Appendix 6

Wave propagation in isotropic plates

A6.1 FIELD EQUATIONS AND CONSTITUTIVE RELATIONS

In an isotropic material there is of necessity no piezoelectricity. Hence the field equations and constitutive relations of the linear theory set out in Appendix 4 reduce (in the absence of body forces) to

$$\begin{aligned} t_{kl,l} &= \rho \ddot{u}_k \\ t_{kl} &= c_{klmn} S_{mn} \\ S_{mn} &= (u_{m,n} + u_{n,m})/2 \end{aligned} \tag{A6.1}$$

where t_{kl} and S_{kl} are the stress and strain tensors, u_k is the mechanical displacement, ρ is the density and the c_{klmn} are the elastic constants. Since in an isotropic material all directions and coordinate systems are equivalent, every coordinate transformation is equivalent to a symmetry operation in the sense of Appendix 5. Therefore the elastic constants must satisfy the identities

$$c_{ikmp} = a_{ij} a_{kl} a_{mn} a_{pq} c_{jlnq}$$

for all orthogonal transformations a_{kl}, which implies that, using the reduced matrix notation of Appendix 5, the matrix of elastic constants must have the simple form

$$\begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{12} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{44} \end{bmatrix}$$

where the additional relation

$$c_{11} = c_{12} + 2c_{44} \tag{A6.2}$$

must also hold.

A6.2 PLANE WAVE SOLUTIONS

As all directions are equivalent in an isotropic material, in considering the propagation of plane waves there is no loss of generality in assuming that the propagation direction is along the x_2 axis. Then in the notation of Section 2.1, the wave normal n_k has components (0,1,0). Taking account of the effect of isotropy on the elastic constants, it then follows from Eqn (2.11) that the tensor L_{km} reduces to diagonal form with $L_{11} = c_{44} = L_{33}$ and $L_{22} = c_{11}$. Consequently Eqn (2.9) separates into three uncoupled wave equations each involving a single displacement component

$$\begin{aligned} c_{44}u_{1,22} &= \rho\ddot{u}_1 \\ c_{11}u_{2,22} &= \rho\ddot{u}_2 \\ c_{44}u_{3,22} &= \rho\ddot{u}_3 \end{aligned} \tag{A6.3}$$

The first and third of these represent transverse or shear waves propagating along x_2 with a phase velocity V_s, whereas the second represents a longitudinal or *dilatational* wave with velocity V_d, where

$$V_s = (c_{44}/\rho)^{1/2} \qquad V_d = (c_{11}/\rho)^{1/2} \tag{A6.4}$$

Since from Eqn (A6.2) $c_{11} > c_{44}$ for positive c_{12}, the longitudinal wave velocity V_d is greater than the shear wave velocity V_s.

In each of the three waves described by Eqns (A6.3), there is only a single non-zero strain, S_6, S_2 and S_4, respectively. In the transverse waves, the corresponding non-zero stresses are T_6 and T_4, respectively, while in the longitudinal wave, all three extensional stress components T_1, T_2 and T_3 are non-zero but there are no shear stresses.

A6.3 BOUNDARY CONDITIONS FOR A PLATE

Consider a plate in the XZ plane with major surfaces at $y = x_2 = \pm h$, so that the plate thickness is $2h$. If the major surfaces are stress free, then the appropriate boundary conditions are that $t_{2k} = 0$ when $y = \pm h$, or in the reduced notation, that $T_6 = T_2 = T_4 = 0$ on the major surfaces. Because of the form of the elastic constant matrix, the vanishing of the shear stresses T_6 and T_4 implies the vanishing of the corresponding shear strains S_6 and S_4, while the vanishing of T_2 implies

$$c_{12}S_1 + c_{11}S_2 + c_{12}S_3 = 0$$

In terms of the displacements $u = u_1$, $v = u_2$ and $w = u_3$, the boundary conditions may be written

$$v_{,z} + w_{,y} = 0$$

$$u_{,y} + v_{,x} = 0 \qquad \text{(A6.5)}$$
$$c_{11}v_{,y} + c_{12}(u_{,x} + w_{,z}) = 0$$

where x, y, z has been written for x_1, x_2, x_3 and Eqns (A6.5) hold at both major surfaces, $y = \pm h$.

A6.4 THICKNESS MODES

As discussed in Section 2.2, pure thickness modes in plates can be regarded as standing wave systems resulting from the reflection at the major surfaces of plane waves propagating along the thickness. In the isotropic case, the analysis is very simple compared to the anisotropic piezoelectric case discussed in Chapter 2. There are just two families of thickness modes, one resulting from the reflections of transverse waves and termed the *thickness shear* or TS family, the other resulting from reflections of longitudinal waves and termed the *thickness extensional* or TE family.

In the TS case, the appropriate solution of the first of Eqns (A6.3) is, assuming a harmonic time dependence and writing u for u_1, y for x_2

$$u = [A\sin(ky) + B\cos(ky)]\exp(j\omega t) \qquad \text{(A6.6)}$$

where k is the wave number along the thickness, and A, B are constants to be chosen to satisfy the boundary conditions. In order for Eqn (A6.6) to satisfy the wave equation, k and ω must satisfy

$$k^2 = \omega^2/V_s^2 \qquad \text{(A6.7)}$$

The appropriate boundary condition is the second of Eqns (A6.5), which in this case reduces to $u_{,y} = 0$ at $y = \pm h$. Hence

$$kA\cos(kh) - kB\sin(kh) = 0$$
$$kA\cos(kh) + kB\sin(kh) = 0$$

Adding and subtracting leads to two sets of conditions for symmetric modes and anti-symmetric modes:

$$\text{Symmetric case: } A = 0, \quad \sin(kh) = 0 \qquad \text{(A6.8)}$$

$$\text{Anti-symmetric: } B = 0, \quad \cos(kh) = 0 \qquad \text{(A6.9)}$$

The conditions on (kh) are equivalent to the single requirement that $\sin(2kh) = 0$, in turn equivalent to

$$k = N\pi/2h \qquad \text{(A6.10)}$$

with N integral. This then leads through Eqn (A6.7) to the frequency equation

$$\omega = (N\pi/2h)V_s \qquad \text{(A6.11)}$$

which is essentially Eqn (2.42). In the frequency equation, odd N values correspond to the anti-symmetric modes, even N values to the symmetric modes.

Precisely the same analysis holds for the TE case provided that the displacement v and phase velocity V_d are used in place of u and V_s. The frequency equation for TE modes is then

$$\omega = (N\pi/2h)V_d \tag{A6.12}$$

A6.5 THICKNESS TWIST (TT) AND FACE SHEAR (FS) WAVES

Consider a transverse wave with particle displacement u along x, propagating in a direction in the YZ plane. If the wave number is k and the unit vector along the direction of propagation has the components $(0, n_y, n_z)$, then for an angular frequency ω an appropriate form for u is

$$u = A\exp\{j[\omega t - k(n_y y + n_z z)]\}$$

where $k^2 = \omega^2/V_s^2$ as before.

A similar wave propagating in the direction $(0, -n_y, n_z)$ would then have the corresponding form

$$u = B\exp\{j[\omega t - k(-n_y y + n_z z)]\}$$

Superimposing these two waves then gives rise to a displacement u

$$u = [A\exp(-jkn_y y) + B\exp(+jkn_y y)]\exp(j\omega t - jkn_z z)$$

or equivalently

$$u = [A'\sin(kn_y y) + B'\cos(kn_y y)]\exp(j\omega t - jkn_z z) \tag{A6.13}$$

This represents a travelling wave in the z direction, with wave number $k_z = kn_z$, a corresponding phase velocity $V = \omega/k_z$, and an amplitude varying with y. Since u depends on both y and z, the non-zero strains are the shear strains S_5 and S_6 and the non-zero stresses the shear stresses T_5 and T_6. From Section A6.3, if $u_{,y}$ can be made to vanish at $y = \pm h$, then Eqn (A6.13) represents a plate wave solution. But the y dependence of u in Eqn (6.13) is just that of u in Eqn (A6.6) provided k is replaced by kn_y, so that the boundary conditions will be satisfied if

$$k_y = kn_y = N\pi/2h \tag{A6.14}$$

The *dispersion relation* Eqn (A6.7) then becomes

$$k^2 = k_y^2 + k_z^2 = \omega^2/V_s^2$$

or

$$w^2/V_s^2 = (N\pi/2h)^2 + k_z^2 \tag{A6.15}$$

where as in the thickness mode case, even N corresponds to modes symmetric in y, odd N to modes anti-symmetric in y.

This can be put into a normalized form by multiplying by $(2h/\pi)^2$ and defining

$$\zeta = 2hk_z/\pi$$
$$\Omega = \omega/\omega_1 \tag{A6.16}$$
$$\omega_1 = (\pi/2h)V_s$$

The normalized dispersion relation is then

$$\Omega^2 = \zeta^2 + N^2 \tag{A6.17}$$

The character of the waves described by Eqn (A6.13) is quite different for the case $N = 0$ and $N > 0$. In the latter case, the particle motion in the wave is, as illustrated in Fig. A6.1, a twisting motion along the direction of propagation: hence these waves are called *thickness twist* or TT waves. In the case $N = 0$, the displacement u is independent of y, and the deformation is a shear in the plane of the plate. Thus the wave is termed a *face shear* or FS wave. Apart from these differences in the particle displacements, the crucial difference between the TT and FS waves lies in the existence of a cut-off frequency for each TT wave.

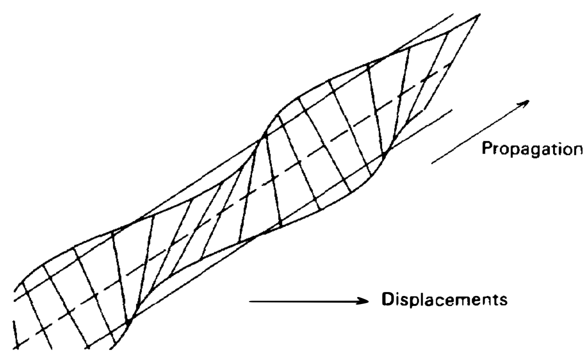

Fig. A6.1 Particle displacements for TT waves.

Analytically, it follows from the normalized dispersion relation that for $N > 0$, the normalized wavenumber ζ is imaginary for all frequencies such that $\Omega < N$. This implies an imaginary k_z, which on substitution into Eqn (A6.13) shows that u decays exponentially with z. Thus for normalized frequencies $\Omega < N$, Eqn (A6.13) represents an *evanescent* rather than a travelling wave. The cut-off frequencies are just the thickness mode frequencies, and in terms of the physical picture of the plate wave as being built up by successive reflections of plane waves at the surfaces of the plate, cut-off corresponds to the limiting case when the plane waves are normally incident on the plate surfaces. As discussed in Chapter 3, as this condition is approached, the phase velocity of the plate wave tends to infinity, while its group velocity tends to zero. The phenomenon of cut-off in elastic wave propagation is closely analogous to the cut-off phenomenon in electromagnetic waveguides.

In the special case $N = 0$, the displacement u depends only on z, so the only strain generated is S_5 and the only stress T_5. But T_5 does not contribute to the surface tractions t_{2k} so that in this case the boundary conditions for the plate are trivially satisfied. Then Eqn (A6.13) actually represents a plane wave solution, and the plate wave velocity coincides with the bulk shear wave velocity.

Figure A6.2 shows a plot of the branches of the dispersion relation Eqn (A6.17) for both real and imaginary k_z (ζ). In the imaginary case, is set equal to $j\bar{\zeta}$, and $\bar{\zeta}$ is plotted to the left of the frequency axis in place of negative ζ. It should be noted that the slope of the TT branches at $\zeta = 0$ is zero, corresponding to the limiting value of zero for the group velocity of these branches.

A6.6 THE RAYLEIGH FREQUENCY EQUATIONS FOR AN ISOTROPIC PLATE

In the previous section, TT waves were built up by combining two bulk plane waves, each with particle displacements in the plane of the plate. If a transverse wave propagating in the XY plane is considered, say along the direction $(n_x, n_y, 0)$, and if the particle displacement is also in the XY plane, then the displacement will have components u and v along both x and y. If the

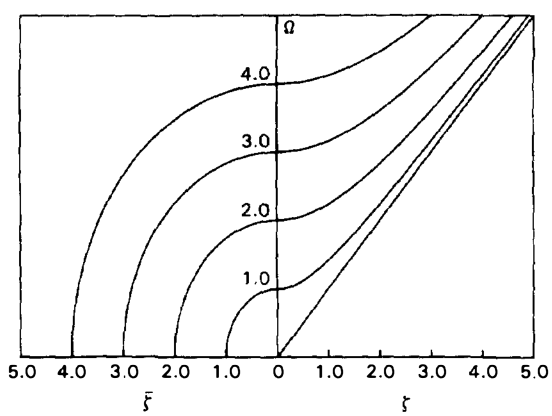

Fig. A6.2 Dispersion relations for TT and FS waves.

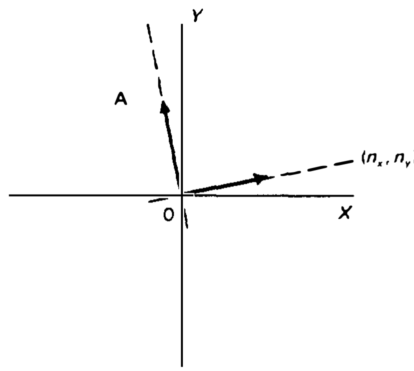

Fig. A6.3 Transverse wave in *OXY* plane.

amplitude of the wave is A, then from Fig. A6.3 the amplitudes of the u and v components will be respectively $-An_y$ and An_x. If k is the wavenumber, then the general expression for the displacements will be

$$
\begin{aligned}
u &= -An_y\exp[j\omega t - jk(n_x x + n_y y)] \\
v &= An_x\exp[j\omega t - jk(n_x x + n_y y)]
\end{aligned}
\tag{A6.18}
$$

where, because of the transverse nature of the wave, the dispersion relation $k^2 = \omega^2/V_s^2$ holds.

Just as in the previous case of TT waves, a second transverse wave of amplitude B and travelling in the direction $(n_x, -n_y, 0)$ can be superimposed on the first to give a resultant wave with displacements of the form

$$
\begin{aligned}
u &= n_y\{-A\exp(-jkn_y y) + B\exp(+jkn_y y)\}\exp(j\omega t - jk_x x) \\
v &= n_x\{+A\exp(-jkn_y y) + B\exp(+jkn_y y)\}\exp(j\omega t - jk_x x)
\end{aligned}
\tag{A6.19}
$$

where $k_x = kn_x$. As before, this is in the form of a travelling wave along the plate, in this case in the x direction, with wavenumber k_x and phase velocity $V = \omega/k_x$, and with an amplitude depending on y.

Two adjustable constants A and B are available to match the boundary conditions. However, whereas in the TT case the boundary conditions reduced to the vanishing of $u_{,y}$ at $y = \pm h$, in this case with both u and v depending on x and y, the conditions to be satisfied are, from Eqns (A6.5)

$$
\begin{aligned}
&u_{,y} + v_{,x} = 0 \\
&c_{11}v_{,y} + c_{12}u_{,x} = 0
\end{aligned}
\tag{A6.20}
$$

Since these have to be satisfied at $y = \pm h$, there are four conditions to satisfy with only two adjustable parameters. The extra parameters required are obtained by also considering a combination of longitudinal waves. Consider then a longitudinal wave, amplitude C, propagating along the direction

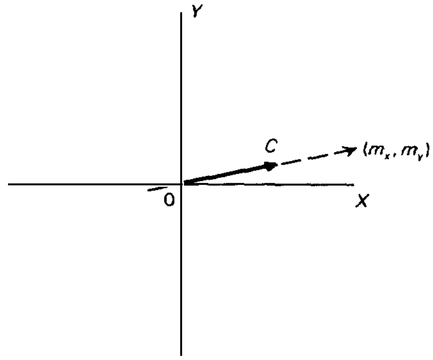

Fig. A6.4 Longitudinal wave in OXY plane.

$(m_x, m_y, 0)$ with wavenumber l. From Fig. A6.4, the u and v displacements will respectively have amplitudes Cm_x and Cm_y, so that

$$\begin{aligned} u &= Cm_x \exp[j\omega t - jl(m_x x + m_y y)] \\ v &= Cm_y \exp[j\omega t - jl(m_x x + m_y y)] \end{aligned} \tag{A6.21}$$

where $l^2 = \omega^2/V_d^2$. Superimposing a second similar wave with amplitude D and direction $(m_x, -m_y, 0)$ then gives in analogy to Eqn (A6.19)

$$\begin{aligned} u &= m_x\{C\exp(-jlm_y y) + D\exp(+jlm_y y)\}\exp(j\omega t - jl_x x) \\ v &= m_y\{C\exp(-jlm_y y) - D\exp(+jlm_y y)\}\exp(j\omega t - jl_x x) \end{aligned} \tag{A6.22}$$

where $l_x = lm_x$.

In order for these expressions to represent a travelling wave along x with the same phase velocity as that represented by Eqns (A6.19), it is necessary to require that the wavenumbers along x match, that is

$$k_x = kn_x = l_x = lm_x \tag{A6.23}$$

Since l and k are known in terms of the frequency and the bulk wave velocities V_d and V_s, this determines $k_y = kn_y$ and $l_y = lm_y$ via

$$\begin{aligned} k_x^2 + k_y^2 &= \omega^2/V_s^2 \\ k_x^2 + l_y^2 &= \omega^2/V_d^2 \end{aligned} \tag{A6.24}$$

If now the sum of the expressions in Eqns (A6.19) and (A6.22) is substituted into the boundary conditions Eqns (A6.20), a set of four homogeneous linear equations in the unknowns A, B, C and D result. The necessary condition for non-trivial solutions to exist is that the determinant of the coefficients of this system of equations vanishes; the vanishing of the determinant in turn leads to a transcendental equation for the wave numbers k_y, l_y which is analogous to the simple trigonometric condition $\sin(2k_y h) = 0$ in the TT case.

The set of four simultaneous equations can be reduced to two independent pairs by the device of adding and subtracting the boundary conditions that apply at $y = h$ and at $y = -h$. This eliminates in turn the anti-symmetric and the symmetric parts of the solutions, and by straightforward but tedious algebra leads to the equations

$$\begin{aligned}(A + B)\cos(k_y h)(k_y n_y - k_x n_x) &= (C - D)\cos(l_y h)(l_y m_x + k_x m_y)\\ (A + B)\sin(k_y h)(c_{12}k_x n_y - c_{11}k_y n_x) &= (C - D)\sin(l_y h)(c_{12}k_x m_x + c_{11}l_y m_y)\end{aligned} \tag{A6.25}$$

and

$$\begin{aligned}(A - B)\sin(k_y h)(k_y n_y - k_x n_x) &= (C + D)\sin(l_y h)(l_y m_x + k_x m_y)\\ (A - B)\cos(k_y h)(c_{12}k_x n_y - c_{11}k_y n_x) &= (C + D)\cos(l_y h)(c_{12}k_x m_x + c_{11}l_y m_y)\end{aligned} \tag{A6.26}$$

To exploit the symmetry of the solutions, first set $A = B$ and $D = -C$, so that Eqns (A6.26) are satisfied trivially, and Eqns (A6.25) reduce to a pair of equations for A and C. The solutions for u and v then become

$$\begin{aligned}u &= \{2jn_y A\sin(k_y y) - 2jm_x C\sin(l_y y)\}\exp(j\omega t - jk_x x)\\ v &= \{2n_x A\cos(k_y y) + 2m_y C\cos(l_y y)\}\exp(j\omega t - jk_x x)\end{aligned} \tag{A6.27}$$

with amplitude ratios determined from Eqns (A6.25), and the transcendental equation that follows from setting the determinant of coefficients in Eqns (A6.25) to zero

$$\tan(l_y h)/\tan(k_y h) = -4k_y l_y k_x^2/(k_y^2 - k_x^2)^2 \tag{A6.28}$$

This family of solutions is termed the *anti-symmetric* family of motions of the isotropic plate. The symmetric family is obtained by setting $B = -A$ and $C = D$ and using Eqns (A6.26) to determine the ratio C/A. The condition for non-trivial solutions differs from Eqn (A6.28) only in the reversal of the two tangent functions,

$$\tan(k_y h)/\tan(l_y h) = -4k_y l_y k_x^2/(k_y^2 - k_x^2)^2 \tag{A6.29}$$

and the displacements take the form

$$\begin{aligned}u &= \{-2n_y A\cos(k_y y) + 2m_x C\cos(l_y y)\}\exp(j\omega t - jk_x x)\\ v &= \{-2jn_x A\sin(k_y y) - 2jm_y C\sin(l_y y)\}\exp(j\omega t - jk_x x)\end{aligned} \tag{A6.30}$$

The results expressed by Eqns (A6.28) and (A6.29) were first obtained by Rayleigh (1889) and are known as the Rayleigh frequency equations. The preceding analysis provides the basic framework for a discussion of the various types of wave that can propagate in an isotropic plate, subject only to the restriction that the particle displacement be in the plane defined by the direction of propagation and the plate normal. Unfortunately, the complex form of the frequency equations and the resulting large variety of solutions make anything approaching a complete discussion out of the question. A more detailed treatment, along with references to the large literature on this subject, may be found in Tiersten (1969). For present purposes, a brief discussion of the anti-symmetric family of solutions that contain as limiting cases the shear thickness modes will suffice.

A6.7 ANTI-SYMMETRIC SOLUTIONS: THICKNESS SHEAR (TS) AND FLEXURAL (F) WAVES

As in the discussion of TT modes, it is convenient to work in terms of normalized quantities, with wavenumbers expressed in terms of the number of half wavelengths that would be contained in a distance equal to the thickness of the plate, and frequencies in terms of the fundamental thickness shear frequency. Define

$$\begin{aligned} \omega_1 &= \pi V_s/2h \\ \Omega &= \omega/\omega_1 \\ \xi &= 2hk_x/\pi \\ \eta_{(1)} &= 2hk_y/\pi \\ \eta_{(2)} &= 2hl_y/\pi \end{aligned} \tag{A6.31}$$

Then Eqns (A6.24) can be written

$$\begin{aligned} \xi^2 + \eta_{(1)}^2 &= \Omega^2 \\ \xi^2 + \eta_{(2)}^2 &= \Omega^2(V_s/V_d)^2 \end{aligned} \tag{A6.32}$$

and the frequency equation (A6.28) becomes

$$\tan(\pi\eta_{(2)}/2)/\tan(\pi\eta_{(1)}/2) = -4\eta_{(1)}\eta_{(2)}\xi^2/(\eta_{(1)}^2 - \xi^2)^2 \tag{A6.33}$$

The general problem is then to determine the relationship between the normalized frequency Ω and the normalized wavenumber ξ in a manner analogous to the determination of the simple Ω versus ζ relationship for TT modes shown in Fig. A6.2. In principle this can be done by substituting for $\eta_{(1)}$ and $\eta_{(2)}$ in Eqn (A6.33) using eqns (A6.32) and solving the resultant equation for Ω in terms of ξ. However, in practice this is far from straightforward.

For present purposes, the complete spectrum is not in any case required. On physical grounds, since the thickness modes already give a reasonable insight into resonator behaviour, it is to be expected that taking into account only slight departures from the pure thickness modes will add considerably to the accuracy of the analysis. Of major interest then is the behaviour of those

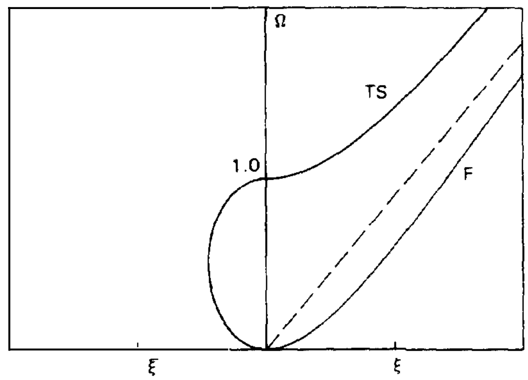

Fig. A6.5 Dispersion relations for TS and F waves.

branches of the spectrum that tend to pure thickness shear modes as the wavenumber ξ tends to zero. In addition, some knowledge of the lowest, flexural mode, branch of the dispersion relation provides an insight into the unwanted coupling of the thickness shear modes in resonators with flexure modes.

Sufficient information for a qualitative understanding of this latter point can be obtained by considering the behaviour of the frequency equation in the limit of both small wavenumbers and low frequencies. This yields the classical dispersion relation for flexural waves in thin plates. This can then be matched to the asymptotic behaviour when both frequency and wavenumber are large, it turning out that the asymptotic phase velocity in this limit is just that of Rayleigh surface waves (Tiersten, 1969). This process is discussed in detail by Tiersten, and the shape of the resulting flexural mode branch is shown in Fig. A6.5.

In the case of the thickness shear modes, the approximate analysis begins by noting that the appropriate thickness mode frequencies do follow from Eqns (A6.32) and (A6.33) in the limit $\xi = 0$. For $\xi = 0$, either $\sin(\pi\eta_{(2)}/2) = 0$ or $\cos(\pi\eta_{(1)}/2) = 0$. The former choice corresponds to TS modes, the latter to TS modes, and adopting this option leads to the conditions

$$\eta_{(1)} = \Omega = N, \qquad N = 1, 3, 5, \ldots .$$

just as in the previous analysis of thickness modes. Considering now modes that only differ marginally from pure thickness modes, set

$$\begin{aligned} \Omega &= N(1 + \epsilon) \\ \eta_{(1)} &= N(1 + \delta) \end{aligned} \tag{A6.34}$$

where both ϵ and δ are $\ll 1$. Then from the first of Eqns (A6.32), to first order in the small quantities

$$\delta = \epsilon - \xi^2/2N^2 \tag{A6.35}$$

From the second of Eqns (A6.32), again to a first approximation

$$\eta_{(2)} = \kappa N \tag{A6.36}$$

where κ has been written for the ratio of shear and dilatational bulk wave velocities

$$\kappa = V_s/V_d \tag{A6.37}$$

With the aid of these approximations, and making use of the expansion

$$\cot(\pi N(1 + \delta)/2) = -\tan(\pi N\delta/2) \sim -\pi N\delta/2 \tag{A6.38}$$

the frequency equation (A6.33) takes on the approximate form

$$\epsilon = \{1 + 16\kappa \cot(\pi\kappa N/2)/\pi N\}\, \xi^2/2N^2 \tag{A6.39}$$

This allows the shape of the thickness shear branches near the axis to be sketched for both real and imaginary values of ξ. The appropriate portions of the dispersion curves are shown in Fig. A6.5, along with the lowest flexural mode branch already discussed. It should be noted that as required from physical considerations, the slope of the curves vanishes at $\xi = 0$, corresponding to zero group velocity at the cut-off frequencies.