APPENDIX 2

ANALYTICAL EXPRESSIONS FOR THE FREQUENCY-TEMPERATURE CHARACTERISTICS

For practical application, as well as for theoretical consideration, it is necessary to define the frequency-temperature behavior quantitatively, introducing some constants—the temperature coefficients. The measured frequency f of a crystal unit as a function of the temperature T can be developed in a power series in the vicinity of the frequency f_o at the arbitrary temperature T_o:

$$\frac{f - f_o}{f_o} = \frac{\Delta f}{f_o} = a_o(\theta)[T - T_o] + b_o(\theta)[T - T_o]^2 + c_o(\theta)[T - T_o]^3 + \cdots, \tag{A2-1}$$

where $a_o(\theta)$, $b_o(\theta)$, and $c_o(\theta)$ are the first-, second- and third-order temperature coefficients of frequency as defined by

$$a_o(\theta) = \frac{1}{f_o}\left(\frac{\partial f}{\partial T}\right)_o, \quad b_o(\theta) = \frac{1}{2f_o}\left(\frac{\partial^2 f}{\partial T^2}\right)_o, \quad c_o(\theta) = \frac{1}{6f_o}\left(\frac{\partial^3 f}{\partial T^3}\right)_o. \tag{A2-2}$$

These constants are functions of the orientation and the other influences mentioned. The temperature coefficient of the frequency is given by

$$Tf = \frac{1}{f_o}\frac{\partial f}{\partial T} = a_o(\theta) + 2b_o(\theta)[T - T_o] + 3c_o(\theta)[T - T_o]^2. \tag{A2-3}$$

For the $-60°$ to $+100°C$ temperature range usually considered, temperature coefficients of order higher than three can be neglected. The three temperature coefficients can be related to the corresponding coefficients of the elastic constants involved and the coefficients of expansion.

Relating the power series to the temperature T_1 instead of the temperature T_o, we can write Eq. (A2-1) as

$$\frac{f - f_1}{f_1} = \frac{\Delta f}{f_1} = a_1(\theta)[T - T_1] + b_1(\theta)[T - T_1]^2 + c_1(\theta)[T - T_1]^3 + \cdots \tag{A2-4}$$

where the coefficients $a_1(\theta)$, $b_1(\theta)$, and $c_1(\theta)$ are referred to $a_o(\theta)$, $b_o(\theta)$, and $c_o(\theta)$ by the following equations

$$a_1 = \frac{1}{p}[a_o + 2b_o(T_1 - T_o) + 3c_o(T_1 - T_o)^2]$$

$$b_1 = \frac{1}{p}[b_o + 3c_o(T_1 - T_o)] \tag{A2-5}$$

$$c_1 = \frac{1}{p} c_o \qquad \textbf{(A2-5)}$$

$$p = 1 + a_o(T_1 - T_o) + b_o(T_1 - T_o)^2 + c_o(T_1 - T_o)^3.$$

A family of frequency-temperature curves is then given by the following expressions, assuming the change of the three temperature coefficients to be linear with angle of orientation:

$$\frac{\Delta f}{f} = a_o(\theta_o)[T - T_o] + b_o(\theta_o)[T - T_o]^2 + c_o(\theta_o)[T - T_o]^3$$
$$+ \left\{ \frac{\partial a_o(\theta)}{\partial \theta}[T - T_o] + \frac{\partial b_o(\theta)}{\partial \theta}[T - T_o]^2 + \frac{\partial c_o(\theta)}{\partial \theta}[T - T_o]^3 \right\}(\theta - \theta_o), \qquad \text{(A2-6)}$$

where $\partial a_o(\theta)/\partial\theta$, $\partial b_o(\theta)/\partial\theta$, and $\partial c_o(\theta)/\partial\theta$ are the derivatives with respect to the angle of the three temperature coefficients, respectively. The linear terms are sufficient for a 1° range; higher terms for the derivatives of the temperature coefficients must be introduced when considering a wider range of orientation.

In the vicinity of a zero angle of orientation for the frequency, when a_o is zero or very small, two types of frequency-temperature behavior may be distinguished.

a. In case where b_o is rather small and c_o large, the frequency-temperature characteristic has a cubic form. An example is the AT-cut where generally b is smaller than $5 \cdot 10^{-9}/(°C)^2$ and c in the order $100 \cdot 10^{-12}/(°C)^3$. Another example is the GT-cut where both the second- and third-order temperature coefficients are very small.

b. In most of the other cuts, the second-order temperature coefficient is predominant, giving a parabolic frequency-temperature characteristic.

Considering first the frequency-temperature characteristics of an AT-type crystal, a typical frequency-temperature curve for an angle of orientation, giving a small negative value for the first-order temperature coefficient of frequency, is shown in Fig. 40. Some characteristic quantities for the frequency-temperature behavior are: maximum and minimum temperature (T_{max}, T_{min}), together with the corresponding maximum and minimum frequency change ($\Delta f/f_{max}$, $\Delta f/f_{min}$); the inflection temperature (T_i), i.e., the temperature for which the derivative of the temperature coefficient of frequency becomes zero; and the temperatures T_a and T_b, where in the case considered $T_a - T_b = 2(T_{min} - T_{max})$.

The analytical expressions for T_{min} and T_{max} follow from Tf = 0:

$$T_{\substack{min \\ max}} - T_o = \frac{-b_o \pm \sqrt{b_o^2 - 3a_oc_o}}{3c_o}. \qquad \text{(A2-7)}$$

The corresponding frequency deviation is given by

$$\frac{(\Delta f)}{f_{\substack{max\\min}}} = \frac{\pm 2\sqrt{b_o^2 - 3a_oc_o}^{\,3} + 2b_o^3 - 9a_ob_oc_o}{27c_o^2}. \qquad \text{(A2-8)}$$

From this follows the expressions for the difference between the maximum and minimum temperatures

$$T_{min} - T_{max} = \frac{2\sqrt{b_o^2 - 3a_oc_o}}{3c_o}, \qquad \text{(A2-9)}$$

and the corresponding total frequency change

$$\left(\frac{\Delta f}{f}\right)_{max} - \left(\frac{\Delta f}{f}\right)_{min} = \frac{4}{27}\,\frac{\sqrt{b_o^2 - 3a_oc_o}^{\,3}}{c_o^2}. \qquad \text{(A2-10)}$$

The inflection temperature T_i is defined by $\frac{\partial Tf}{\partial T} = \frac{\partial^2 f}{\partial T^2} = 0$, hence

$$T_i - T_o = -\frac{b_o}{3c_o}. \qquad \text{(A2-11)}$$

The following equations are of practical interest:

$$\frac{\left(\frac{\Delta f}{f}\right)_{max} - \left(\frac{\Delta f}{f}\right)_{min}}{(T_{min} - T_{max})^3} = \frac{c_o}{2}, \qquad \text{(A2-12)}$$

and

$$\frac{\left(\frac{\Delta f}{f}\right)_{max} - \left(\frac{\Delta f}{f}\right)_{min}}{T_{min} - T_{max}} = \frac{2}{9}\cdot\frac{b_o^2 - 3a_oc_o}{c_o}. \qquad \text{(A2-13)}$$

Introducing the inflection temperature T_i as reference temperature instead of T_o, where $T_i - T_o = -b_o/3c_o$, then $b_i = 0$ and Eq. (A2-1) simplifies to

$$\frac{f - f_i}{f_i} = \frac{\Delta f}{f} = a_i(T - T_i) + c_i(T - T_i)^3, \qquad \text{(A2-14)}$$

where

$$
\begin{aligned}
a_i &= \frac{1}{p}\,\frac{3a_o c_o - b_o^2}{3c_o} \\
c_i &= \frac{1}{p}\,c_o \qquad\qquad \text{(A2-15)} \\
p &= 1 + \frac{2\,b_o^3 - 9a_o b_o c_o}{27c_o^2}\,.
\end{aligned}
$$

Equations (A2-7) and (A2-8) then simplify to

$$T_{\substack{min\\max}} - T_i = \pm\sqrt{\frac{-\,a_i}{3c_i}}\,, \qquad \text{(A2-16)}$$

and

$$\left(\frac{\Delta f}{f}\right)_{\substack{max\\min}} = \pm\,2\sqrt{\frac{-\,a_i^3}{27c_i}}\,. \qquad \text{(A2-17)}$$

For Equations (A2-9) and (A2-10), we obtain

$$T_{min} - T_{max} = 3\sqrt{\frac{-\,a_i}{3c_i}}\,, \qquad \text{(A2-18)}$$

and

$$\left(\frac{\Delta f}{f}\right)_{max} - \left(\frac{\Delta f}{f}\right)_{min} = 4\sqrt{\frac{-\,a_i^3}{27c_i}}\,. \qquad \text{(A2-19)}$$

Equations (A2-12) and (A2-13) simplify to

$$\frac{\left(\frac{\Delta f}{f}\right)_{max} - \left(\frac{\Delta f}{f}\right)_{min}}{\left(T_{min} - T_{max}\right)^3} = \frac{c_i}{2}\ ; \quad \frac{\left(\frac{\Delta f}{f}\right)_{max} - \left(\frac{\Delta f}{f}\right)_{min}}{T_{min} - T_{max}} = -\frac{2}{3}\,a_i\,. \qquad \text{(A2-20)}$$

The above formulae describe the frequency-temperature behavior of an AT cut with sufficient accuracy in the temperature range considered.

For the AT-type resonator, there is one special angle of orientation θ_J for which for the inflection temperature T_J, the first-order temperature coefficient a_J is zero, so that for this special orientation the frequency-temperature characteristic is represented by a single term

$$\frac{\Delta f}{f} = c_J(\theta_J)\,[T - T_J]^3\,, \quad a_J(\theta_J) = 0\,, \quad b_J(\theta_J) = 0\,. \qquad \text{(A2-21)}$$

This equation may be called the main frequency-temperature characteristic.

In case of the BT-type quartz resonator and similar cuts with a parabolic frequency-temperature characteristic, the relations given above hold. In the temperature range considered, only one maximum of the frequency occurs. Instead of the inflection temperature T_i, the temperature T_{max} at which the frequency maximum occurs is a significant temperature.

The frequency-temperature characteristic related to its maximum is given by

$$\frac{\Delta f}{f} = b_m(\theta) [T - T_{max}]^2 + c_m(\theta) [T - T_{max}]^3 . \tag{A2-22}$$

From Equation (A2-5), we obtain

$$\begin{aligned} a_m &= \frac{1}{p} [a_o + 2b_o (T_{max} - T_o) + 3c_o T_{max} - T_o)^2] = 0 \\ b_m &= \sqrt{b_o^2 - 3a_o c_o} \\ c_m &= c_o . \end{aligned} \tag{A2-23}$$

Further, from Equation (A2-7),

$$\begin{aligned} T_{max} - T_o &= \frac{b_m - b_o}{3c_o} \\ &= -\frac{a_o}{b_o + b_m} \approx -\frac{a_o}{2b_o} . \end{aligned} \tag{A2-24}$$

T_{max} as a function of the angle of orientation of the plate can be obtained from the derivatives of the temperature coefficients $a_o(\theta)$, $b_o(\theta)$, and $c_o(\theta)$ or from the derivatives of the temperature coefficients $b_m(\theta)$ and $c_m(\theta)$, with respect to the angle of orientation.