5
MECHANICAL VIBRATIONS

INTRODUCTION

In this chapter we discuss the ideal case of free vibrations of solid bodies. Although the quartz crystal unit is not a freely vibrating system, many of the pertinent features can be understood by considering the simpler case of free vibrations. The effects of frictional and driving forces will be considered later.

FUNCTION OF THE PIEZOID IN OSCILLATOR CIRCUITS

The quartz crystal unit serves the same purpose in an oscillator circuit as does the balance wheel and hair spring in a watch. In both cases the function is to provide isochronous (same time) vibrations, i.e., to mark off equal intervals of time.

The time required for the balance wheel of a watch to complete one vibration is given by

$$T = 2\pi \sqrt{\frac{I}{h}}$$

where I is the moment of inertia of the balance wheel and h is the stiffness coefficient of the hair spring. The equation above applies only to a freely vibrating system, i.e., one which is frictionless and not being driven. Of course, no vibrating system is frictionless, so if sustained oscillations are required, energy must be supplied to replace that lost in friction and other damping effects. The energy to replace that lost through damping is supplied by the main spring through the escapement.

The function of the quartz crystal unit in an oscillator circuit is likewise to mark off equal intervals of time or, in other words, to stabilize the frequency of oscillation. As with all mechanical vibrating systems, some energy is dissipated in the vibration of the crystal and this energy must be replaced if oscillations are to be sustained. The energy is supplied to the vibrating crystal through the piezoelectric effect, which is therefore analogous to the escapement of the watch.

Just as the period or frequency of the balance wheel of the watch is affected by friction and driving forces, so is the frequency of the quartz resonator similarly affected. The effect is small in both cases but especially so in a well-made quartz resonator where the fraction of the energy dissipated per cycle may be as low as one millionth.

Isochronous mechanical vibrations could presumably be excited in almost any solid, elastic material. However, the piezoelectric effect provides an extremely simple and convenient method of exciting vibrations in a material such as quartz in which a linear mechanical strain is created by an electric field. The properties of quartz which make it the most satisfactory material for use as a frequency-stabilizing element are its hardness, low internal dissipation, durability, uniformity, freedom from flaws, and not least, the combination of characteristics which makes possible the design of piezoids in which the resonant frequencies are substantially independent of temperature. Other crystals are known which satisfy some of these requirements but none, except quartz, satisfies all of them.

ONE-DIMENSIONAL VIBRATIONS

Any solid body may be set into vibration in many different ways, that is, it has many resonant frequencies. Various types of vibrations are used in piezoelectric resonators. In every case a natural resonant frequency of the mechanical system is used to control the frequency of the associated electric system. The simplest type of ideal vibrating system is one in which the vibrating body has only one dimension. Naturally no such device exists, but several systems may be treated as one-dimensional, at least to a high degree of approximation. Whenever the controlling dimension is very large or very small compared with the other dimensions, the vibration can be considered to be approximately one-dimensional.

Probably the simplest example of a one-dimensional system is provided by a thin, flexible string stretched between two fixed points. Another example is a long, thin elastic rod vibrating in such a way that the displacement is parallel to the length of the rod. A long, slender organ pipe is a fairly good approximation to a one-dimensional system. A plate in which the length and width are very large compared with the thickness can be treated as a one-dimensional system for waves traveling in the thickness direction.

The differential equation describing the free vibration of a one-dimensional system is

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial x^2} \tag{27}$$

where

ψ = displacement of a point in the vibrating body relative to its equilibrium position,
v = velocity with which a disturbance corresponding to the displacement ψ is propagated in the body,
x and t = space and time coordinates, respectively.

Equation (27) is applicable to one-dimensional systems in which the displacement ψ is either parallel or perpendicular to the direction of wave propagation.

To find a solution of Eq. (27) we assume that a solution can be found having the form

$$\psi(x, t) = [T(t)]\,[X(x)] \tag{28}$$

where T is a function only of time and X is a function of the space coordinate x. We take the partial derivatives of $\psi(x, t)$ with respect to t and x and substitute these into Eq. (27). After dividing both sides by XT, we obtain

$$\frac{1}{T}\frac{d^2 T}{dt^2} = \frac{v^2}{X}\frac{d^2 X}{dx^2} \tag{29}$$

The expression on the left side of Eq. (29) is a function only of t and that on the right only of x. It is necessary, therefore, that each be equal to the same constant which we will take as $-\omega^2$.

We now have

$$\frac{1}{T}\frac{d^2 T}{dt^2} = -\omega^2 \quad \text{and} \quad \frac{v^2}{X}\ \frac{d^2 X}{dx^2} = -\omega^2$$

The solution of the equation in t is

$$T = A \sin(\omega t) + B \cos(\omega t)$$

and that of the equation in x is

$$X = D \sin\left(\frac{\omega x}{v}\right) + E \cos\left(\frac{\omega x}{v}\right)$$

The product of these two equations is a solution of Eq. (27). The values of the arbitrary constants, A, B, C, D, and ω are determined by the mechanical constraints imposed upon the vibrating system. Such constraints are called *boundary conditions.*

Before the solution can be carried further, it is necessary to define the nature of the vibrating system. Let us suppose that we have a thin string of length L rigidly attached to points at $x = 0$ and $x = L$. We shall further assume that $\psi = 0$ when $t = 0$, which is equivalent to deciding when we are to start measuring time. Mathematically, the boundary conditions can be expressed as

1. $\psi = 0$ for $x = 0$
2. $\psi = 0$ for $x = L$
3. $\psi = 0$ for $t = 0$

Condition 1 requires that $E = 0$. Condition 2 requires that

$$\frac{\omega L}{v} = n\pi \qquad n = 1, 2, 3, \ldots$$

and condition 3 requires that $B = 0$.

A solution of Eq. (27) consistent with the three boundary conditions is therefore

$$\psi_n = A_n \sin\left(\frac{n\pi v}{L} t\right) \sin\left(\frac{n\pi}{L} x\right)$$

For our purpose it is significant to note that the string may vibrate in many different modes depending on the value of n which must be an integer. The frequency of the vibration in each of these modes is found by setting

$$\omega_n = \frac{n\pi v}{L} = 2\pi f_n \qquad \text{or} \qquad f_n = n\frac{v}{2L} \tag{30}$$

The quantity $f_1 = (v/2L)$ is called the fundamental frequency or the first harmonic. The frequency $f_2 = 2f_1$ is called the second harmonic, etc. Thus f_n, the frequency of the nth harmonic, is exactly n times the frequency of the fundamental and the "eigenfrequencies" or resonant frequencies of the simple one-dimensional system are related as the integers 1, 2, 3, ... Not only may the string vibrate at any one of these frequencies, it may also vibrate at all of them simultaneously. The characteristic tones of string musical instruments are the result of simultaneous vibrations at a number of different frequencies.

The displacement of the string vibrating in modes corresponding to $n = 1$, 2, and 3 is illustrated in Fig. 5.1. The relative amplitudes and phases of the various modes depend on the way in which the string is set into motion but the frequencies are determined by Eq. (30).

The modes of motion of the string are determined by the conditions that the ends of the string must be nodal points (points of zero displacement) and that the length of the string be an integral number

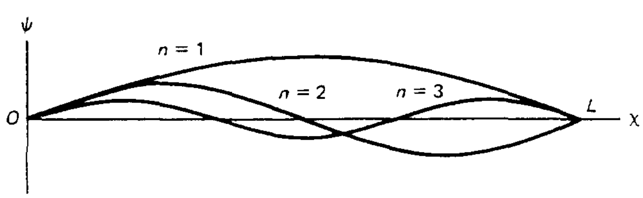

Fig. 5.1. Modes of vibration of a stretched string.

of half-wavelengths of the wave which travels from one end of the string to the other. These conditions, of course, serve to determine the resonant frequencies. The most complicated vibration which a string may execute is made up of a superposition of a set of modes. The amplitudes of the various modes as well as their phase relationships may be determined by writing the Fourier series for the shape of the string at the time it is set into motion.[1]

THE VIBRATING BAR

The solution of the problem of the longitudinal vibration of a long, thin bar is carried out in the same way done for the string. However, vibrating bars of quartz are usually supported at the midpoint rather than at the ends. The displacement is parallel to the length of the bar instead of at right angles as in the case of the string. The differential equation of motion is the same [Eq. (27)] and the solution is carried out in the same way except that the boundary conditions are different. Since the bar is clamped at its midpoint, that point must be a nodal point and the ends of the bar, being unconstrained, are antinodes. With these boundary conditions, the solution requires that

$$f_n = n\frac{v}{2L} \qquad n = 1, 3, 5, \ldots$$

and the resonant frequencies are related as the odd integers.

It is somewhat difficult to visualize the motion of the longitudinally vibrating bar. One way to do so is to imagine a series of dots to be placed on the surface of the bar before it is set into vibration. When the bar vibrates the dots appear to be stretched into lines by the phenomenon of persistence of vision. The lines would be longest at the antinodes and shortest near the nodal points. Figure 5.2 illustrates how the dots would appear when the bar is vibrating in modes $n = 1$, 3, and 5.

The length of the bar is $\lambda/2$, $3\lambda/2$, $5\lambda/2$, etc. No modes corresponding to $n = 2, 4, 6, \ldots$ exist.

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[1] See R. L. Churchill, *Fourier Series and Boundary Value Problems.* McGraw-Hill, New York, 1969.

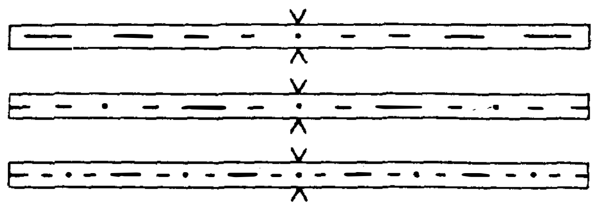

Fig. 5.2. Modes of motion of a vibrating bar clamped at its midpoint. The dots are nodal points.

Precise measurements of the resonant frequencies of actual vibrating systems reveal that they are not exactly integral multiples of the fundamental frequency, because actual vibrating systems are not perfectly one-dimensional. Every string has a finite thickness and every rod or bar has a finite width and thickness. If the length is very large compared with the lateral dimensions, the resonant frequencies closely approach the harmonic values.

TWO-DIMENSIONAL VIBRATING SYSTEMS

The CT- and DT-cut family of resonators consist of either a square or circular plate of quartz vibrating in a face-shear mode. The centerpoint of the plate is a nodal point and the plate is supported by attaching lead wires at that point. Such resonators closely approximate two-dimensional vibrating systems. It is instructive to investigate the vibration of such systems as a basis for understanding these devices and to illustrate further the principles of vibrating solids.

The differential equation of motion of a two-dimensional system, in rectangular coordinates, is

$$\frac{\partial^2 \psi}{\partial t^2} = v^2\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) \tag{31}$$

where ψ is the displacement, either in the plane of the plate or perpendicular to it. The wave velocity v is the velocity of the wave traveling in the plane of the plate. The velocity v depends on the density and the applicable elastic constants. The quantities x, y, and t are the usual space and time coordinates. Equation (31) may be solved in exactly the same way that Eq. (27) was solved for the one-dimensional case.

If we assume that the plate is a rectangle with dimensions a and b and that the center of the plate is constrained to be a nodal point of displacement, then the solution of Eq. (31) consistent with the boundary conditions above is

$$\psi_{nm} = A_{nm}\left(\sin \frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\omega_{nm} t\right)$$

where

$$\omega_{nm} = \pi v \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}} \qquad n, m = 1, 3, 5, \ldots \tag{32}$$

or

$$f_{nm} = \frac{v}{2}\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$$

The relationship between the frequencies of the modes of the two-dimensional system is not as simple as in the one-dimensional case. The mode of lowest frequency is the mode in which $n = 1$ and $m = 1$. This is the fundamental frequency. The shape assumed by the plate when vibrating in the 1 1 (read one one) mode is shown in Fig. 5.3b. Because of the boundary conditions, the length and width of the plate must be an odd number of half-wavelengths of the wave propagated along the same direction. Hence the modal index 1 1 indicates that the length and width of the plate are each one half-wavelength.

The mode next higher in frequency depends on the length and width of the plate. The frequencies can be readily computed using

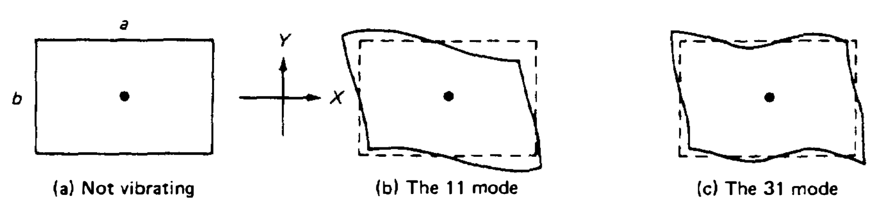

Fig. 5.3. Modes of motion of a thin plate vibrating in face shear. (*a*) not vibrating; (*b*) the 11 mode; (*c*) the 31 mode.

the formula derived above. For example, if $a = 3$ and $b = 2$, the relative frequencies of the first few modes are shown in Table 5.1.

From Table 5.1 it can be seen that some of the overtones are harmonically related to the fundamental mode but other modes are not. In the example given, the first overtone mode is the 31 mode, but its frequency is not an integral multiple of the frequency of the 11 mode. However, the frequency of the 33 mode is exactly three times that of the 11 mode.

Many quartz crystal units are made from square blanks vibrating in the face-shear mode. It is readily seen, from Eq. (32), that the frequency of the nm mode is given by

$$f_{nm} = \frac{v}{2a}\sqrt{n^2 + m^2} \qquad m, n = 1, 3, 5, \ldots \tag{33}$$

Hence the frequencies of the nm and mn modes are equal. In such cases the modes are said to be *degenerate*.

So far we have been discussing the vibration of isotropic plates. In crystals, however, the wave velocity is likely to differ with direction in the crystal. If we let the wave velocity in the Y direction be b times that in the X direction which is taken as v, then we have for a square anisotropic plate

$$f_{nm} = \frac{v}{2a}\sqrt{n^2 + b^2 m^2} \quad m, n = 1, 3, 5, \ldots \tag{34}$$

Piezoids, made from circular plates of quartz, can also be excited in face-shear modes. Since the solution of the equation of motion in polar coordinates involves Bessel's functions and is rather complicated,

Table 5.1. Relative Frequencies of the Overtone Modes of a Plate Vibrating in Face Shear.*

Mode	Relative Frequency
11	0.601
31	1.118
13	1.536
33	1.803
55	3.005

*Length =3, width = 2. To obtain actual frequencies, multiply by $v/2$.

we omit the details. The results show that a circular plate may also vibrate in many different modes, the frequencies of which are not harmonically related. The nodal lines are circular and radial with a node at the central point where the blank is supported.

The behavior of two-dimensional systems differs from that of one-dimensional systems in two respects. The number of modes is much larger in two-dimensional systems and the frequencies of the modes are not harmonically related as in the one-dimensional system. The inharmonic modes are responsible for the characteristic musical tones of bells and cymbals. They are also responsible for a number of problems in the design of quartz crystal resonators. These problems are discussed later.

Either an X-cut or a Y-cut plate of quartz can be excited into vibration in a face-shear mode by imposing an electric field of the appropriate frequency in the thickness direction of the plate. The X-cut plate is excited through the coefficient d_{14} of Eq. (26), which relates the field in the X-direction to a shear strain about the X-axis. The Y-cut plate is excited through the coefficient $d_{25} = -d_{14}$, which relates a field in the Y-direction to a shear strain about the Y-axis. Neither of these piezoids is particulary useful, because the frequencies are not temperature-independent. By rotating the plane of the Y-cut about the X-axis, two positions are found at which the frequency of the face-shear vibration is substantially independent of temperature. These are known as the CT- and DT-cuts. Quartz crystal units employing these cuts are extensively used in the frequency range between 100 and 1000 kHz.

THREE-DIMENSIONAL VIBRATING SYSTEMS (RECTANGULAR PLATES)

Most quartz resonators are made in the form of thin plates. Circular plates have certain advantages, including ease of production, and are preferred for most applications. The theory, however, is simpler in the case of rectangular plates, so we will use it to illustrate the important features of the three-dimensional vibrating system.

X-cut plates of quartz can be set into extensional vibration in the thickness direction through coefficient d_{11} of Eq. (26). Such plates are sometimes used as ultrasonic transducers but the temperature coefficient of frequency which is about $-20 \times 10^{-6}/°C$ makes them undesirable for use as frequency controlling elements.

Y-cut plates of quartz may be excited into a thickness-shear mode of vibration through coefficient $d_{26} = -2d_{14}$ [Eq. (26)]. The temperature coefficient of frequency of the Y-cut plate is about $+100 \times 10^{-6}/°C$ but by rotating the plane of the blank about the X-axis, two positions are found at which the frequency is nearly independent of temperature. The resulting piezoids are called the AT- and BT-cuts.

The mode of motion of the X-cut plate is equivalent to a longitudinal wave traveling in the thickness direction of the plate. The displacement in the Y-, AT-, and BT-cut plates is equivalent to a transverse wave traveling in the thickness direction of the plate. The equation of motion in both cases is

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}\right] \qquad (35)$$

where the displacement ψ is taken to be parallel to the direction of wave propagation in the X-cut and perpendicular to the direction of propagation in the case of the Y-cut.

The solution of Eq. (35) is carried out by assuming that a solution in the form

$$\psi(x, y, z, t) = [X(x)]\ [Y(y)]\ [Z(z)]\ [T(t)]$$

can be found.

If we let the length l be in the X direction; the width w be in the Z direction; the thickness e in the Y direction and assume that the plate is not clamped, then the midpoint of the plate must be a node and the boundary conditions are

$$\begin{aligned} \psi &= 0 \quad \text{at } x = 0 \\ \psi &= 0 \quad \text{at } y = 0 \\ \psi &= 0 \quad \text{at } z = 0 \end{aligned}$$

The solution of Eq. (35) satisfying these boundary conditions is

$$\psi_{nmp} = A_{nmp} \sin\left(\frac{n\pi y}{e}\right) \sin\left(\frac{m\pi x}{l}\right) \sin\left(\frac{p\pi z}{w}\right) \sin\left(\omega_{nmp} t\right) \qquad (36)$$

where

$$\omega_{nmp} = \pi v \sqrt{\frac{n^2}{e^2} + \frac{m^2}{l^2} + \frac{p^2}{w^2}} \qquad n, m, p = 1, 3, 5, \ldots \tag{37}$$

The modes of motion thus consist of standing waves in the X, Y, and Z directions with the thickness, length, and width each being equal to an odd number of half-wave lengths. The corresponding frequencies given by Eq. (37) are, in general, not harmonically related. If l and w are both very large compared with the thickness e, then Eq. (37) can be written as

$$\omega_n = \frac{n\pi v}{e} \qquad \text{or} \qquad f_n = \frac{n\, v}{2e} \tag{38}$$

The wave-propagation velocity v is given by $v = \sqrt{c_{ij}/\rho}$, where $c_{ij} = c_{11}$ for the X-cut and c_{66} for the Y-cut. ρ is the density.

The resonant frequency of an infinitely large X-cut plate is therefore given by

$$f_n = \frac{n}{2e}\sqrt{\frac{c_{11}}{\rho}} = n\,\frac{2849}{e} \qquad (X\text{-cut}) \qquad (39)^1$$

and for an infinitely large Y-cut plate

$$f_n = \frac{n}{2e}\sqrt{\frac{c_{66}}{\rho}} = n\,\frac{1954}{e} \qquad (Y\text{-cut}) \qquad (40)^1$$

The overtone frequencies of an infinite plate would be odd-integral multiples of the fundamental frequency, but in plates of finite size, this is not the case. Furthermore, plates of finite size display many more modes of vibration, the frequencies of which depend upon the length and width of the plate and on the values of m and p. The mode of lowest frequency is the 1 1 1 mode. If, as is usually the case,

[1] In Eqs. (39) and (40), e is in meters and f is in hertz.

l and w are larger than e, then a series of modes appear, all of which have frequencies higher than the 111 mode. These modes have indices 131, 113, 133, 151, 153, 155, etc.

The inharmonic modes are the *spurs* (spurious modes) which are so troublesome in the design and fabrication of piezoids for use in high-frequency filters. Like the harmonic modes they are thickness-shear modes, controlled by the same elastic coefficients and having the same temperature coefficient of frequency. Figure 5.4 gives a qualitative picture of the distribution and relative amplitudes of the various modes of vibration of an AT-cut plate. In a later section we consider the steps which may be taken in the design and fabrication to minimize the effect of the overtone modes.

The inharmonic modes are not so important in the design of piezoids for oscillator applications. The harmonic modes are much easier to excite and the oscillator usually selects the strongest mode. There exist, however, many other modes of vibration which have different temperature coefficients of frequency. Occasionally, as the temperature of the piezoid is changed, the frequency of one of these modes may coincide with that of the oscillator frequency. The resulting extra dissipation of energy reduces the amplitude of oscillation (activity) and the result is called an *activity dip.* Sometimes the oscillator "jumps" to one of the inharmonic modes with a "skip" in frequency.

In a later section we discuss the problem of coupled modes and the steps which may be taken in the design of piezoids to minimize their effects.

In older designs for quartz crystal units, rectangular AT- and BT-cut plates were often placed between metal plates which served to clamp the plate around its edges or at its corners. These changes in the boundary conditions are such that the modal indices m and p may be even as well as odd integers and consequently modes such as

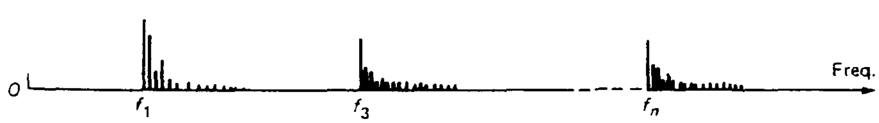

Fig. 5.4. Overtone spectrum of an AT-cut plate (qualitative).

112, 121, 123, etc., are permitted. The result is to increase greatly the number of inharmonic overtone modes.

THREE-DIMENSIONAL VIBRATING SYSTEMS (CIRCULAR PLATE)

For reasons which will be discussed later, most high-frequency quartz resonators are now made using circular plates. Although the general features of three-dimensional vibrating systems are similar for rectangular and circular plates, it is interesting and instructive to study them in some detail.

The complete solution of the equation of motion of a three-dimensional plate cut from an anisotropic crystal is extremely complex. The greatest mathematical difficulties arise from the uncertain nature of the boundary conditions. By neglecting the effects of anisotropy and idealizing the boundary conditions, a soluble problem is defined, and even though the solution yields numerical results which are only fair approximations, yet certain useful and important results are obtained.

Circular AT-cut plates are usually contoured to give one or both of the major surfaces a spherical shape. Sometimes a flat circular region is left in the center of the plate, as shown in Fig. 5.5*d*. Electrodes smaller than the quartz plate are deposited on the major surfaces concentric with the plate. Under these conditions the vibrating region is confined to a circular region which may be smaller, equal to, or larger than the area of the electrodes. The annular ring of quartz surrounding the vibrating area serves to confine the energy of vibration, giving rise to the "energy-trapping" concept which is treated in Chap. 10.

The problem is solved by writing the wave equation in cylindrical coordinates, separating the variables in the usual way, and applying

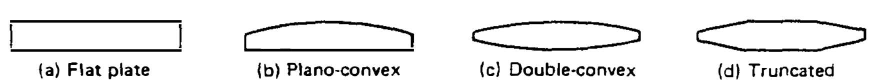

Fig. 5.5. Contoured quartz plates. (*a*) Flat plate; (*b*) plano-convex; (*c*) double convex; (*d*) truncated.

the boundary conditions described above. In this way the eigenfrequencies of the various normal modes are found to be

$$\omega_{nmk} = v\sqrt{\frac{n^2\pi^2}{e^2} + \frac{\chi^2_{mk}}{a^2}} \qquad \begin{aligned} n &= 1, 3, 5, \ldots \\ m &= 0, 1, 2, 3, \ldots \\ k &= 1, 2, 3, \ldots \end{aligned} \tag{41}$$

and

$$f_{nmk} = \frac{\omega_{nmk}}{2\pi} \doteq \frac{nv}{2e} + \frac{v\,e\,\chi^2_{mk}}{4n(\pi a)^2} \tag{42}$$

In the equations above χ_{mk} is the kth root of Bessel's function of order m, values of which are found in tables of mathematical data.

The mode of the lowest frequency is the one for which $n = 1, m = 0$, and $k = 1$. If the radius of the plate a is very large compared with the thickness e, the frequency of the 101 mode is approximately $v/2e$, which is the same as that of the infinitely large rectangular plate. Obviously, if the boundary is infinitely far away, it does not matter whether it is circular or rectangular. Likewise, the frequencies of the 301, 501, etc., modes are approximately 3, 5, . . .times that of the fundamental mode, so we have

$$f_n = \frac{nv}{2e} = n\,\frac{K}{e} \qquad \text{where} \qquad K = \frac{v}{2} = \frac{1}{2}\sqrt{\frac{c_{ij}}{\rho}} \tag{43}$$

From Eq. (42) the frequencies of the inharmonic modes differ from that of the fundamental mode by terms which are proportional to the squares of the roots of Bessel's functions. The differences are also inversely proportional to the square of the radius of the vibrating area a. Therefore, if a is made smaller by contouring and/or depositing smaller electrodes, the frequency separation between the fundamental and inharmonic overtones is increased and this fact is utilized extensively in the design of crystal units for high-frequency filter applications.

It should not be expected that the frequencies of a given plate would be given exactly by Eq. (42) because of the approximations made in the definition of the problem. It is not surprising, however, that a large number of inharmonic modes may be observed. It is easy to measure the frequencies of the various modes by using the crystal unit as an element in a passive network and exciting it by a suitable oscillator, as shown in Fig. 5.6.

It is also possible to "observe" the displacement of the surface of the quartz plate by permitting some fine dust to fall on the surface of the quartz while it is excited piezoelectrically. The dust is driven from the regions of maximum motion to collect along the nodal lines. The resulting "dust patterns" clearly show the complicated nature of the vibration and the differences between the various modes of motion.

Figure 5.7 shows the actual overtone spectrum of a 3.1-MHz circular AT-cut plate having a diameter of 1.397 cm. The surfaces of the plate were lapped to have a radius of curvature of 25 cm (using a 2.0 diopter optical lap). Although the frequencies of the modes are not given precisely by Eq. (42), the qualitative validity of the simple theory is indicated by the way in which the frequencies of the overtone modes may be arranged in three series and the modal indices assigned, as shown in Fig. 5.7.

The abscissa of Fig. 5.7 is the frequency range between 3.10 and 3.75 MHz. Within this range the crystal unit has 13 modes of vibration, including the 101, or fundamental, mode. The first overtone mode is the 111 mode at 3.22 MHz. The second overtone is the 102 mode and it occurs at 3.27 MHz. All the modes can be arranged in series corresponding to $k = 1, 2, 3, \ldots$ and values of $m = 0, 1, 2, \ldots$. A plot of χ^2_{mk} shows a similar arrangement.

When the plate vibrates in the $n01$ mode, all points on the surface move in phase, so the surface charge has the same sign at all points. This condition is illustrated in the drawing in Fig. 5.8. In filter applications the $n11$ mode is the most troublesome because it is the first mode above the fundamental, or desired, mode. The index

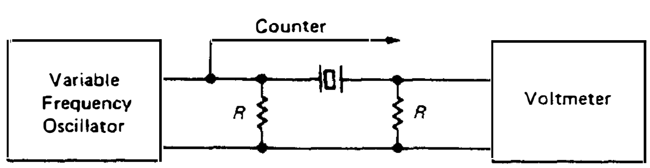

Fig. 5.6. Passive network for observing the frequencies of the modes of vibration of a quartz crystal unit.

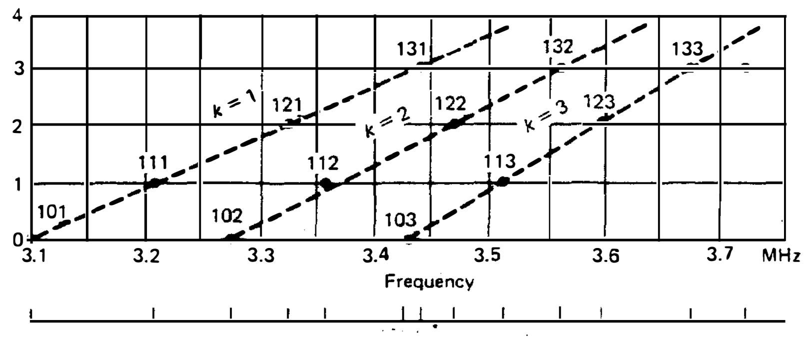

Fig. 5.7. Overtone spectrum of a 3.1-MHz AT-cut plate (experimental).

$m = 1$ indicates that one nodal line exists along a diameter of the plate. The polarization charges on opposite sides of the line have opposite signs, as indicated in Fig. 5.8. The index $m = 2$ indicates that two radial nodal lines exist, etc. The index $k = 1$ indicates that only one circular nodal line exists, i.e., the one surrounding the vibrating area. The index $k = 2$ indicates that two concentric nodal lines exist, etc. The electric charge developed on the two sides of any nodal line must be different, since the phase of the strain is opposite on the two sides.

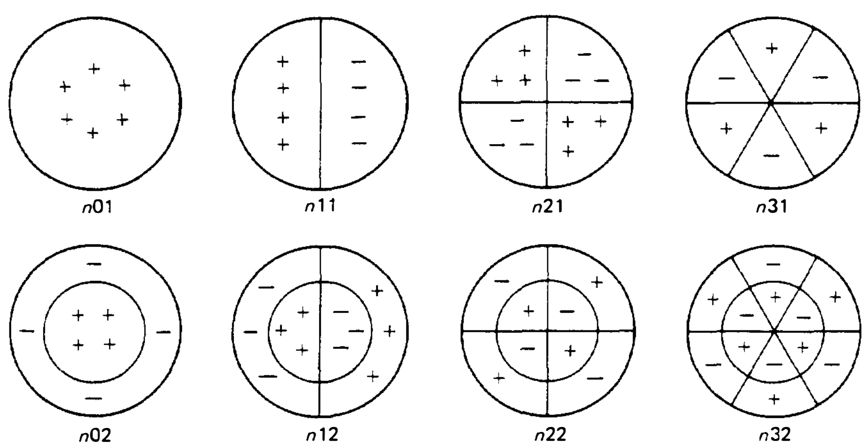

Fig. 5.8. Relative phase and charge on the surface of a thickness-shear plate vibrating in different modes *nmk*.

If the quartz plate and electrodes are perfectly symmetric with respect to the nodal line, then the $n11$ mode should not be excited, since the polarization charges would exactly cancel. The same thing is true for all modes having values of m other than $m = 0$, because the center of the plate is a nodal point with the charges developed symmetrically around it (neglecting the effects of anisotropy).

If the plate and electrodes are perfectly symmetrical and concentric with the vibrating area, then only modes having the modal indices $n0k$ could be excited. The first troublesome mode would then be the $n02$ mode in which two circular nodal lines are present. The charge inside the inner nodal line is opposite in sign to that developed on the surface between the two nodal lines. Conceivably it might be possible to suppress the $n02$ mode without losing the $n01$ mode by adjusting the electrode size so that the charges developed in the two zones would exactly cancel. Of course, the problem is greatly complicated by the anisotropy of the quartz.

SUMMARY

In this chapter we have tried to show that the vibration of a quartz piezoid is not unlike that of any other vibrating solid. The principle difference is the manner in which the vibrations are excited. The piezoelectric effect provides a particularly convenient method for setting the mechanical system into resonance. The overtone modes, some harmonic and some inharmonic, are characteristic of all vibrating bodies. There is nothing "spurious" about them, that is, the term should not imply that they are in some way unnatural or illegitimate.

We should recall, too, that we have been discussing idealized situations for the sake of simplicity. The complete problem of the vibrating piezoid including all the effects of anisotropy and with realistic boundary conditions is practically insoluble. We have neglected damping and driving forces which further complicate the problem. However, from a discussion of the idealized problem, it is possible to see the reasons for many of the phenomena which are observed and to suggest steps which may be taken to achieve desired design goals.

The idealized drawings of Fig. 5.8 must be understood to be illustrative but not of all features of the vibration. Nevertheless, the strain and charges on the surfaces of actual resonators do exhibit nodal lines as shown by dust patterns and other methods. Some overtone modes can be suppressed by careful attention to design of the quartz plate and to the deposition of the electrodes. It is apparent that the most meticulous care must be given to the geometry of the plate and to the symmetry of the electrodes in order to provide resonators which are suitably free of undesired modes for filter applications. It is clear that this can be done more exactly with round than with rectangular plates.

The material presented in this chapter should indicate that the design of a quartz resonator, taking into account, damping, anisotropy, overtone modes, mechanical imperfections, and coupled modes of vibration is not a simple process. The fabrication of a piezoid to a given design requires meticulous attention to details and care in every step of the process in order to realize the device as the designer intended.

The inharmonic overtone modes of thickness-shear resonators and the problem which they present to the designer of high-frequency crystal units for filter applications are discussed further in Chap. 10. The several measures which may be taken to minimize their effects including, particularly, contouring and energy trapping are treated in some detail.

In conclusion it may be worthwhile to describe a simple method for investigating the modes of motion of the vibrating piezoid. The piezoid is excited in one of its resonant modes by means of a self-excited oscillator or driven by means of an oscillator, as shown in Fig. 5.6. While it is vibrating, the surface is probed with a sharp probe made from a sliver of polyethylene or some similar material. Probing the surface at nodal points and along nodal lines has little or no effect on the damping, whereas touching the surface at an antinodal point has maximum effect. In this way a map may be made of the vibrating surface showing the nodes and antinodes of the mode of vibration. Much useful information about the nature of the vibration may be obtained in this way.