6
EQUIVALENT CIRCUIT OF THE QUARTZ RESONATOR

INTRODUCTION

From the point of view of the crystal engineer, the quartz crystal unit is a carefully prepared geometric solid, excited through the piezoelectric effect to vibrate at one of its normal resonance modes. To the circuit engineer, however, it is a two-terminal passive network having highly desirable characteristics, not realizable in any other way. The crystal resonator is really an electromechanical transducer, i.e., a device for converting electric to mechanical energy, and vice versa. Other examples are loudspeakers, microphones, electrically driven tuning forks, and magnetostriction resonators.

Several years before the discovery of the piezoelectric resonator, Dye and Butterworth showed that any mechanically vibrating system driven by means of an electromagnetic field could be represented by the electric network of Fig. 6.1*a* and that one driven by means of an electrostatic field could be represented by the network of Fig. 6.1*b*. The relationship between the parameters of the electric network and the physical properties of the piezoid were developed by Van Dyke, Butterworth, and others in the mid-twenties after the discovery of the piezoelectric resonator.[1]

In this chapter we attempt to show, in the simplest possible way, how the resistance, inductance, and capacitances of the equivalent

[1] D. W. Dye, *Proc. Phys. Soc.*, **38**: 399-457 (1926); S. Butterworth, *Proc. Phys. Soc.*, **27**: 410-424 (1915); K. S. Van Dyke, *Proc. I.R.E.*, **16**: 742-764 (1928).

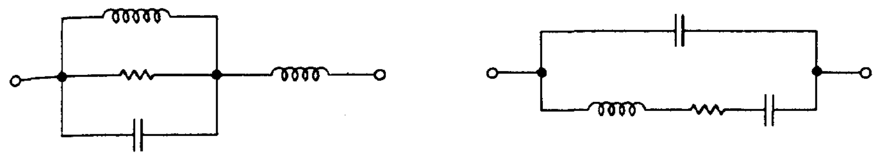

Fig. 6.1. Equivalent circuits of electromechanical transducers. (*a*) Electromagnetic transducer; (*b*) electrostatic transducer.

circuit are related to the dimensions, density, and elastic, dielectric, and piezoelectric constants of the piezoid. The equivalent circuit which we derive is an "exact" equivalent circuit, meaning that the parameters are independent of frequency, at least in the neighborhood of the resonant frequency and so long as only one mode of vibration is present. Simpler "limited" equivalent circuits may be used to represent the crystal unit at a single frequency.

In order to simplify the problem as much as possible we consider an infinite plate, vibrating in a thickness mode, thereby limiting the problem to one dimension. We assume a single mode of vibration and assume that all effects are linear, i.e., independent of amplitude. Other simplifications will be introduced at the appropriate time.

THE QUARTZ RESONATOR AS A VIBRATING CAPACITOR[2]

A piece of quartz or other piezoelectric crystal placed in an electric field will, in general, experience strains which are proportional to the strength of the field and change signs if the direction of the field is reversed. At the same time the quartz serves as the dielectric in a capacitor. The crystal unit may thus be thought of as a capacitor in which the dielectric vibrates in response to the applied field.

Let the piece of quartz be in the shape of a thin plate cut with its thickness in the Y-direction. Imagine electrodes to be plated on the surfaces. A difference of potential is established between the electrodes. Two types of strain exist due to the resulting electric field. One strain may be described as a pure shear in the XZ-plane, the other as a pure shear in the XY-plane. The first, or face-shear, strain is used in the fabrication of the low-frequency resonators -

[2] The reader may find it useful to review App. I at this point.

the CT- and DT-cut plates. The second type of strain is used in the Y-cut family, which includes the AT- and BT-cut plates. In this chapter we are exclusively concerned with the high-frequency or thickness-shear strain (although it should not be forgotten that the face-shear strain is also present).

Since the mechanical strain is proportional to and therefore reverses sign with the applied field, an alternating field causes an alternating strain. Unless the applied frequency is equal to that of a resonant frequency of the plate, the strain is small; but if the frequency of the applied field is nearly equal to that of the resonant frequency, the strain may become very large. In fact, it is quite easy to cause the strain to exceed the elastic limit and to break the quartz.

The amount of energy required to overcome frictional losses and to maintain the plate in vibration at constant amplitude is very small, so the normal resonance frequency of the plate is influenced very little by the driving forces applied through the piezoelectric effect. With appropriate connections to an amplifier, the mechanical vibration of the quartz resonator can be made to control the frequency of oscillation in the electric circuit. It can also be used as a passive component in an electric-filter network.

At frequencies far from the resonant frequency, the quartz plate is equivalent to a simple parallel-plate capacitor having a capacitance given by

$$C_0 = k \epsilon_0 \frac{A}{e}$$

in mks units, where A is the area of the electrodes and e is the thickness of the plate. The dielectric constant k must be determined for the direction of the field, and unless the plate is infinitely large, correction must be made for fringing of the field. For the moment, we will not distinguish between the clamped and unclamped dielectric constant, although it should be pointed out that a piezoelectric crystal has two dielectric constants depending on whether the crystal is clamped or not. If the crystal is constrained in such a way that the strain is zero, the effective dielectric constant is not the same as if the crystal is completely free with the applied stress equal to zero. Fortunately the two values are not very different.

The electric conductivity of quartz is extremely low at ordinary temperatures, so that electrically the quartz plate and its electrodes may be represented by a pure capacitance. At frequencies other than those at which mechanical resonance occurs, the device is simply a capacitor and the current into it is simply the charging current, which is the displacement current due to the polarization of the dielectric just as in any other capacitor. This current leads the applied voltage by 90°; therefore the impedance is given by[3]

$$Z = -\frac{j}{\omega C_0}$$

where, as usual, $j = \sqrt{-1}$ and ω is the angular frequency $2\pi f$.

At frequencies near that of one of the normal modes of vibration of the piezoid, the strain is greatly increased. The strain associated with the vibration results in further polarization through the piezoelectric effect. The current flowing into the resonator then includes two components – the dielectric displacement current which was discussed earlier and the piezoelectric displacement current. The piezoelectric displacement current may be hundreds of times greater than the dielectric displacement current. The resonator current is the vector sum of the two.

The component due to the dielectric displacement current is 90° out of phase with the applied voltage, as in any perfect capacitor; and insofar as this component of the current is concerned, the resonator may be represented as an ordinary capacitor. The component of current which is due to the piezoelectric strain may either lead or lag the applied voltage, depending on whether the applied frequency is greater or less than that of the resonant frequency of the resonator. If the applied frequency is exactly equal to the mechanical resonant frequency, the strain and therefore the piezoelectric displacement current is in phase with the applied voltage, just as in the case of a resistor. The resonator then appears to be a pure resistance in parallel with a capacitor, as shown in Fig. 6.2. The current into the resonator is therefore the vector sum of the currents i_0 and i.

At frequencies above the resonant frequency, the piezoelectric current lags behind the applied voltage, i.e., the device behaves like an inductance and the current is the vector sum of i_0 and i', as shown in Fig. 6.3*a*. At frequencies below the resonant frequency, the pi-

[3] In this, and following chapters, extensive use will be made of complex notation. Familiarity with complex numbers is a prerequisite. See App. II.

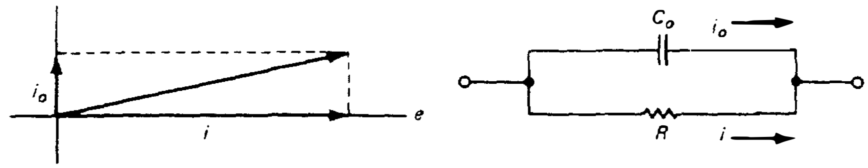

Fig. 6.2. The equivalent circuit of a quartz resonator at the frequency of mechanical resonance (currents not to scale).

ezoelectric current into the resonator leads the applied voltage and the device behaves as if it were a capacitor and the current is the vector sum of i_0 and i'', as shown in Fig. 6.3*b*.

If we consider a series circuit consisting of an inductance, a capacitance, and a resistance, we see that the current varies with frequency in the same way that the piezoelectric displacement current varies with frequency in the resonator. In both cases the current and voltage are in phase at the resonant frequencies. At higher frequencies the inductive reactance predominates and the current lags behind the voltage. At frequencies below the resonant frequency the capacitive reactance is predominant and the voltage lags behind the current. In each case the current into the resonator is the vector sum of the two currents, as indicated in Fig. 6.3.

DERIVATION OF THE EQUIVALENT CIRCUIT

We have just shown qualitatively that the piezoelectric resonator is equivalent electrically to a circuit consisting of a capacitor C_0 in parallel with the series circuit L, C, and R, as shown in Fig. 6.4.

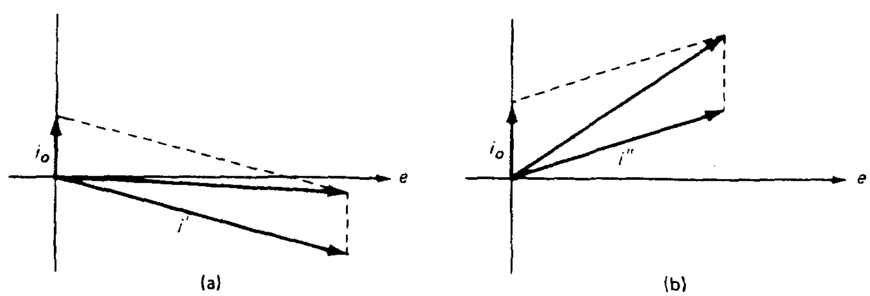

Fig. 6.3. Dielectric and piezoelectic currents at frequencies (*a*) below and (*b*) above the frequency of resonance.

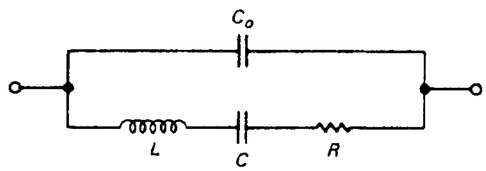

Fig. 6.4. The equivalent circuit of the piezoelectic resonator.

The capacitance $C_0 = k\epsilon_0 A/e$, where A is the electrode area, e is the thickness, and k is the applicable dielectric constant. The values of L, C, and R are much more difficult to compute, depending as they do upon the piezoelectric effect for the coupling between the mechanical strain in the resonator and the parameters of the equivalent circuit. The complete treatment of the problem including finite-boundary conditions, anisotropy, and nonlinear effects would be prohibitively complex, but a simplified treatment which takes into account only the most important factors yields results which are of considerable value in the design of quartz crystal units. For the complete treatment, the reader is referred to Cady's* *Piezoelectricity* (Chap. 14).

The steps which we follow in deriving the expressions for the parameters of the equivalent circuit are the following:

1. Solve the equation of motion, including the effects of damping, assuming that the damping forces are proportional to the particle velocity.
2. Apply appropriate boundary conditions to determine the displacement at all points in the vibrating plate.
3. Determine the strain by differentiating the displacement equation.
4. Determine the polarization by multiplying the strain by the appropriate piezoelectric constant.
5. Compute the space charge density by taking the negative of the gradient of the polarization,

$$q_v(y) = -\frac{\partial P_y}{\partial_y}$$

*W.G. Cady, *Piezoelectricity*, McGraw-Hill (1946) Chap. 14.

6. Use Poisson's equation:

$$V'' = \frac{q_v}{k\epsilon_0}$$

to obtain the potential at each point in the plate. (See symbols below.)
7. Evaluate the potential function at the surface of the plate.
8. Compute the field intensity at the surface from the relation

$$E = -\frac{\partial V}{\partial y}$$

(See symbols below.)
9. Determine the current density from

$$i_s = k\epsilon_0 \dot{E}$$

10. Divide the piezoelectric current by the applied voltage to obtain the electric admittance of the resonator, thereby obtaining the parameters of the equivalent circuit.

In order to keep the problem as simple as possible, we make the following assumptions.

1. The piezoid is an infinitely large, thin plate. This reduces the problem to one dimension.
2. The displacement is considered to be a wave traveling in the thickness direction. (However, no distinction between transverse and longitudinal waves is required except in the constants.)
3. Anistropy effects are neglected.
4. The electrodes are deposited directly on the plates, have negligible mass, and are perfectly conducting. The complication of an air gap is unnecessary, since few, if any, such devices are produced.
5. A single mode of vibration with no coupling to other modes is assumed.
6. The thickness of the plate is taken to be in the y direction, with the origin of coordinates at the center of the plate. The thickness of the plate is e and the surfaces are therefore at $y = \pm\frac{1}{2}e$.
7. The strain is considered to be constant in a plane parallel to the plane $y = 0$.

The following symbols are required.

A = Area of the plate
e = Thickness of the plate
c = Appropriate elastic constant
d = Piezoelectric strain constant
D = Electric displacement
$E = E(y, t)$ = Electric-field intensity, $\dot{E} = \partial E/\partial t$
f = Frequency (Hz, kHz, MHz, etc.)
i_s = Current density
k = Dielectric constant
P = Polarization
q_v = Electric-charge density
r = Damping coefficient
t = Time
ν = Wave velocity
$V = V(y, t)$ = Potential
V_m = Amplitude of driving voltage
ϵ = Piezoelectric stress constant
ϵ_0 = Susceptibility of space = 8.85×10^{-12} $C^2/N \cdot m^2$ (or $C/V \cdot m$)
γ = Propagation constant
ρ = Density of quartz = 2.65 g/cm^3 = 2650 kg/m^3
ψ = Mechanical displacement
$\psi' = \dfrac{\partial \psi}{\partial y}$ = Strain
$\omega = 2\pi f$ = Angular frequency (rad/sec)

$$\psi'' = \frac{\partial^2 \psi}{\partial y^2} \qquad V'' = \frac{\partial^2 V}{\partial y^2}$$

$$\dot{\psi} = \frac{\partial \psi}{\partial t} \qquad V' = \frac{\partial V}{\partial y}$$

$$\ddot{\psi} = \frac{\partial^2 \psi}{\partial t^2} \qquad \dot{V} = \frac{\partial V}{\partial t}$$

The equation of motion of each point in the plate, including the effect of damping, is

$$\rho \ddot{\psi} - c\psi'' + r\dot{\psi} = 0 \tag{44}$$

The assumption that the damping force is proportional to the velocity is justified by the results (and the not inconsiderable fact that it results in an equation which can be solved). Little theoretical justification is available for the assumption; it may be that some of the observed variations of frequency with amplitude are due to nonlinear damping effects. It seems much more likely, however, that other effects are responsible.

The generalized stiffness constant c is used when considering the vibration of plates, whereas the compliance constant s is used in the problem of a vibrating bar. This distinction is necessary because the longitudinal vibration of a bar involves only the extensional stresses, the lateral stresses being negligible. In a thin plate, the lateral strains are zero; the lateral stresses are, in general, unknown.

The solution of Eq. (44) may be found by the usual method of assuming that the displacement ψ is the product of a function $A(y)$ and a function $\exp(j\omega t)$, i.e.,

$$\psi = A(y)\exp(j\omega t) \tag{45}$$

Substitution in Eq. (44) leads to the following solution.

$$\psi = [C_1 \sin(\gamma y) + C_2 \cos(\gamma y)]\exp(j\omega t) \tag{46}$$

where γ is the complex quantity

$$\gamma = \left[\frac{\omega^2}{\nu^2} - \frac{j\omega r}{\rho\nu^2}\right]^{\frac{1}{2}} \tag{47}$$

and the wave velocity $\nu = (c/\rho)^{\frac{1}{2}}$.

In case the damping coefficient $r = 0$, then $\gamma = \omega/\nu$ and the solution is the same as the one obtained earlier. [See Eq. (30)].

From the boundary condition that $\psi = 0$ at the center of the plate where $y = 0$, it follows that $C_2 = 0$, leaving

$$\psi = C_1 \sin(\gamma y)\exp(j\omega t) \tag{48}$$

To evaluate the constant C_1 we make use of the fact that the stress at the surface is due to the electric field alone; the mechanical stress due to internal tractive forces is zero. This is not the case in the interior of the plate, where stress (and strain) results from the combined mechanical and piezoelectric stresses. At the surface where $y = +\frac{1}{2}e$, the strain

$$\psi' = dE = d\,\frac{V_m}{e}\exp(j\omega t)$$

where $V_m \exp(j\omega t)$ is the applied voltage, assumed to be sinusoidal.

Differentiating Eq. (48) and evaluating at $y = \frac{1}{2}e$, we have

$$\psi' = \gamma C_1 \cos(\tfrac{1}{2}\gamma e)\exp(j\omega t) \tag{49}$$

Equating the last two expressions for ψ' and solving for C_1 gives

$$C_1 = \frac{d\,V_m}{\gamma\,e\cos(\frac{1}{2}\gamma e)}$$

The solution of Eq. (44) consistent with the boundary conditions including the driving voltage $V_m \exp(j\omega t)$ is

$$\psi = \frac{d\,V_m}{\gamma e\cos(\frac{1}{2}\gamma e)}\sin(\gamma y)\exp(j\omega t) \tag{50}$$

The polarization $P = \epsilon\,\psi'$.

The charge density

$$q_v = -\frac{\partial P}{\partial y} = -\epsilon\,\psi''$$

Differentiating Eq. (50) twice with respect to y gives

$$\psi'' = -\frac{\gamma V_m d}{e\cos(\frac{1}{2}\gamma e)}\sin(\gamma y)\exp(j\omega t)$$

From Poisson's equation (See App. I),

$$V'' = -\frac{q_v}{k\,\epsilon_0} = \frac{\epsilon}{k\,\epsilon_0}\,\psi''$$

so that

$$V'' = -\frac{\epsilon\gamma\,d\,V_m}{k\,\epsilon_0\,e\cos(\frac{1}{2}\gamma e)}\sin(\gamma y)\exp(j\omega t) \tag{51}$$

Integrating once,

$$V' = \frac{\epsilon d V_m}{k \epsilon_0 e \cos(\frac{1}{2}\gamma e)} \cos(\gamma y) \exp(j\omega t) + C_3$$

and integrating again,

$$V = \frac{\epsilon d V_m}{\gamma k \epsilon_0 e \cos(\frac{1}{2}\gamma e)} \sin(\gamma y) \exp(j\omega t) + C_3 y + C_4 \quad (52)$$

From the boundary condition that $V = 0$ when $y = 0$, we have that $C_4 = 0$. The other boundary condition is

$$V = \tfrac{1}{2} V_m \exp(j\omega t) \qquad \text{at } y = +\tfrac{1}{2} e$$

from which

$$\tfrac{1}{2} V_m \exp(j\omega t) = \frac{\epsilon d \sin(\frac{1}{2}\gamma e)}{k \epsilon_0 e \cos(\frac{1}{2}\gamma e)} V_m \exp(j\omega t) + C_3 (\tfrac{1}{2} e)$$

and

$$C_3 = \left[\frac{1}{e} - \frac{2\epsilon d}{\gamma k \epsilon_0 e^2} \tan(\tfrac{1}{2}\gamma e)\right] V_m \exp(j\omega t)$$

Substituting in Eq. (52) gives the expression for the potential $V(y)$ within the plate.

$$V = \left[\frac{\epsilon d \sin(\gamma y)}{\gamma k \epsilon_0 e \cos(\frac{1}{2}\gamma e)} + \frac{y}{e} - \frac{2\epsilon d}{\gamma k \epsilon_0 e^2} \tan(\tfrac{1}{2}\gamma e) y\right] V_m \exp(j\omega t)$$

The electric field

$$E(y) = -\frac{\partial V}{\partial y}$$

and thus

$$E = \left[\frac{\epsilon d \cos(\gamma y)}{k \epsilon_0 e \cos(\frac{1}{2}\gamma e)} + \frac{1}{e} + \frac{2\epsilon d}{\gamma k \epsilon_0 e^2} \tan(\tfrac{1}{2}\gamma e)\right] V_m \exp(j\omega t)$$

At the surface $y = +\frac{1}{2}e$:

$$E_{y=\frac{1}{2}e} = \left[\frac{1}{e} + \frac{\epsilon d}{k\epsilon_0 e} + \frac{2\epsilon d}{\gamma k \epsilon_0 e^2}\tan(\tfrac{1}{2}\gamma e)\right] V_m \exp(j\omega t)$$

The current density at the surface of the resonator is

$$i_s = \frac{\partial D}{\partial t}$$

or since $D = k\epsilon_0 E$,

$$i_s = k\epsilon_0 \frac{\partial E}{\partial t}$$

and therefore the current density at the surface of the resonator is

$$i_s = j\omega\left[\frac{k\epsilon_0}{e} + \frac{\epsilon d}{e} + \frac{2\epsilon d}{\gamma e^2}\tan(\tfrac{1}{2}\gamma e)\right] V_m \exp(j\omega t)$$

The total current into the resonator, assuming the current density (current per unit area) to be constant over the surface, is

$$i = \left[j\omega A\left(\frac{k\epsilon_0}{e} + \frac{\epsilon d}{e}\right) + j\omega A\,\frac{2\epsilon d}{\gamma e^2}\tan(\tfrac{1}{2}\gamma e)\right] V_m \exp(j\omega t) \quad (53)$$

The two terms inside the bracket of Eq. (53) have the dimensions of an admittance, since each is a current divided by a voltage. The first term

$$j\omega A\left[\frac{k\epsilon_0}{e} + \frac{\epsilon d}{e}\right]$$

is the admittance of a capacitor having capacitance

$$C_0 = (k\epsilon_0 + \epsilon d)\frac{A}{e}$$

If the piezoelectric constants were zero or the plate completely clamped so that all mechanical strains were zero, then

$$C_0 = k\epsilon_0 \frac{A}{e}$$

which is the capacitance of an ordinary parallel-plate capacitor. The term ϵd is about 1 percent of $k\epsilon_0$ so that the capacitance of a parallel-plate capacitor having a mechanically free quartz dielectric would be reduced by about 1 percent by clamping the plate to suppress the strain.

The second term in Eq. (53) represents another admittance in parallel with C_0. This admittance is obviously complex, since the quantity γ is complex. Therefore, the quartz plate with its electrodes is electrically equivalent to a capacitor in parallel with a complex admittance whose value depends upon the frequency. It is now necessary to investigate the nature of this admittance.

From Eq. (47) γ represents the quantity

$$\gamma = \left(\frac{\omega^2}{\nu^2} - \frac{j\omega r}{\rho\nu^2}\right)^{\frac{1}{2}}$$

The second term in the expression for γ is very small, so we may write with a high degree of approximation

$$\gamma = \frac{\omega}{\nu} - j\frac{r}{2\rho\nu} \tag{54}$$

For convenience in notation, we let $r/2\rho\nu = \beta$ so that Eq. (54) may be written

$$\gamma = \frac{\omega}{\nu} - j\beta$$

From the identity

$$\tan(x - jy) = \frac{\sin(2x) - j\sinh(2y)}{\cos(2x) + \cosh(2y)}$$

$$\tan\tfrac{1}{2}\gamma e = \tan\left[\frac{\omega e}{2\nu} - j\frac{\beta e}{2}\right] = \frac{\sin(\omega e/\nu) - j\sinh(\beta e)}{\cos(\omega e/\nu) + \cosh(\beta e)} \tag{55}$$

The angular frequency ω may be written as $\omega_0 - \Delta\omega$, where $\Delta\omega$ is always very small (almost always less than 0.1 percent) compared with ω or ω_0. This is because appreciable mechanical response occurs only when the applied frequency ω is very nearly equal to the resonant frequency ω_0.

The quantity $(\omega e/\nu)$ may be written

$$\frac{\omega e}{\nu} = \left(\frac{\omega_0 e}{\nu} - \frac{\Delta\omega e}{\nu}\right) = \left(\frac{2\pi f_0 e}{\nu} - \frac{\Delta\omega e}{\nu}\right) = \left(\pi - \frac{\Delta\omega e}{\nu}\right)$$

With this substitution Eq. (55) may be written

$$\tan\left(\tfrac{1}{2}\,\gamma e\right) = \frac{\sin(\pi - \Delta\omega e/\nu) - j\sinh(\beta e)}{\cos(\pi - \Delta\omega e/\nu) + \cosh(\beta e)}$$

which is equal to

$$\tan\left(\tfrac{1}{2}\,\gamma e\right) = \frac{\sin(\Delta\omega\, e/\nu) - j\sinh(\beta e)}{-\cos(\Delta\omega\, e/\nu) + \cosh(\beta e)} \tag{56}$$

But since $(\Delta\omega\ e/\nu)$ is a very small angle (equal to $\Delta\omega\pi/\omega_0$), it is permissible to make the following substitutions:

$$\left(\frac{\Delta\omega e}{\nu}\right) \quad \text{for} \sin\left(\frac{\Delta\omega e}{\nu}\right)$$

$$1 - \frac{1}{2}\left(\frac{\Delta\omega e}{\nu}\right) \quad \text{for} \cos\left(\frac{\Delta\omega e}{\nu}\right)$$

$$(\beta e) \text{ for } \sinh(\beta e)$$

$$1 + \frac{1}{2}(\beta e)^2 \text{ for } \cosh(\beta e)$$

With these approximations and a considerable amount of simple algebra, Eq. (56) may be written as

$$\tan\left(\tfrac{1}{2}\,\gamma e\right) = \frac{2\nu}{e}\,\frac{\Delta\omega - j\alpha}{\Delta\omega^2 + \alpha^2} \qquad \text{where } \alpha = \frac{r}{2\rho} \tag{57}$$

The second term in the bracket of Eq. (53) may now be written as

$$Y_m = \frac{4j\,\nu\,\omega A\,\epsilon d}{\gamma e^3}\;\frac{\Delta\omega - j\,\alpha}{\Delta\omega^2 + \alpha^2}$$

Using the approximation,

$$\gamma = \omega_0/\nu = \pi/e,$$

$$Y_m = \frac{4j\,\omega\,\omega_0\,A\,\epsilon d}{\pi^2 e}\;\frac{\Delta\omega - j\,\alpha}{\Delta\omega^2 + \alpha^2}$$

The reciprocal of the admittance Y_m is the impedance Z_m, which is

$$Z_m = \frac{\pi^2 e}{4\,\omega\,\omega_0 A \epsilon d}\ (\alpha - j\Delta\omega)$$

The resistance part of Z_m is

$$R_m = \frac{\pi^2 er}{8\,\omega_0^2 A \epsilon d \rho} \tag{58}$$

Equation (58) may be simplified by expressing d in terms of ϵ and the elastic constant c. In matrix notation

$$\begin{aligned} x &= dE && \text{from Eq. (15)} \\ -X &= \epsilon E && \text{from Eq. (20)} \\ -X &= cx && \text{from Eq. (13)} \\ -x &= c^{-1} X. \end{aligned}$$

From the four equations above we have $x = c^{-1}\epsilon E = dE$ and therefore

$$d = c^{-1}\epsilon$$

In a simple one-dimensional case c^{-1} is equivalent to $1/c$, so that we have $d = \epsilon/c$. Substituting for d and recalling that

$$f_0 = \frac{1}{2e}\sqrt{\frac{c}{\rho}}$$

we obtain

$$R_m = \frac{\pi^2 erc}{8\,\omega_0^2\, A\, \epsilon^2 \rho}$$

$$R_m = \frac{e^3 r}{8A\epsilon^2} \tag{59}$$

The expression above for R_m shows that the motional resistance in the equivalent network is directly proportional to the damping constant r and to the cube of the thickness of the plate. R_m is inversely proportional to the area A and to the square of the piezoelectric stress constant ϵ. It is not possible to compute the numerical value of R, since no direct measurement of the value of r is possible. It must be recalled that r is assumed to include all dissipative forces and that they are proportional to the particle velocity in the quartz. Other sources of damping, in addition to internal dissipation, are air damping, surface friction, and mounting losses. In most cases these are greater than the internal losses and it is unlikely that each is exactly proportional to the particle velocity. Possibly a part of the variation of resistance with amplitude which is sometimes observed may be due to nonlinear damping effects. The expression for R is valid even though r should be frequency-dependent; and if the dependence were known, the variation of the value of R with frequency could be calculated.

Despite its semiquantitative nature, Eq. (59) shows why the equivalent resistance of a low-frequency plate is so high compared with that of a high-frequency plate of the same area.

The reactive part of Z_m is given by

$$X_m = -\frac{\Delta\omega\pi^2 e}{4\omega_0^2\, d\,\epsilon A}$$

Eliminating d as was done in the derivation of R gives

$$X_m = -\frac{\Delta\omega\rho e^3}{4A\epsilon^2}$$

Thus X_m varies directly with $-\Delta\omega$, the difference between the frequency of resonance and the operating frequency. This is precisely the behavior of a circuit consisting of an inductance and a capacitance in series. This may be shown in the following way.

The reactance of a circuit consisting of L and C in series is

$$X = \omega L - \frac{1}{\omega C} = \frac{\omega^2 LC - 1}{\omega C} = \frac{\omega^2 LC - \omega_0^2 LC}{\omega C}$$

$$= \frac{(\omega + \omega_0\,(\omega - \omega_0)}{\omega} L \doteq -\frac{2\omega\Delta\omega}{\omega} L = -2\Delta\omega L^\dagger$$

† Note that $\Delta\omega = (\omega_0 - \omega)$ (see page 94).

Therefore the reactance X_m behaves as if it were an inductance L having the value

$$L = \frac{\rho e^3}{8A\,\epsilon^2} \tag{60}$$

in series with a capacitance having the value

$$C = \frac{8A\,\epsilon^2}{\pi^2\,e\,c} \tag{61}$$

We have shown that a quartz plate with its electrodes is the equivalent of the electric circuit shown in Fig. 6.5, in which the parameters have the following values.

$$C_0 = k\,\epsilon_0\,\frac{A}{e}$$

$$R_m = R = \frac{e^3 r}{8A\,\epsilon^2}$$

$$L_m = L = \frac{e^3 \rho}{8A\,\epsilon^2}$$

$$C_m = C = \frac{8A\,\epsilon^2}{\pi^2\,e\,c}$$

In deriving the equations for R, L, and C it has been assumed that the strain distribution is uniform over the surface of the plate A. The strain over the surface of an actual resonator is, of course, not constant. The results obtained by the use of Eqs. (60) and (61) must be expected to be approximately valid and to differ from the measured values by factors of the order of unity. The equations do show how R, L, and C depend upon the physical properties of quartz and the dimensions of the plate and they therefore point the direction which must be taken to obtain desired variations in the parameters L and C.

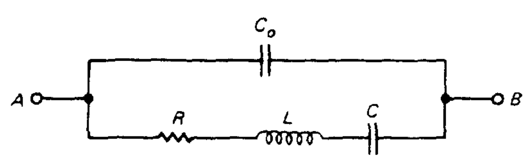

Fig. 6.5. The equivalent circuit of the piezoelectric resonator. The parameters R, L, and C are often called R_m, L_m, and C_m to call attention to the fact that they are associated with the motion of the plate.

In order to calculate the numerical values of L and C one must make a more elaborate calculation, taking into account the boundary conditions on a finite plate. Such a calculation is difficult. Alternatively, one may make some assumption concerning the distribution of the strain over the surface. Cady assumed the strain to be sinusoidal in the case of rectangular plates. The result agrees fairly well with measured values.

To investigate the consequences of the approximations made in deriving the expressions for C_0, C_m and L_m we may consider a certain 10.8-MHz unit made with an AT-cut piezoid having the following dimensions:

Diameter of blank	8.0×10^{-3} m	(315 mils)
Electrode diameter	2.5×10^{-3} m	(100 mils)
Blank thickness e	1.54×10^{-4} m	(6.06 mils)
Electrode area A	4.91×10^{-6} m^2	(0.010 in^2)
Angle of cut θ	$35\frac{1}{4}°$	

We must determine the value of the applicable piezoelectric coefficient ϵ'_{26} and the applicable stiffness coefficient c'_{66}. For the AT-cut these are

$$\epsilon'_{26} = (\epsilon_{14} s - \epsilon_{11} c)\, c$$

and $$c'_{66} = c_{44} s^2 + 2\, c_{14}\, sc + c_{66}\, c^2$$

where $s = \sin 35\frac{1}{4}°$ and $c = \cos 35\frac{1}{4}°$. These values are

$$\epsilon'_{26} = 9.65 \times 10^{-2}\ \text{C/m}^2 \qquad \text{and} \qquad c'_{66} = 29.3 \times 10^{9}\ \text{N/m}^2$$

Substituting these values in the expressions for C_0, C_m, and L_m and remembering to use mks units for all quantities, we obtain the theoretical values to be compared with the measured values with the following results:

	Theoretical	Measured
C_0	1.2 pF (plus holder capacitance)	2.2 pF (including holder)
C_m	8.2×10^{-15} F	6.6×10^{-15} F
L_m	0.026 H	0.033 H

The theoretical value of C_m is too large by about 25 percent and the theoretical value of L_m is too small by about the same amount. This is to be expected, since we have assumed that the strain is constant over the entire electrode area. This is impossible in a finite plate, of course. Nevertheless, the computed values are close enough to the actual measured values to be useful in predicting the parameters to be expected from a given design.