3
STRESS AND STRAIN RELATIONSHIPS IN QUARTZ

When a solid is subjected to forces which change or tend to change its shape or the distance between elements of the body, the body is said to be subjected to stress. The actual stresses which occur in solids are often quite complex, but it can be shown that the most complex stress is a combination of compressional, tensile, and shear stresses. A compressional stress tends to change the volume of a body. Tension tends to change the length. A shear stress tends to change the shape of the body. We will have no need to consider the effect of pressure, since few piezoelectric devices are produced which operate by subjecting a crystal to hydrostatic pressure. However, both tensile and shear stresses are common in piezoids and a clear understanding of their nature is essential to the design and operation of the quartz crystal unit.

DEFINITIONS OF STRESS AND STRAIN

A stress is defined as a force per unit area and therefore has the dimensions of pounds per square inch, newtons per square meter, dynes per square centimeter, etc. A tensile stress is the value of the tensile force divided by the area to which the force is applied. A shear stress is also a force divided by the area over which it is applied. In the case of a tensile stress the force is applied in a direction perpendicular to the area, while in a shear stress the direction of the force is parallel to the plane of the area. The two types of stresses are illustrated in Figure 3.1,

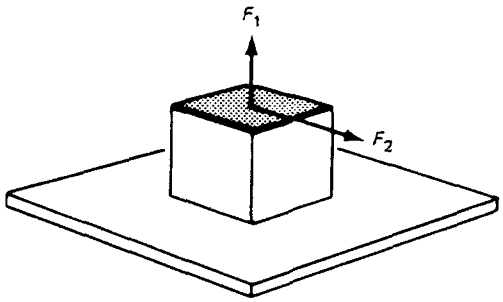

Fig. 3.1. F_1 **produces a longitudinal or tensile stress;** F_2 **produces a shear stress.**

where a solid body in the form of a cube is attached to a fixed surface and has a rigid plate attached to its upper surface. The area of the upper surface and of the rigid plate is A. The force F_1, perpendicular to A, produces a tensile stress F_1/A. The force F_2, parallel to A, produces a shear stress F_2/A.

Strain is the deformation associated with stress. Sometimes it is convenient to consider strain to be the result of a stress, but mathematically, as well as physically, it is quite logical to consider either as the cause and the other as the effect. The measure of a strain is the degree of the deformation. For instance, if the deformation consists of a change of length due to a tensile stress, the strain is the change in length divided by the initial length. The strain is therefore a dimensionless quantity.

In Fig. 3.2*a* the tensile stress F_1/A has resulted in an elongation of the dimension OQ by an amount QQ'. The longitudinal strain is thus given by QQ'/OQ. Some quartz piezoids vibrate in this way.

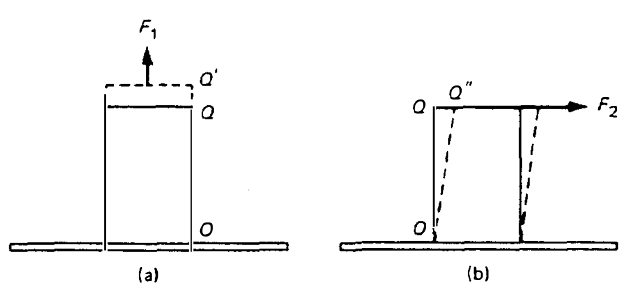

Fig. 3.2. **(*a*) Longitudinal strain and (*b*) shear strain.**

Figure 3.2*b* shows a shear strain. The shear stress F_2/A has produced a displacement QQ'' and a shear strain QQ''/OQ. Shear strains, like longitudinal strains, are very small in quartz, the maximum value being set by the elastic limit.[1] Since the shear displacement QQ'' is very small, it is customary to speak of the angle of shear and to define the angle (in radians) as QQ''/OQ.

Modes of vibration employing shear stresses and strains are used in many quartz piezoids. It is necessary, therefore, to expand upon the concept of shear stress and strain. A complete treatment of this topic is quite involved, but for our purpose it will be sufficient to distinguish between a simple and a pure shear and to note the type of displacement involved in each.

The concept of a simple shear can be understood from Fig. 3.3*a* in which the shaded area $OABC$ represents a square unstrained area. A simple shear strain X_y results in the deformation of $OABC$ into the shape $OAB'C'$. The simple shear Y_x distorts $OABC$ into the shape $OA''B''C$ as shown in *b*. Thus a simple shear strain is produced by holding one side of the square fixed while distorting it into a rhombus. A simple shear can be transformed into a pure shear by a change of coordinates, i.e., the deformation of the body is the same in both

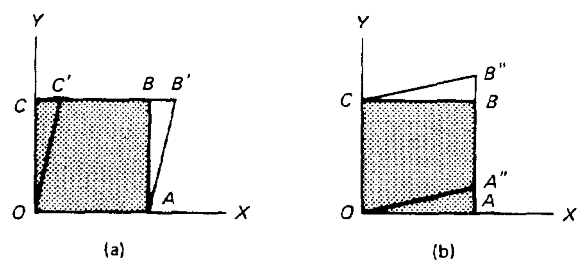

Fig. 3.3. Simple shear strains.

[1] The elastic limit, or breaking strain, in a crystal such as quartz is very hard to determine. The measured elastic limit, or fracture point, is far smaller than the theoretical value in quartz and other high-purity crystals such as silicon and germanium. It is believed that these crystals fail by fracture caused by stress concentration at minute surface cracks. The stress-strain ratio in quartz appears to be linear up to the fracture point but, of course, not necessarily to the true elastic limit. Some phenomena which have been attributed to nonelastic properties of quartz are probably due to other effects.

In any case, the strains in piezoelectric devices used in the communication field should be very much less than the elastic limit.

simple and pure shears. Physically, the difference depends upon the manner in which the body is supported and the forces applied. In the ideal case of a thin square plate subjected to a shear strain in the plane of the plate, the strain may be regarded as a simple shear if the plate is clamped at one edge, as shown in Fig. 3.4*a*. Exactly the same deformation will occur if the plate is supported at its center while equal and opposite forces are applied, as shown in *b*. It should also be apparent that the same deformation can be produced by the application of compressive forces F_2 along one diagonal while extensional forces F_3 are applied along the other diagonal, as shown in *c*.

A pure shear is equivalent to two longitudinal strains at right angles to each other. This can be seen by considering the square *ABCD*, Fig. 3.5, marked on the plane surface S_1. When the surface is distorted by a shear strain in the same plane, the square becomes the rectangle *A'B'C'D'*. In the same way a circle is distorted into an ellipse. The equivalence of a pure shear strain to two extensional strains at right angles is important in the design of certain oscillator plates, especially those designated as *CT* and *DT*. These plates vibrate in such a way that the diagonals (or perpendicular diameters) become alternately longer and shorter. Such a mode of motion is called the *face-shear* mode.

It is customary, in discussing shear strains and stresses, to speak of the axis of stress or strain or of a "shear about a given axis." These terms refer to the direction normal to the plane in which the shear occurs. This is also the axis about which rotation occurs in a simple shear. The axis of a shear strain is analogous mathematically to the direction of an extensional strain.

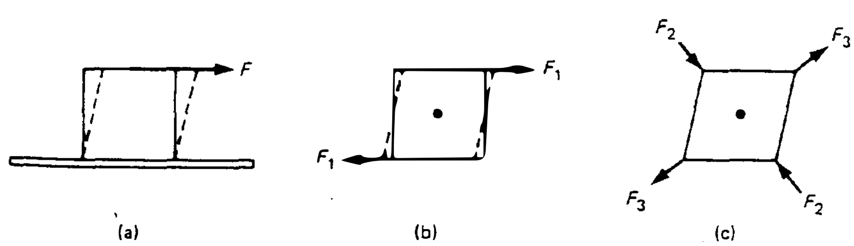

Fig. 3.4. Pure shear strain and its relation to simple shear strain.

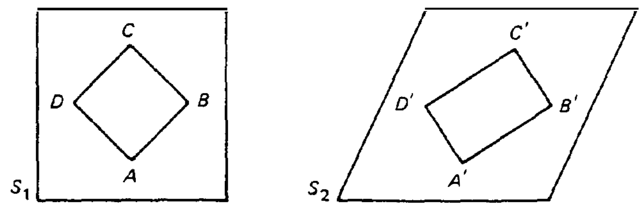

Fig. 3.5. The shear strain is equivalent to two orthogonal extensional strains

DEFINITIONS OF THE STRESS-STRAIN RELATIONSHIPS

According to Hooke's law the strain which a solid body experiences is linearly proportional to the applied stress. Like many of the laws of physics, Hooke's law is an approximation which is valid only under certain conditions. A very well made spring "obeys" Hooke's law over a wide range of strain, but the law eventually fails as the spring reaches its maximum extension. A rubber band does not even approximately obey Hooke's law.

Hooke's law may be used to describe the stress-strain relationship in many solids provided the strains are small enough. If the stresses and strains are small enough that their ratio may be considered to be a constant, independent of the magnitude of the stress, we may define a quantity called the *modulus of elasticity*, or *elastic modulus*. The most-common example is Young's modulus, which is used to describe the elastic behavior of one-dimensional objects such as thin wires. The Young's modulus (YM) of a thin wire is given by

$$\mathrm{YM} = \frac{F/A}{e/L} = \frac{F\,L}{A\,e}$$

where

A = area of cross section of wire
L = initial length
e = extension resulting from the application of force F

The dimensions of a modulus are those of force/area.

The modulus of shear is defined as the ratio of the shear stress, Fig. 3.2*b*, to the resulting strain defined as the angle of shear.

The bulk modulus is defined as the ratio of the hydrostatic pressure divided by the fractional change of volume. We have no need of the bulk modulus in this book but the concept of the shear modulus is most important.

In work in piezoelectricity it is more common to speak of the stress/strain ratio as a *stiffness coefficient*. The inverse ratio, strain/stress, is called a *compliance coefficient*. The term *constant* is often used in place of *coefficient* in both definitions, and the stiffness coefficients are commonly called the *elastic constants*.

THE GENERALIZED STRESS-STRAIN RELATIONSHIPS

For small displacements it can be shown that the most general strain in a solid can be resolved into a linear combination of an extensional and a pure shear strain. Any extensional strain can be resolved into three components parallel to a set of orthogonal axes. Likewise, any shear strain can also be resolved into three shear strains about the same three axes.

It is easily understood that a longitudinal stress in an isotropic solid will produce not only a longitudinal strain parallel to the stress but also lateral strains perpendicular to it. The ratio of the lateral to the longitudinal strain is called Poisson's ratio. It is not conceivable, however, that a tensile stress in an isotropic material should produce any shear strains.

The situation in anisotropic (crystalline) materials is more complex. An extensional stress produces extensional strain and lateral contraction, and shear strains may also exist. For example, if a tensile stress is applied to a thin rod of quartz cut with its length parallel to the X-axis, not only does the rod become longer and thinner, but it also tends to rotate about its longitudinal axis.

It follows therefore that to completely describe the stress-strain relationships in crystalline materials, any component of stress must be considered to be capable of producing any component of strain. Since we have 6 components of stress and 6 of strain, 36 constants must be employed to describe the behavior in the general case. Fortunately, many of the constants are zero and simplifying relations exist among the others so that the number is greatly reduced. In quartz, for example, 18 of the 36 are zero and only 6 of the remainder are independent.

For a crystalline material the linear stress-strain relationship has the form:

$$
\begin{aligned}
-x_x &= s_{11}X_x + s_{12}Y_y + s_{13}Z_z + s_{14}Y_z + s_{15}Z_x + s_{16}X_y \\
-y_y &= s_{21}X_x + s_{22}Y_y + s_{23}Z_z + s_{24}Y_z + s_{25}Z_x + s_{26}X_y \\
-z_z &= s_{31}X_x + s_{32}Y_y + s_{33}Z_z + s_{34}Y_z + s_{35}Z_x + s_{36}X_y \\
-y_z &= s_{41}X_x + s_{42}Y_y + s_{43}Z_z + s_{44}Y_z + s_{45}Z_x + s_{46}X_y \\
-z_x &= s_{51}X_x + s_{52}Y_y + s_{53}Z_z + s_{54}Y_z + s_{55}Z_x + s_{56}X_y \\
-x_y &= s_{61}X_x + s_{62}Y_y + s_{63}Z_z + s_{64}Y_z + s_{65}Z_x + s_{66}X_y
\end{aligned}
\tag{12}
$$

where x_x, y_y, and z_z are the extensional strain components parallel to the X, Y, and Z axes, respectively, and y_z, z_x, and x_y are the shear strain components about the X, Y, and Z axes, respectively. The negative signs result from the convention that a stress which tends to compress a body is called positive, while a decrease in the length is called a negative strain. Therefore, a positive stress is associated with a negative strain.

The terms X_x, Y_y, and Z_z refer to extensional stresses and the terms Y_z, Z_x, and X_y refer to the shear stresses. In a pure shear, as we have seen, X_y is mathematically equivalent to Y_x, thereby reducing the number of stress-strain components from 9 to 6 and the number of terms in the matrix from 81 to 36.

Sometimes it is desirable to express the stresses in terms of the strains. In this case we have the following set of equations:

$$
\begin{aligned}
-X_x &= c_{11}x_x + c_{12}y_y + c_{13}z_z + c_{14}y_z + c_{15}z_x + c_{16}x_y \\
-Y_y &= c_{21}x_x + c_{22}y_y + c_{23}z_z + c_{24}y_z + c_{25}z_x + c_{26}x_y \\
-Z_z &= c_{31}x_x + c_{32}y_y + c_{33}z_z + c_{34}y_z + c_{35}z_x + c_{36}x_y \\
-Y_z &= c_{41}x_x + c_{42}y_y + c_{43}z_z + c_{44}y_z + c_{45}z_x + c_{46}x_y \\
-Z_x &= c_{51}x_x + c_{52}y_y + c_{53}z_z + c_{54}y_z + c_{55}z_x + c_{56}x_y \\
-X_y &= c_{61}x_x + c_{62}y_y + c_{63}z_z + c_{64}y_z + c_{65}z_z + c_{66}x_y
\end{aligned}
\tag{13}
$$

Equations (12) and (13) may be considered the generalized form of Hooke's law for crystalline materials.

In matrix notation the stress-strain relations, Eqs. (12) and (13), may be written, respectively,

$$-x = s\,X \tag{12a}$$

and

$$-X = c\,x \tag{13a}$$

where s is the square matrix

$$s = \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{21} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ s_{31} & s_{32} & s_{33} & s_{34} & s_{35} & s_{36} \\ s_{41} & s_{42} & s_{43} & s_{44} & s_{45} & s_{46} \\ s_{51} & s_{52} & s_{53} & s_{54} & s_{55} & s_{56} \\ s_{61} & s_{62} & s_{63} & s_{64} & s_{65} & s_{66} \end{pmatrix}$$

The c matrix is entirely analogous.

The s's in Eq. (12) are called the *compliance* coefficients and the c's in Eq. (13) are called the *stiffness* coefficients, a choice of notation which obviously was not chosen for its mnemonic value.

The principle of conservation of energy requires that $c_{ij} = c_{ji}$ and that $s_{ij} = s_{ji}$, thus reducing the number of independent coefficients in each set from 36 to 21. The symmetry conditions require that many of the coefficients be zero and that simple relationships exist between others. In a cubic crystal only three independent elastic coefficients exist, but in quartz the number of nonzero, independent coefficients is six.

By performing rotations of the c and s matrices as permitted by the holoaxial, trigonal symmetry of quartz, it may be shown that the compliance matrix reduces to the matrix

$$s = \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} & 0 & 0 \\ s_{12} & s_{11} & s_{13} & -s_{14} & 0 & 0 \\ s_{13} & s_{13} & s_{33} & 0 & 0 & 0 \\ s_{14} & -s_{14} & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{44} & 2s_{14} \\ 0 & 0 & 0 & 0 & 2s_{14} & 2(s_{11} - s_{12}) \end{pmatrix} \tag{12b}$$

The c matrix is found in the same way to be

$$c = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{12} & c_{11} & c_{13} & -c_{14} & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ c_{14} & -c_{14} & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & c_{14} \\ 0 & 0 & 0 & 0 & c_{14} & \frac{1}{2}(c_{11} - c_{12}) \end{pmatrix} \tag{13b}$$

It is necessary to understand clearly the meaning of each term in the s and c matrices. The term s_{11} is the ratio[2] of an extensional strain in the X direction to a tensile stress in the same direction. The terms s_{12} and s_{13} relate a tensile strain in the X direction with extensional stresses in the Y and Z directions, respectively. The ratio s_{12}/s_{11} and the ratio s_{13}/s_{11} are analogous to the quantity known as Poisson's ratio in isotropic solids. The algebraic signs of the various terms must, of course, be appropriate to the conventions used in defining the quantities.

The meaning of s_{14} is somewhat more difficult to visualize. A nonzero value of s_{14} means that a shear stress about the X axis results in an extensional strain in the X direction. Such effects do not occur in isotropic materials but are common in crystals. One may visualize the effect by considering the change of length of a helical spring as it is wound or unwound. The zero values of s_{15} and s_{16}, in quartz, mean that shear stresses about the Y- and Z-axes produce no extensional strain in the X-direction.

In the same manner, s_{21} relates an extensional strain in the Y direction to a tensile stress in the X direction. The numerical value of s_{21} must be equal to that of s_{12} in order to satisfy the law of conservation of energy. The value of s_{22}, which relates the extensional strain in the Y direction to the tensile stress in the Y direction, is equal to s_{11} in quartz. (It may be worth noting that when a subscript ij is a multiple of 11, the stress and strain are parallel. But it must be noted that a subscript in a matrix term should not be read as a number. For instance s_{45} should be read "s four five" and not "s forty-five.")

The remaining coefficients of the s matrix have similar interpretations. The terms in the c matrix are interpreted in the same way. The zero terms are especially worthy of some attention because they tell which stresses and strains are unrelated. Conversely, the nonzero value of $s_{56} = 2s_{14}$ is particularly troublesome, since it shows that a shear stress about the Z axis produces a shear strain about the Y axis. Examples such as this one are termed *mechanical coupling*. The importance of this type of coupling will be discussed in the section on "Activity Dips."

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[2] The coefficient s_{ij} is not a ratio in the mathematical sense, since it is the quotient of a strain by a stress and is therefore not a dimensionless quantity.

Many interesting and important relationships between the coefficients can be observed. For example, $s_{11} = s_{22}$ but not s_{33}. $s_{44} = s_{55}$ but not s_{66}.

It can be seen in a semiquantitative way why $s_{66} = 2(s_{11} - s_{12})$. Consider a Z-cut section of quartz as shown in Fig. 3.6. The Z-axis is perpendicular to the page. The sides of the square $ABCD$ are parallel to the X- and Y-axes. If the quartz is subjected to an X_y strain, the square $ABCD$ is distorted into rhombus $A'B'C'D'$ and the square $MNOP$ is distorted into the rectangle $M'N'O'P'$. The strains are equivalent. The stresses involved in distorting $MNOP$ into a rectangle involve the coefficients s_{11} and s_{12}, which are independent of direction in the XY-plane. The number 2 arises from the fact that the longitudinal strain is referred to the distance MN, while the pure shear strain is measured by the angular rotation of line BC about point N and not about point B.

THERMAL STRESSES AND STRAINS

Strains in a solid such as quartz may be produced by internal as well as external forces. One method by which strain is induced is change of temperature.

A change of temperature produces not only changes of dimensions but also, in certain cases, changes of angle. The thermal stresses can be calculated by imagining the crystal to be clamped in such a way that neither its size nor shape can change. We might, for example, imagine it to be encased in a perfectly rigid casting. The stresses could

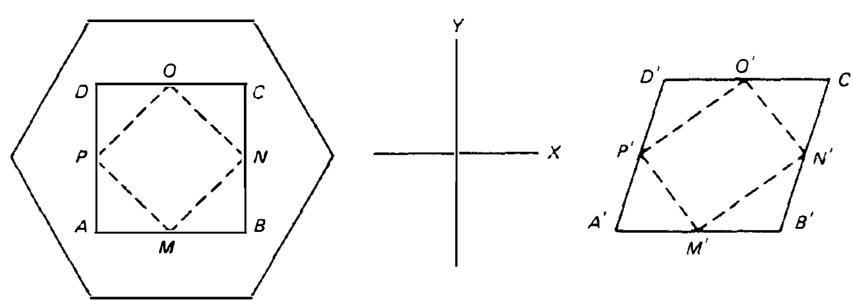

Fig. 3.6. The effect of X_y shear.

be calculated by first calculating the thermal strain and then the mechanical stress required to suppress that strain.

As a very simple one-dimensional example, suppose that the temperature of an X-cut plate is increased 1°. The thickness would increase by 14.3×10^{-6} cm for each centimeter of thickness. Now imagine that a force is applied sufficiently large to compress the blank back to its initial thickness. The elastic modulus c_{11} is 86.05×10^9 N/m². Hence a compressional stress of 86×10^9 N/m² $\times 14.3 \times 10^{-6} = 1.2 \times 10^6$ N/m² would be required to prevent thermal expansion resulting from a temperature increase of 1°. This is equivalent to a pressure of approximately 12 atm.

From considerations of this kind it is clear that extremely large stresses may result if different parts of the crystal are heated to different temperatures. Thermal gradients of only a few tenths of a degree per centimeter can produce stresses great enough to exceed the fracture point and cause the crystal to break. Another effect of stress, whether induced mechanically or thermally or by a combination of the two, is to produce electrical twinning in the quartz crystal.

A quartz crystal must never be heated or cooled rapidly. The larger the crystal, the less rapidly may the temperature be safely changed. The thermal conductivity of quartz is low and strongly anisotropic. Therefore, the time required for heat flow to equalize the temperature throughout the crystal is long. For example, a quartz crystal having a mass of 250 g (0.55 lb) should not be heated or cooled at a rate exceeding a few (say 5) degrees per minute.

Internal stresses and strains can also be produced in quartz by the application of an electric field. This is called the piezoelectric effect which is the subject of the next chapter.

Note that in Table 3.1 opposite signs are given to c_{14} and s_{14} by the two authorities. All older authorities, including Mason, Cady, and Vigoreux have followed Voigt in assigning c_{14} a positive sign.

A list of the equations required for computing the values of the effective elastic and piezoelectric coefficients for a rotation of θ about the X-axis has been given by Sykes[3]. These equations require that c_{14} be assigned a positive sign. For example, the equation for c'_{66},

[3]Raymond A. Heising, *Quartz Crystals for Electrical Circuits.* New York: Van Nostrand, 1946, pp. 247–248.

Table 3.1. The Elastic Coefficients of Alpha Quartz.

	Stiffness			Compliance	
	Bechmann*	Mason†		Bechmann*	Mason†
c_{11}	86.74	86.05 × 10^9 N/m²	s_{11}	12.77	12.79 × 10^{-12} m²/N
c_{12}	6.99	5.05	s_{12}	-1.79	-1.535
c_{13}	11.91	10.45	s_{13}	-1.22	-1.10
c_{14}	-17.91	18.25	s_{14}	+4.50	-4.46
c_{33}	107.2	107.1	s_{33}	9.60	9.56
c_{44}	57.94	58.65	s_{44}	20.04	19.78
c_{66}	39.88	40.5	s_{66}	29.12	28.65
	10^9 N/m² = 10^{10} dyn/cm²			10^{-12} m²/N = 10^{-13} cm²/dyn	

*R. Bechmann, *Phys. Rev.*, **110**, (5): 1060–1061 (June 1958).

†W. P. Mason, *Piezoelectric Crystals and their Applications to Ultrasonics*, New York: Van Nostrand, 1950, p. 84.

which is the effective stiffness coefficient for the AT- and BT-cut piezoids, is

$$c'_{66} = c_{44}\, s^2 - 2c_{14}\, sc + c_{66}\, c^2$$

where $c = \cos\theta$ and $s = \sin\theta$. In order to agree with the observed values, c_{14} must be assigned a positive sign.

However, computation of the rotation matrix for c'_{66} gives

$$c'_{66} = c_{44}\, s^2 + 2c_{14}\, sc + c_{66}\, c^2$$

which requires c_{14} to have a negative sign.

Later authorities including Beckmann et al. and Mindlin et al.[4] assign the minus sign to c_{14} and the plus sign to s_{14}. The origin of the discrepancy is not clear but in this book, following Bechmann, the value of c_{14} is taken to be negative.

[4]R. Bechmann, A. D. Ballato, and T. J. Lukaszek, *Proc. I.R.E.*, 50:1816 (1962); R. D. Mindlin, and D. C. Gazis, *ibid.*