Appendix 3
Continuum mechanics

A3.1 DEFORMATION AND STRAIN

Let B be a body that occupies a volume V_0 bounded by a surface S_0 at a time t_0. Let X be a point of B which at time t_0 has coordinates X_K with respect to a rectangular cartesian coordinate system Ox_K. Then the X_K can be regarded as labelling the point X. A motion of the body B can be described by giving the coordinates x_k of X at time $t > t_0$ as functions f_k of X_K and t

$$x_k = f_k(X_K, t) \tag{A3.1}$$

It is assumed that the functions f_k can be inverted to give an alternative description of the motion in the form

$$X_K = F_K(x_k, t) \tag{A3.2}$$

The necessary and sufficient condition for the inverse functions F_K to exist is that the Jacobian determinant $J = \det\{\partial f_k/\partial X_K\}$ be non-zero in V_0.

From Eqn (A3.1) follows

$$\mathrm{d}x_k = (\partial f_k/\partial X_K)\,\mathrm{d}X_K = x_{k,K}\,\mathrm{d}X_K \tag{A3.3}$$

where the notation $x_{k,K} = \partial f_k/\partial X_K$ is used. Similarly $X_{K,k}$ may be written for $\partial F_K/\partial x_k$, and it then follows by the chain rule for differentiation that

$$x_{k,K}X_{K,l} = \delta_{kl} \qquad \text{and} \qquad X_{K,k}x_{k,L} = \delta_{KL} \tag{A3.4}$$

Taking determinants in Eqn (A3.4) leads to

$$\det\{x_{k,K}\}\det\{X_{K,l}\} = 1$$

so that the non-vanishing of the Jacobian $\mathrm{J} = \det\{x_{k,K}\}$ implies the non-vanishing of the Jacobian $\mathrm{J}^{-1} = \det\{X_{K,k}\}$ of the inverse functions F_K in Eqn (A3.2).

Equation (A3.3) shows how an elementary vector $\mathrm{d}X_K$ joining neighbouring points of B is transformed into a vector $\mathrm{d}x_k$ by the motion. If $\mathrm{d}L$ and $\mathrm{d}l$ are the magnitudes of $\mathrm{d}X_K$ and $\mathrm{d}x_k$, respectively, then $\mathrm{d}L^2 = \mathrm{d}X_K\mathrm{d}X_K$ and $\mathrm{d}l^2 = \mathrm{d}x_k\mathrm{d}x_k$ and by Eqn (A3.3)

$$\mathrm{d}l^2 = x_{k,K}x_{k,L}\,\mathrm{d}X_K\mathrm{d}X_L = C_{KL}\,\mathrm{d}X_K\mathrm{d}X_L \tag{A3.5}$$

where

$$C_{KL} = x_{k,K}x_{k,L} \tag{A3.6}$$

The C_{KL} form the components of a symmetric second rank tensor known as *Green's deformation tensor*. In a motion such that $C_{KL} = \delta_{KL}$ the distance between pairs of neighbouring points does not change, since

$$dl^2 = C_{KL}dX_K dX_L = \delta_{KL}dX_K dX_L = dL^2$$

Hence B moves as a rigid body when $C_{KL} = \delta_{KL}$.

Generally, the difference in the squared lengths dl^2 and dL^2 is

$$dl^2 - dL^2 = (C_{KL} - \delta_{KL})dX_K dX_L = 2S_{KL}\,dX_K dX_L \tag{A3.7}$$

where

$$S_{KL} = (C_{KL} - \delta_{KL})/2 \tag{A3.8}$$

The S_{KL} are the components of the symmetric *Lagrangian finite strain tensor*, which vanishes in a rigid body motion.

Now let $dX_K^{(1)}$ and $dX_K^{(2)}$ be two elementary vectors defining an element of area dA_{KL} in B (cf. Section A2.3). The dA_{KL} are given by

$$dA_{KL} = 2\,dX^{(1)}_{[K}\,dX^{(2)}_{L]}$$

The element of area is transformed by the motion into the element da_{kl}, where

$$da_{kl} = 2\,dx^{(1)}_{[k}\,dx^{(2)}_{l]}$$

and $dx_k^{(1)}$, $dx_k^{(2)}$ are related to $dX_K^{(1)}$, $dX_K^{(2)}$ by Eqn (A3.3). Therefore

$$da_{kl} = x_{k,K}x_{l,L}dA_{KL} \tag{A3.8}$$

In terms of the associated vector elements of area da_k and dA_K defined by $da_k = \frac{1}{2}\epsilon_{klm}\,da_{lm}$ and $dA_K = \frac{1}{2}\epsilon_{KLM}\,dA_{LM}$ (Eqn (A2.17))

$$da_k = JX_{K,k}\,dA_K \tag{A3.9}$$

If $dX_K^{(3)}$ is a third elementary vector, then $dX_K^{(1)}$, $dX_K^{(2)}$ and $dX_K^{(3)}$ together define an element of volume dV in B given by (Eqn (A2.18))

$$dV = \epsilon_{KMP}\,dX_K^{(1)}\,dX_M^{(2)}\,dX_P^{(3)}$$

which is transformed in the motion to a volume element dv, where

$$dv = J\,dV \tag{A3.10}$$

In summary, Eqns (A3.1), (A3.3), (A3.9) and (A3.10) show how points, line elements, area elements and volume elements are transformed by the motion of B.

A3.2 KINEMATICS AND CONSERVATION OF MASS

From Eqn (A3.1) and velocity v_k of the particle X is just the partial derivative of $f_k(X_K, t)$ with respect to t, keeping X_K constant. As such v_k is to be regarded as a function of X_K and t. However, v_k can equally well be regarded as a function of x_k and t by substituting $X_K(x_k, t)$ for X_K. Thus

$$v_k = v_k(X_K, t) = v_k(X_K(x_k, t), t) = v_k(x_k, t)$$

Here v_k is being used to denote two distinct functions along with their values. To avoid potential confusion the following conventions are adopted:

(a) the partial time derivative of v_k when regarded as a function of X_K and t is indicated by a superimposed dot and termed the *material time derivative*;
(b) the partial time derivative of v_k when regarded as a function of x_k and t is written $\partial v_k/\partial t$ in the conventional manner;
(c) the notation $v_{k,l}$ is used to indicate the partial derivative of v_k with respect to x_l with v_k being regarded as a function of x_k and t;
(d) the notation $v_{k,K}$ is used to indicate the partial derivative of v_k with respect to X_K with v_k being regarded as a function of X_K and t.

It follows that the material time derivative of v_k is

$$\dot{v}_k = v_l v_{k,l} + \partial v_k/\partial t \tag{A3.11}$$

If the same conventions are applied to any other field quantity $f = f(x_k, t) = f(x_k(X_K, t), t) = f(X_K, t)$, then it also follows that

$$\dot{f} = v_l f_{,l} + \partial f/\partial t \tag{A3.12}$$

Taking the material time derivative of Eqn (A3.3) gives

$$\dot{\overline{dx_k}} = \dot{\overline{x_{k,K}}}\,dX_K \tag{A3.13}$$

By changing the order of differentiation, $\dot{\overline{x_{k,K}}} = v_{k,K}$, and by the chain rule, $v_{k,K} = v_{k,l}x_{l,K}$. Therefore $\dot{\overline{dx_k}} = v_{k,l}x_{l,K}dX_K$ and by Eqn (A3.3)

$$\dot{\overline{dx_k}} = v_{k,l}\,dx_l \tag{A3.14}$$

Differentiating Eqn (A3.6), the material time derivative of C_{KL} is

$$\dot{C}_{KL} = 2x_{k,K}x_{l,L}\,v_{(k,l)} \qquad (= 2\dot{S}_{KL}) \tag{A3.15}$$

Therefore from Eqn (A3.5)

$$\dot{\overline{dl^2}} = 2v_{(k,l)}\,dx_k\,dx_l \tag{A3.16}$$

The Jacobian $J = \det\{x_{k,K}\}$ can be written

$$J = (1/3!)\epsilon_{kmp}\epsilon_{KMP}x_{k,K}\,x_{m,M}x_{p,P}$$

Differentiating and using the result $v_{k,K} = v_{k,l}x_{l,K}$ leads directly to

$$\dot{J} = J v_{k,k} \tag{A3.17}$$

Differentiation of the identity $X_{K,k}x_{k,L} = \delta_{KL}$ leads to

$$\dot{\overline{X_{K,k}}} = -X_{K,l}v_{l,k} \tag{A3.18}$$

Equations (A3.17) and (3.18) can then be used in the differentiation of (A3.9) and (A3.10) to give

$$\dot{\overline{da_k}} = v_{l,l}da_k - v_{l,k}da_l \tag{A3.19}$$

and

$$\dot{\overline{dv}} = v_{k,k}dv \tag{A3.20}$$

Equations (A3.14), (A3.19) and (A3.20) give the material time derivatives of elements of length, area and volume, and can be used to determine the rates of change of physical quantities defined as integrals over material lines, surfaces and volumes. In particular, if the quantity F is expressed as an integral over V of $f dv$, where f is the density of F, then

$$dF/dt = (d/dt)\int_V f dv$$

The integral over the moving region V can be transformed into an integral over the stationary region V_0 by using the rule for change of variables in a multiple integral (Eqn (A2.25), and then the time differentiation can be taken inside the integral sign

$$(d/dt)\int_V f dv = (d/dt)\int_{V_0} f J dV = \int_{V_0} (\dot{f}J + f\dot{J})dV$$

Using Eqns (A3.10), (A3.17) and (A3.20) it follows that

$$(d/dt)\int_V f dv = \int_V (\dot{f}dv + f\dot{\overline{dv}}) = \int_V \dot{\overline{fdv}} = \int_V (\dot{f} + fv_{k,k})dv$$

In the particular case where $F = M$, the total mass of B, and $f = \rho$, the mass density, then the principle of conversation of mass can be stated as

$$dM/dt = \int_V (\dot{\rho} + \rho v_{k,k})dv = 0$$

Since this must hold for all parts of B, then

$$\dot{\rho} + \rho v_{k,k} = 0 \tag{A3.21}$$

In a similar fashion, if f_k is a vector field defined over B and F is the *flux* of f_k through a material surface S, so that

$$F = \int_S f_k \mathrm{d}a_k$$

then

$$\mathrm{d}F/\mathrm{d}t = (\mathrm{d}/\mathrm{d}t) \int_S f_k \mathrm{d}a_k = \int_S \overline{f_k \mathrm{d}a_k} = \int_S (\dot{f}_k + f_k v_{l,l} - f_l v_{k,l}) \mathrm{d}a_k$$

Thus

$$\mathrm{d}F/\mathrm{d}t = \int_S \overset{*}{f}_k \mathrm{d}a_k \tag{A3.22}$$

where the *convected time derivative* of f_k is defined by

$$\overset{*}{f}_k = \dot{f}_k + f_k v_{l,l} - f_l v_{k,l} \tag{A3.23}$$

or alternatively

$$\overset{*}{f}_k = \partial f/\partial t + v_k f_{l,l} + (v_l f_k)_{,l} - (f_l v_k)_{,l}$$

A3.3 CONSERVATION OF MOMENTUM AND THE STRESS TENSOR

The forces acting in the interior of a material are conventionally divided into short and long range. Typical of the latter is gravitation, which acts throughout the body of the material in such a way that the force on any given element of the body is independent of that on any other element. Consequently the net force on the body due to gravity is simply the sum of all the elemental forces. Short-range forces on the other hand are envisaged as acting at the molecular or atomic level. In a phenomenological approach the precise details of the atomic scale interactions are ignored, and the assumption is made that their effects can be adequately described by the hypothesis of forces, or *surface tractions*, acting across surfaces drawn in the material.

Consider the body B occupying a volume V bounded by a surface S. The external forces acting on the body are in general a combination of long-range or body forces acting throughout V and surface forces or tractions acting over the surface S. (An obvious example is a piece of material at rest on a table or bench, where the body force is that due to gravity and the surface traction is the normal reaction of the table or bench to the weight of the body.) In equilibrium, the sum of the external forces applied must be zero whatever the internal forces in the material, whereas in the dynamical case the sum of the external forces must equal the rate of change of linear momentum of the body

$$(\mathrm{d}/\mathrm{d}t) \int_V \rho v_k \mathrm{d}v = \int_V \rho f_k \mathrm{d}v + \oint_S t_k \mathrm{d}a \tag{A3.24}$$

where f_k is the body force per unit mass, t_k are the surface tractions and da is the magnitude of the vector element of area da_k. If n_k is a unit vector normal to da, then $da_k = n_k da$. Equation (A3.24) expresses the principle of the conversation of linear momentum for the body B.

Now suppose that in equilibrium a surface S_1 be drawn in the body so that the volume V is divided into two portions V_1 and V_2. In general the external forces acting on either portion of the body alone will no longer be balanced, so that it is necessary to assume the existence of forces acting across S_1 to maintain equilibrium. The material in V_1 is taken to exert a force through the surface S_1 on the material in V_2 with an equal and opposite force exerted by the material in V_2 on the material in V_1.

Let da be a small element of area in S_1 with unit normal n_k. Suppose n_k is drawn from V_1 towards V_2. Then the force exerted by the material in V_2 on the material in V_1 through the area da is assumed to be proportional to da and to depend on the orientation n_k of da. Thus the force can be written $t_k(n_l)da$, where the t_k are the components of the *stress vector* in the material. In particular, when da is normal to the coordinate axis Ox_l and the normal n_l lies along the positive Ox_l axis, the components of the stress vector are written t_{kl}.

Now suppose that S_1 is a closed surface drawn entirely within the body and that V_1 denotes the volume enclosed by S_1. At each point of S_1 let the unit normal be drawn pointing out of V_1. The net force acting on the material in V_1 now consists of the integral over V_1 of the body forces plus the integral over S_1 of the stress vector t_k. In equilibrium this force must vanish; in the dynamical case the force must equal the rate of change of momentum of the material in V_1. In either case, if the linear dimensions of V_1 are imagined to decrease without limit, so that V_1 shrinks to a point, it is clear that the volume integrals of momentum and body force will decrease according to the cube of the linear dimensions, whereas the surface integral decreases only as the square. Hence

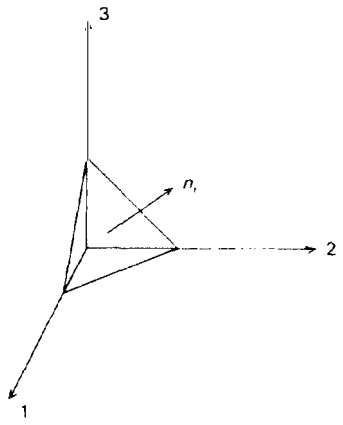

Fig. A3.1 Elementary tetrahedron.

in the limit the surface integral must vanish independently of the volume integrals. For the special case where V_1 is a tetrahedron oriented as in Fig. A3.1, this means that the sum of the forces acting on the faces of the tetrahedron must vanish as the length of the sides tends to zero. If the oblique face has area da and unit normal n_l the face perpendicular to the Ox_l axis has area $\mathrm{d}a_l = n_l \mathrm{d}a$. The forces acting on the tetrahedron across its faces are $t_k(n_l)\mathrm{d}a$ on the oblique face and $-t_{kl}\mathrm{d}a_l$ on the remaining faces, the negative sign arising from the fact that the outward pointing normals on these faces are in the negative direction of the coordinate axes. It follows therefore that

$$t_k(n_l) = t_{kl}n_l \tag{A3.25}$$

Since the product $t_{kl}n_l\mathrm{d}a = t_{kl}\mathrm{d}a_l$ is the force acting across da, it is a vector quantity. From Section A2.3, da_l is a relative vector of weight $+1$, and it therefore follows that the t_{kl} are the components of a second rank relative tensor of weight -1, known as the *stress tensor*. Knowledge of the stress tensor at all points of a material body is sufficient to allow the determination of the internal forces acting across any arbitrarily oriented element of area within the body.

By the same argument used to demonstrate the existence of the stress tensor, it can easily be shown that on the surface S of B at time t the external tractions t_k must be related to the stress tensor through $t_k = t_{kl}n_l$. Then the divergence theorem (Section A2.4) can be used to transform the surface integral in Eqn (A3.24) to a volume integral, giving

$$(\mathrm{d}/\mathrm{d}t)\int_V \rho v_k \,\mathrm{d}v = \int_V (\rho f_k + t_{kl,l})\,\mathrm{d}v \tag{A3.26}$$

On carrying out the time differentiation and using the conservation of mass as expressed in Eqn (A3.21), this becomes

$$\int_V \rho \dot{v}_k \,\mathrm{d}v = \int_V (\rho f_k + t_{kl,l})\,\mathrm{d}v$$

Since this must apply equally to every part of B, it follows that

$$\rho \dot{v}_k = \rho f_k + t_{kl,l} \tag{A3.27}$$

which is the differential form of the conservation of momentum Eqn (A3.24).

A3.4 CONSERVATION OF ANGULAR MOMENTUM

In particle mechanics, the conservation of angular momentum follows from the conservation of linear momentum. This is not the case in continuum mechanics, where the angular momentum principle has to be adopted as an additional law. In the notation of the previous section, it is

$$(\mathrm{d}/\mathrm{d}t)\int_V \epsilon_{kmp}x_m(\rho v_p)\,dv = \int_V (\epsilon_{kmp}x_m(\rho f_p)+L_k)\,\mathrm{d}v + \oint_S \epsilon_{kmp}x_m t_p\,\mathrm{d}a \qquad \text{(A3.28)}$$

The left side is the rate of change of the moment of linear momentum about the origin O. The first term in the volume integral on the right is the moment of the body forces about O, and the surface integral represents the moment of the surface tractions. The second term in the volume integral allows for the possibility of *body couples* L_k. The existence of such couples is directly related to the symmetry of the stress tensor.

Substituting $t_p = t_{pq}n_q$ in the surface integral and using the divergence theorem allows the right side of Eqn (A3.28) to be written

$$\int_V [\epsilon_{kmp}x_m(\rho f_p + t_{pq,q}) + (L_k - \epsilon_{kpq}t_{pq})]\,\mathrm{d}v$$

The left side reduces to

$$\int_V \epsilon_{kmp}x_m(\rho \dot{v}_p)\,\mathrm{d}v$$

Since conservation of linear momentum demands

$$\rho\dot{v}_p = \rho f_p + t_{pq,q}$$

Equation (A3.28) reduces to

$$\int_V (L_k - \epsilon_{kpq}t_{pq})\,\mathrm{d}v = 0$$

and since this must hold for all parts of B, finally

$$L_k - \epsilon_{kpq}t_{pq} = 0 \qquad \text{(A3.29)}$$

This is the differential expression of the conservation of angular momentum and implies in particular that, in the absence of body couples L_k, the stress tensor t_{pq} is symmetric.

A3.5 ELECTRODYNAMICS

The motion of a purely elastic material is determined by the conservation laws of mass, momentum and angular momentum and the specific stress–strain relationships that happen to hold for the particular material in question. In a piezoelectric material, the situation is complicated by the interaction of electromagnetic and mechanical phenomena, and the conservation laws have to be complemented by the electromagnetic field equations.

Fortunately, in most situations of practical interest it is permissible to make the *quasistatic* approximation and to neglect the coupling of the electric and magnetic fields that results from the time dependent terms in Maxwell's equations. The justification for this neglect essentially lies in the several orders of magnitude difference between the velocities of elastic waves on the one hand and electromagnetic waves on the other, and amounts to assuming that the electric field is completely described by the static field equations.

In the quasistatic approximation the relevant equations are

$$\oint_S D_k \mathrm{d}a_k = Q \tag{A3.30}$$

$$\oint_C E_k \mathrm{d}l_k = 0 \tag{A3.31}$$

$$D_k = \epsilon_0 E_k + P_k \tag{A3.32}$$

where S is a closed surface bounding a volume V, and C is a closed curve in V. The quantities D_k, E_k, and P_k are, respectively, the electric displacement, the electric field and the polarization, and Q is the total free charge contained in V. ϵ_0 is the permittivity of free space. The polarization P_k is introduced to represent the electrical properties of any dielectric materials present, and is zero in free space. If the total charge Q is taken to be a scalar quantity, then since the element of area $\mathrm{d}a_k$ is a relative vector of weight $+1$, it follows from Eqn (A3.30) that D_k must be a relative vector of weight -1. From Eqn (A3.31) and the vector nature of the line element $\mathrm{d}l_k$ it similarly follows that E_k is an absolute vector. Additionally from Eqn (A3.32) follows that P_k is a relative vector of weight -1 and ϵ_0 is a relative scalar of weight -1.

If the field quantities are sufficiently smooth in V, the divergence theorem and Stokes' theorem can be applied to Eqns (A3.30) and (A3.31), respectively, to give the differential field equations

$$D_{k,k} = q \tag{A3.33}$$

$$\epsilon_{kmp} E_{m,p} = 0 \tag{A3.34}$$

where it has been assumed that the total charge Q is the integral over V of a free charge density q. It follows from (A3.34) that the field E_k is the gradient of a scalar ϕ known as the electric potential

$$E_k = -\phi_{,k} \tag{A3.35}$$

Equation (A3.33) can be rewritten in terms of E_k and P_k by using Eqn (A3.32)

$$\epsilon_0 E_{k,k} = q - P_{k,k} \tag{A3.36}$$

or in terms of the potential

$$\epsilon_0 \phi_{,kk} = -q + P_{k,k} \tag{A3.37}$$

If the region V contains a surface of discontinuity S_d then the differential

field equations cannot be obtained by direct application of the integral theorems. Suppose that S_d divides V into two parts V_1 and V_2, and the unit normal n_k at each point on S_d points from V_1 to V_2. Then application of the divergence theorem to V_1 and V_2 separately leads to the conclusion that Eqn (A3.33) must be satisfied in each region, but that in addition the change in D_k on crossing S_d must satisfy

$$(D_k^+ + D_k^-)n_k = w \tag{A3.38}$$

where w is the surface density of free charge on S_d, and D_k^+, D_k^- are the limiting values of D_k as S_d is approached from within V_2 and within V_1, respectively. Similar arguments applied in Eqn (A3.31) lead to the conclusion that Eqn (A3.34) must be satisfied on either side of S_d, but that in crossing S_d the electric field must additionally satisfy

$$(E_k^+ - E_k^-)m_k = 0 \tag{A3.39}$$

where m_k is any unit vector tangent to S_d. A sufficient condition for this is to require that the potential ϕ be continuous across S_d.

Equation (A3.38) can be rewritten in terms of E_k and P_k

$$\epsilon_0(E_k^+ - E_k^-)n_k = w - (P_k^+ - P_k^-)n_k \tag{A3.40}$$

In the particular case where S_d is the boundary of a dielectric body and P_k^+ is zero, then

$$\epsilon_0(E_k^+ - E_k^-)n_k = w + P_k n_k \tag{A3.41}$$

where P_k is the limiting value of the polarization on the surface S_d. Equations (A3.36) and (A3.41) show clearly the equivalence of the polarized dielectric to a volume charge density $-P_{k,k}$ and a surface charge density $P_k n_k$.

If the charge densities q and w, and the polarization P_k are given, the equations above are sufficient to determine the electric field and displacement E_k and D_k. In a real case, generally the polarization is not given but is rather a quantity to be determined as part of the solution to a given problem. Then in order to solve the field equations it is necessary to assume some functional relationship between P_k and the other field variables. The precise form of the relationship will depend upon the particular problem, specifically on the type of material being considered. In elementary texts, attention is usually restricted to linear isotropic materials and the assumption is made that P_k is proportional to E_k, that is $P_k = kE_k$, where k is a constant. Whatever assumption is made, subsequent discussion is restricted in application to the particular class of materials defined. For present purposes, it is advantageous to delay making any such assumptions and to regard P_k as an independent quantity, except insofar as it is constrained by the field equations themselves.

As in earlier sections of this Appendix, consider then a body B occupying a volume V bounded by a surface S at time t. As before, the points of B are labelled by their coordinates X_K at time $t = t_0$, when B occupied the volume V_0

bounded by the surface S_0. Additionally, let P_k be a polarization field defined in V, let q be the free charge density in V, and let w be the free charge density on S. In terms of the potential ϕ and the field $E_k = -\phi_{,k}$ the field equations are Eqn (A3.37) in V and (A3.41) on S, with ϕ being assumed continuous across S.

$$\epsilon_0 \phi_{,kk} = -q + P_{k,k} \tag{A3.37}$$

$$\epsilon_0 (E_k^+ - E_k^-) n_k = w + P_k n_k \tag{A3.41}$$

In terms of the total volume and surface charge densities q_t and w_t defined by $q_t = q - P_{k,k}$ and $w_t = w + P_k n_k$ these become

$$\epsilon_0 \phi_{,kk} = -q_t \tag{A3.42}$$

$$\epsilon_0 (E_k^+ - E_k^-) n_k = w_t \tag{A3.43}$$

The physical interpretation of the fields E_k and ϕ is that $E_k(x_l)$ represents the force per unit charge that would be experienced by an infinitesimally small test charge at the position x_l, and that $\phi(x_l)$ represents the potential energy per unit charge. The potential energy can be regarded as equal to the work done per unit charge, against the electric field, in bringing up a test charge to the point x_l from infinity, or, more.precisely, from any point where the fields are zero. According to the field-theoretic view, the mutual potential energy of a system of charges is to be regarded as residing in the field itself. Since the field is determined through the field equations by the current distribution of charges, it follows that the precise details of the way in which the charges were assembled are irrelevant to their potential energy.

Consider then a virtual process in which the body B is rigidly clamped so that no physical displacements can occur, and in which the surface and volume charge densities w_t and q_t are incremented by amounts Δq_t and Δw_t. Then the work done in this process, which can be equated to the change ΔU_F in the potential energy of the system, is

$$\Delta U_F = \int_V \phi \Delta q_t \, \mathrm{d}v + \oint_S \phi \Delta w_t \, \mathrm{d}a$$

Writing ΔE_k for the change in the field E_k resulting from the virtual process, then it follows from the field equations Eqns (A3.42) and (A3.43) that

$$\Delta U_F = \int \epsilon_0 E_k \, \Delta E_k \, \mathrm{d}v$$

from which it follows straightforwardly that the total potential energy U_F of the system of charges q_t and w_t is

$$U_F = (\epsilon_0/2) \int E_k^2 \, \mathrm{d}v \tag{A3.44}$$

where the integral is understood to be taken over all space.

Although derived by a consideration of a particular virtual process, the result, Eqn (A3.44), is independent of that process. At least in the quasistatic approximation, U_F in Eqn (A3.44) can be regarded as the mutual potential energy, or the electrostatic field energy, even in the case where the dielectric body B is undergoing an arbitrary motion. Then the changes in U_F have to be related to both the mechanical work done by the electrical forces acting on the material and to the changes in *internal energy* of the material resulting from changes in its state of polarization.

The integral in Eqn (A3.44) is to be taken over all space. In calculating the rate of change of U_F it is necessary to take account of the discontinuities in E_k on crossing the surface S of B, and the space integral has to be split into an integral over the volume V of B and an integral over the rest of space V^*. Generally, V and V^* will be time-dependent owing to the motion of B, so it is convenient to be able to transform the integrals to integrals over stationary regions. In the case of V, the equations $x_k = f_k(X_K, t)$ allow this to be done, but in the region V_0^* exterior to V_0 this mapping is not defined. Nevertheless, it is useful to extend the functions f_k arbitrarily over V_0^*, with the sole proviso that they be continuous over the surface S_0.

The rate of change of U_F can then be written

$$\mathrm{d}U_F/\mathrm{d}t = (\mathrm{d}/\mathrm{d}t)\int (\epsilon_0/2)E_k^2\,\mathrm{d}v = (\epsilon_0/2)\int \overline{E_k^2}\,\mathrm{d}v$$

Therefore

$$\mathrm{d}U_F/\mathrm{d}t = \epsilon_0 \int E_k\dot{E}_k\,\mathrm{d}v + (\epsilon_0/2)\int E_k^2 v_{l,l}\mathrm{d}v \tag{A3.45}$$

Using Eqn (A3.32) the first term on the right can be written as

$$\epsilon_0 \int E_k\dot{E}_k\,\mathrm{d}v = \int E_k\dot{D}_k\,\mathrm{d}v - \int E_k\dot{P}_k\mathrm{d}v \tag{A3.46}$$

If it is assumed that free charge is conserved in the motion of B, then the total free charge Q in S will be constant. Therefore by Eqn (A3.30) the surface integral of D_k over S will be constant, and by Eqn (A3.22) the convected time derivative $\overset{*}{D}_k$ of D_k will vanish

$$\overset{*}{D}_k = \dot{D}_k + D_k v_{l,l} - D_l v_{k,l} = 0$$

Therefore

$$\int E_k\dot{D}_k\,\mathrm{d}v = -\int E_k D_k v_{l,l}\mathrm{d}v + \int E_k D_l v_{k,l}\mathrm{d}v \tag{A3.47}$$

The second term in Eqn (A3.46) involves $\dot{P}_k$, the material time derivative of the polarization. This can be rewritten in terms of the polarization per unit mass π_k, with $P_k = \rho\pi_k$. This allows the separation of changes in P_k which are only due to the motion of the material from intrinsic changes in P_k which can be regarded as changes in the state of polarization of the dielectric. In terms of π_k, $\dot{P}_k = \overline{\dot{\rho}\pi_k} = \dot{\rho}\pi_k + \rho\dot{\pi}_k$, and from Eqn (A3.21)

$$\dot{P}_k = -P_k v_{l,l} + \rho\dot{\pi}_k \tag{A3.48}$$

The rate of change of the field energy U_F can now be written

$$dU_F/dt = \int \{[E_k D_l - (\epsilon_0/2)E_m^2 \delta_{kl}]v_{k,l} - \rho E_k \dot{\pi}_k\} dv$$

or

$$dU_F/dt = \int \{[t_{kl}^M + E_k P_l]v_{k,l} - \rho E_k \dot{\pi}_k\} dv \tag{A3.49}$$

where the *Maxwell stress tensor* $t_{kl}^M = [\epsilon_0 E_k E_l - (\epsilon_0/2)E_m^2 \delta_{kl}]$ has been introduced. The first two terms on the right of Eqn (A3.49) can be identified with the rate at which mechanical work is done against the electrostatic forces, and the negative of the third term can be identified with the rate at which work is done in changing the polarization state of the material.

A3.6 CONSERVATION OF ENERGY

Consider again a body B in motion under the influence of external body forces f_k per unit mass and surface tractions t_k. Assuming that B consists of a polarized dielectric material carrying volume and surface free charge densities q and w, it will be associated with an electrostatic field E_k having an energy U_F given by Eqn (A3.44). B will also have kinetic energy U_K equal to the integral over V of the kinetic energy density $\rho v_k^2/2$. In addition to the field energy U_F and the kinetic energy U_K it is assumed that B has an internal energy U_I. The internal energy is taken to be the integral over the body of an energy density Σ per unit mass, and it is further assumed that the energy density is a function of the deformation gradients $x_{k,K}$, the polarization per unit mass π_k and the *entropy* per unit mass σ. Then the total energy U of the system consisting of B and its associated electrostatic field is $U = U_I + U_K + U_F$, where

$$U_I = \int_V \rho\Sigma(x_{k,K}, \pi_k, \sigma) dv$$

$$U_K = \int_V [(\rho v_k^2)/2] dv$$

$$U_F = \int_{V+V^*} [(\epsilon_0 E_k^2)/2] dv$$

The principle of *conservation of energy* states that the rate of change of the total system energy U must equal the rate at which external work is done on the system plus the rate at which thermal energy is supplied to the system. The former is just the rate of working of the *external* body forces and surface tractions, and if dissipation is neglected, the latter is the rate at which heat is transmitted through S, that is, the integral over S of the heat flux vector h_k. Therefore

$$dU/dt = \int_V \rho f_k v_k dv + \oint_S t_k v_k da + \oint_S h_k da_k \tag{A3.50}$$

Note that in Eqn (A3.50) the forces f_k and surface tractions t_k are the *external* influences acting, and in particular do not include the electrostatic forces on the material. Hence f_k and t_k do not denote the same quantities as they do in Eqns (A3.24) and (A3.27).

The integral of h_k over S in Eqn (A3.50) can be transformed into a volume integral by using the divergence theorem. Then $(h_{k,k}\mathrm{d}v)$ can be interpreted as the rate at which heat or thermal energy is supplied to the volume element $\mathrm{d}v$. In a reversible process in the thermodynamic sense, this can be equated to the product of the absolute temperature θ and the rate of increase of the entropy of the volume element $\mathrm{d}v$, $\overline{\rho\sigma\mathrm{d}v}^{\cdot}$. Since mass is conserved, it follows that

$$\oint_S h_k \,\mathrm{d}a_k = \int_V \rho\theta\dot{\sigma}\mathrm{d}v \tag{A3.51}$$

The rate of change of the electrostatic field energy U_F is given by Eqn (A3.49). The rate of change of the kinetic energy U_K is

$$\mathrm{d}U_K/\mathrm{d}t = (\mathrm{d}/\mathrm{d}t)\int_V (\rho v_k^2/2)\mathrm{d}v = \int_V \rho v_k \dot{v}_k \,\mathrm{d}v \tag{A3.52}$$

The rate of change of the internal energy U_I is

$$\mathrm{d}U_I/\mathrm{d}t = (\mathrm{d}/\mathrm{d}t)\int_V \rho\Sigma\mathrm{d}v = \int_V \rho\dot{\Sigma}\,\mathrm{d}v$$

But $\dot{\Sigma}$ is

$$\dot{\Sigma} = \frac{\partial\Sigma}{\partial x_{k,K}}\overline{\dot{x}_{k,K}} + \frac{\partial\Sigma}{\partial\pi_k}\dot{\pi}_k + \frac{\partial\Sigma}{\partial\sigma}\dot{\sigma}$$

and since $\overline{\dot{x}_{k,K}} = v_{k,l}x_{l,K}$,

$$\rho\dot{\Sigma} = t^L_{kl}v_{k,l} + \rho\frac{\partial\Sigma}{\partial\pi_k}\dot{\pi}_k + \rho\frac{\partial\Sigma}{\partial\sigma}\dot{\sigma}$$

where

$$t^L_{kl} = \rho x_{l,K}\frac{\partial\Sigma}{\partial x_{k,K}} \tag{A3.53}$$

Therefore

$$\mathrm{d}U_I/\mathrm{d}t = \int_V \left\{ t^L_{kl}v_{k,l} + \rho\frac{\partial\Sigma}{\partial\pi_k}\dot{\pi}_k + \rho\frac{\partial\Sigma}{\partial\sigma}\dot{\sigma}\right\}\mathrm{d}v \tag{A3.54}$$

Transforming the first term by using the divergence theorem gives

$$\mathrm{d}U_I/\mathrm{d}t = \int_V \left\{ -t^L_{kl,l}v_k + \rho\frac{\partial\Sigma}{\partial\pi_k}\dot{\pi}_k + \rho\frac{\partial\Sigma}{\partial\sigma}\dot{\sigma}\right\}\mathrm{d}v + \oint_S t^L_{kl}v_k n_l \mathrm{d}a \tag{A3.55}$$

The first term of Eqn (A3.49) can be similarly transformed, with the proviso that since the integral in Eqn (A3.49) extends over all space, the resulting surface integral involves the 'jump' in $t^M_{kl} + E_k P_l$ in crossing S

$$dU_F/dt = \int_{V+V^*} \{-[t^M_{kl,l} + (E_k P_l)_{,l}]v_k - \rho E_k \dot{\pi}_k\} dv - \oint_S [\![t^M_{kl} + E_k P_l]\!] v_k n_l da \tag{A3.56}$$

where the notation $[\![f]\!]$ is used to indicate the 'jump' in a quantity f in crossing a surface of discontinuity.

Equations (A3.51), (A3.52), (A3.55) and (A3.56) can now be substituted into Eqn (A3.50) to give an equation involving the sum of three volume integrals and a surface integral

$$\int_{V+V^*} [\rho \dot{v}_k - \rho f_k - t^L_{kl,l} - t^M_{kl,l} - (E_k P_l)_{,l}] v_k \, dv +$$
$$\oint_S \{-t_k + t^L_{kl} n_l - [\![t^M_{kl} n_l + E_k P_l n_l]\!]\} v_k \, da +$$
$$\int_V \left[\rho \frac{\partial \Sigma}{\partial \pi_k} - \rho E_k \right] \dot{\pi}_k \, dv +$$
$$\int_V \left[\rho \frac{\partial \Sigma}{\partial \sigma} - \rho \theta \right] \dot{\sigma} \, dv$$

This sum must vanish for all admissible velocity, polarization and entropy fields and therefore the terms multiplying $v_k, \dot{\pi}_k$ and $\dot{\sigma}$ can be separately set equal to zero, leading to the field equations

$$\rho \dot{v}_k = \rho f_k + t^L_{kl,l} + t^M_{kl,l} + (E_k P_l)_{,l} \tag{A3.57}$$

$$\frac{\partial \Sigma}{\partial \pi_k} = E_k \tag{A3.58}$$

$$\frac{\partial \Sigma}{\partial \sigma} = \theta \tag{A3.59}$$

satisfied in V,

$$t^M_{kl,l} = 0 \tag{A3.60}$$

satisfied in V^*, and

$$t_k = t^L_{kl} n_l - [\![t^M_{kl} + E_k P_l]\!] n_l \tag{A3.61}$$

satisfied on S.

Comparing Eqn (A3.57) to Eqn (A3.27) leads naturally to the physical interpretation of t^L_{kl} as the local 'elastic' stress in the material, related to the deformation by Eqn (A3.53), and to the interpretation of $t^M_{kl} + E_k P_l$ as the

electrostatic stress. From the definition of the Maxwell stress tensor t^M_{kl} and the field Eqns (A3.34) and (A3.36) it follows that

$$t^M_{kl,l} + (E_k P_l)_{,l} = qE_k + P_l E_{k,l}$$

The first term is just the force exerted by the field E_k on the free charge density q, and the second is the force on an electric dipole P_l in an electric field E_k.

It should be noted that for a dipole in an electric field, not only is there a force $P_l E_{k,l}$ acting, but there is also a couple $\epsilon_{klm} P_l E_m$. This couple constitutes a body couple in the sense of Section A3.4, and its existence implies that the local stress t^L_{kl} must be asymmetric.

This can be confirmed by considering the dependence of the internal energy density Σ on its arguments $x_{k,K}$, π_k, and σ. Σ is an absolute scalar and therefore invariant under orthogonal transformations of coordinates. However, the deformation gradients and the polarization are vector quantities and transform according to the appropriate law under orthogonal transformation. It is therefore clear that Σ cannot be an arbitrary function of its arguments. It can in fact be shown (Weyl, 1939) that any scalar function of N vectors must be expressible as a function of the scalar products of the N vectors taken two at a time, so that in the present case Σ must be expressible as a function of the quantities $x_{k,K}x_{k,L}$, $x_{k,K}\pi_k$, and $\pi_k\pi_K$. The quantities $x_{k,K}x_{k,L}$ are just the components of the deformation tensor C_{KL}, and if P_K is written for $x_{k,K}\pi_k$ and π^2 for $\pi_k\pi_k$, then Σ must be expressible as a function of C_{KL}, P_K, and π^2. The defining relation for P_K can be inverted to give $\pi_k = P_K X_{K,k}$, from which it follows that

$$\pi^2 = X_{K,k}X_{L,k}P_K P_L = C^{-1}_{KL}P_K P_L$$

where $C^{-1}_{KL} = X_{K,k}X_{L,k}$ is the inverse of the deformation tensor. Thus π^2 is itself a function of the C_{KL} and the P_K and so can be dropped from the argument list of Σ, leading to the conclusion that Σ must be expressible as a function of the C_{KL} and P_K (in addition to the scalar quantity σ). The partial derivatives of Σ with respect to $x_{k,K}$ and π_k can then be expressed as

$$\frac{\partial \Sigma}{\partial x_{k,K}} = \frac{\partial \Sigma}{\partial C_{LM}} \frac{\partial C_{LM}}{\partial x_{k,K}} + \frac{\partial \Sigma}{\partial P_L} \frac{\partial P_L}{\partial x_{k,K}}$$

$$\frac{\partial \Sigma}{\partial \pi_k} = \frac{\partial \Sigma}{\partial P_L} \frac{\partial P_L}{\partial \pi_k}$$

Then from Eqns (A3.53) and (A3.58) it follows that

$$t^L_{kl} = \rho (x_{l,M}x_{k,L} + x_{k,M}x_{l,L}) \frac{\partial \Sigma}{\partial C_{LM}} + P_k E_l$$

so that the antisymmetric part $t^L_{[kl]}$ is just

$$t^L_{[kl]} = P_{[k}E_{l]}$$

just as required by the conservation of angular momentum Eqn (A3.29).

A3.7 SUMMARY OF EQUATIONS FOR THE ELASTIC DIELECTRIC

The results of the previous sections can be summarized as follows. For an elastic, heat-conducting, dielectric body B, without dissipation, occupying a volume V bounded by a surface S and acted upon by external forces f_k per unit mass and surface tractions t_k per unit area, the field equations in V are

$$D_{k,k} = q \tag{A3.33}$$

$$\rho \dot{v}_k = \rho f_k + t^L_{kl,l} + (t^M_{kl} + E_k P_l)_{,l} \tag{A3.57}$$

while on the boundary surface S

$$[\![D_k n_k]\!] = w \tag{A3.38}$$

$$[\![\phi]\!] = 0 \tag{A3.39}$$

$$t_k = t^L_{kl} n_l - [\![t^M_{kl} + E_k P_l]\!] n_l \tag{A3.61}$$

where $[\![f]\!]$ denotes the 'jump' in a quantity f in crossing S. In the region V^* exterior to V,

$$D_{k,k} = 0 \tag{A3.62}$$

In the above q and w are volume and surface free charge densities in V and on S. E_k, D_k and P_k are respectively the electric field, displacement and polarization, with

$$D_k = \epsilon_0 \mathrm{E}_k + \mathrm{P}_k \tag{A3.32}$$

where ϵ_0 is the permittivity of free space. P_k is identically zero in V^*. The electric field E_k is obtained from the potential ϕ via

$$E_k = -\phi_{,k} \tag{A3.35}$$

The Maxwell stress tensor is defined in terms of the electric field by

$$t^M_{kl} = \epsilon_0 E_k E_l - (\epsilon_0/2) E_m{}^2 \delta_{kl} \tag{A3.63}$$

The mass density of B is ρ, and v_k is the velocity of a point of B. The body B is assumed to have an internal energy density Σ per unit mass, given as a function of the deformation gradients $x_{k,K}$, the polarization per unit mass $\pi_k = P_k/\rho$, and the entropy per unit mass σ. Then the *constitutive relations* for the local elastic stress t^L_{kl}, the electric field E_k and the absolute temperature which follow from the conservation of energy are

$$t^L_{kl} = \rho x_{l,K} \frac{\partial \Sigma}{\partial x_{k,K}} \tag{A3.53}$$

$$E_k = \frac{\partial \Sigma}{\partial \pi_k} \tag{A3.58}$$

$$\theta = \frac{\partial \Sigma}{\partial \sigma} \tag{A3.59}$$

Given the form of the internal energy function Σ, these equations are sufficient to determine the motion of B subject to the external influences f_k and t_k and the surface and volume charge densities q, w, which are assumed to be conserved in the motion.