors become too large and too small but even in such cases the diagram serves as a useful guide to the computations and it also provides many interesting and valuable insights into the behavior of the circuit in the neighborhood of the resonant frequency. We now examine the admittance diagram for such a circuit.

ADMITTANCE DIAGRAM OF THE PIEZOELECTRIC RESONATOR

The admittance of the equivalent circuit of the piezoelectric resonator (Fig. 6.5) is the vector sum of the admittances of the two branches. That is,

$$Y_{AB} = Y + Y_0$$

where $Y = g - j\,b$ and $Y_0 = -j\,b_0 = -j\omega C_0$. In case there is a parallel load capacitance C_x then $C_t = C_0 + C_x$ must be used instead of C_0.

To construct the admittance circle diagram of the circuit of Fig. 6.5 we let the point 0 (Fig. 7.8) be the origin of coordinates in the complex plane. A convenient scale factor s_a is chosen making the line $OO' = s_a \cdot \omega\, C_0$ and draw the line OO', representing the susceptance of C_0, in the upward direction. This convention results in making the vectors which represent capacitive susceptances point upward and those representing inductive susceptances to point downward.

The actual length of OO' varies with the frequency but over the narrow range near f_R in which we are interested it may be considered to be constant.

The admittance circle having diameter $O'A' = 1/(s_a R)$ and the frequency line $A'P$ are drawn. The frequency scale factor $s_f = 4\pi\, L/s_a R^2$ (cm/Hz) is calculated and the frequency line is calibrated by making $A'P = s_f(f - f_R)$ with point A' corresponding to the resonant frequency f_R. As the frequency increases the point P moves *downward* along the line AP and the point Q moves clockwise around the admittance circle. At any given frequency the vector sum $(OO' + O'Q) = OQ$

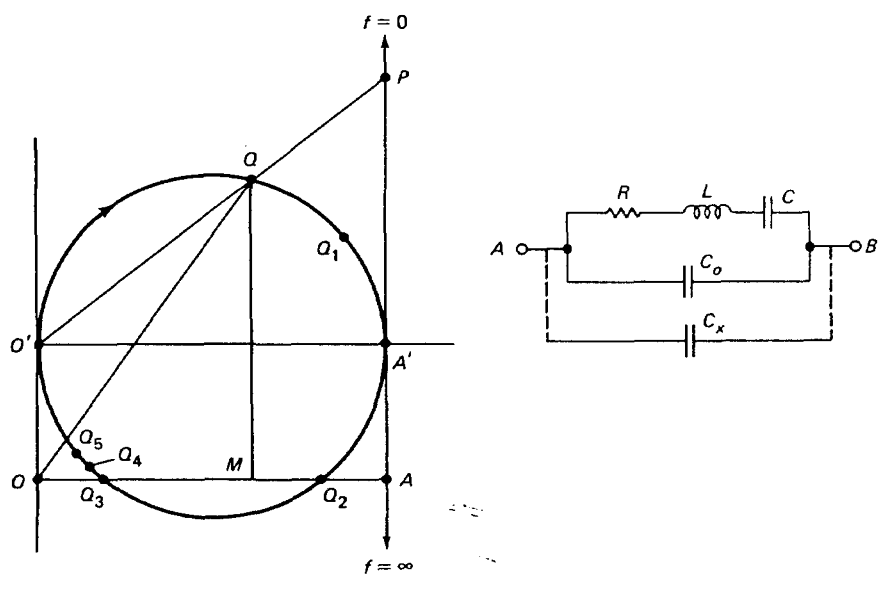


Fig. 7.8. The admittance diagram of the piezoelectric resonator.

represents the admittance of the circuit, OM represents the conductance and QM represents the susceptance. We therefore have

$$Y_{AB} = s_a \, OQ$$
$$g = s_a \, OM$$
$$-b' = -(b + b_0) = s_a \, QM$$
$$\tan \phi = (QM/OM)$$

The point Q_1 on the admittance circle is the point at which the line OQ has the maximum length. The corresponding frequency is therefore the frequency of maximum admittance or minimum impedance. As we have seen before the frequency of minimum impedance is slightly below f_R: The difference between the two frequencies is given by $f_m - f_R = A'P_1/s_f$.

The point Q_2 is the lower frequency of zero phase f_r and the point Q_3 is f_a the upper frequency of zero phase. Both frequencies are higher than f_R since frequencies increase downward on the frequency line and clockwise around the admittance circle. Between f_r and f_a the susceptance is negative which means that the reactance is positive or that the crystal unit develops an inductive reactance.

Point Q_4 is the frequency of parallel resonance f_A and Q_5, where the line OQ is shortest, is the frequency of minimum admittance or maximum impedance. The frequency scale in this region is so compressed that it is usually not practical to make geometric measurements.

The actual frequencies corresponding to points $Q_1, Q_2 \ldots$ can be determined by extending the lines $O'Q_n$ to intersect the frequency line, measuring the distance $A'P_n$ and using $A'P_n = s_f \Delta f$ where $\Delta f = (f - f_R)$. The phase angle at the corresponding frequency is found either by direct measurement or from

$$\tan \phi = QM/OM$$

The use of the admittance diagram to represent the quartz resonator may be illustrated with the example of the 5 MHz resonator in which

$R = 20$ ohms	$C = 3.3 \times 10^{-14}$ F
$L = 0.031$ H	$C_t = 32 \times 10^{-12}$ F

We let the impedance scale factor s_i = 0.4 ohms/cm. This makes the admittance scale factor $s_a = s_i/R^2$ = 0.001 mhos/cm. The length of the line OO' which represents the susceptance of C_t is therefore $\omega\, C_t/s_a$ = 1.00 cm.

The diameter of the admittance circle $O'A' = R/s_i$ or $(R \cdot s_a)^{-1}$ = 50 cm.

The frequency scale factor $s_f = 4\pi\, L/s_i$ = 0.97 cm/Hz.

We draw the line OO' representing the susceptance of C_t; the line $O'A'$ which is the diameter of the admittance circle; and calibrate the frequency line $A'P$.

To illustrate how the diagram may be used we suppose that it is desired to know the admittance and phase angle of the circuit at a frequency 10 Hz below the frequency f_R. The point P is marked making $A'P$ = 9.7 cm with P above point A'. The line $O'P$ is drawn and the point Q marked on the admittance circle. QM and OQ are drawn to form triangle OQM. Then by direct measurement it is found that OQ = 49.5 cm, OM = 48.4 cm and QM = 10.4 cm from which Y_{AB} = 0.049 mhos, g = 0.048 mhos and b' = 0.010 mhos. The phase angle ϕ measured by use of a protractor or computed from tan $\phi = QM/OM$ is 12°. Since the susceptance is positive the circuit has a capacitive reactance at this frequency.

It may be surprising to find that the phase angle changes so rapidly in the neighborhood of the series resonant frequency f_R but it is this characteristic which enables the crystal unit to control the frequency of an oscillator circuit with such great accuracy.

As the value of C_t is increased the length of the line OO' increases and the points Q_2 and Q_3 move closer together finally meeting at a point where the admittance circle is tangent to the line OA. In this case length of OO' is equal to the radius of the admittance circle and therefore

$$C_t^*/s_a = \frac{1}{2}\,(1/s_a R)$$

or

$$C_t^* = 1/2\,\omega R = QC/2$$

which is the same result obtained by algebraic methods in Eq. (82). C_t^* is the maximum value of C_t for which the crystal unit is able to develop an inductive reactance.

Many other interesting and instructive characteristics of the quartz crystal resonator may be found by considering the admittance circle diagram.