Part 3
The crystal as a circuit element

6 Equivalent circuit analysis

6.1 EQUIVALENT CIRCUITS

Figure 6.1 shows the conventionally accepted equivalent circuit of a crystal resonator at a frequency near its main mode of vibration. The circuit elements L_1, C_1, R_1 are the electrical equivalents of the inertia, stiffness, and internal losses of the mechanical vibrating system. If the crystal were clamped in such a way that no vibration were possible, this arm would be absent, and hence L_1, C_1 and R_1 are known as the *motional parameters* of the crystal. The element C_0 represents the capacitance of the capacitor formed by the electrodes of the crystal and the quartz dielectric. It can be measured as the effective capacitance of the crystal unit at frequencies far removed from resonance, and is known as the *static capacitance*.

One of the principal characteristics of a quartz resonator as compared to LC circuits or other types of mechanical resonator, is that the Q factor of the motional arm is extremely high. For commercially available units, typical values range from 20 000 to several hundred thousand, while specially designed units can have Q values of several million. These compare with Qs of less than 1000 for the best wound inductors.

The *motional capacitance* C_1 for typical AT-cut fundamental mode resonators is typically in the range 10 to 30 fF (1 fF = 10^{-15} F). For overtone resonators, C_1 reduces in inverse proportion to the square of the overtone order, so that for a third overtone typical values range from 1 to 3 fF.

As indicated above, the static capacitance C_0 is essentially determined by the electrode size and separation and is thus independent of the overtone order. Typical values for AT-cuts range from 1 to 7 pF. More important in

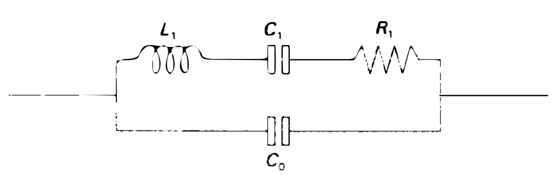

Fig. 6.1 Equivalent circuit of crystal resonator.

practice is the ratio of C_0 to C_1 because this ratio effectively determines the sensitivity of the crystal frequency to changes in external circuit parameters. As shown in Chapter 2, the capacitance ratio is intimately connected with the electromechanical coupling factor for the particular crystal cut in question. For fundamental mode ATs above about 10 MHz, the ratio is approximately 200, whereas for overtones it increases roughly in proportion to the square of the overtone order. For low frequency fundamentals where the crystal blank is partially or fully contoured, C_0/C_1 again increases because the contour limits C_1 but not C_0. The 'pulling' sensitivity is thus greatest for the higher frequency fundamental mode units.

6.2 LOSSLESS CIRCUIT ANALYSIS

6.2.1 Characteristic frequencies

Precise analysis of the equivalent circuit reveals several characteristic frequencies. In most practical cases, because of the very high resonator Q, it is sufficient to consider only two. These are the *series resonance* frequency f_r and the *anti-resonance* or *parallel* resonance frequency f_a. These correspond to the *natural modes* of the resonator under short-circuit and open-circuit conditions, respectively. Expressions for these frequencies can easily be derived by analysis of the lossless circuit obtained by neglecting the motional resistance R_1 as shown in Fig. 6.2.

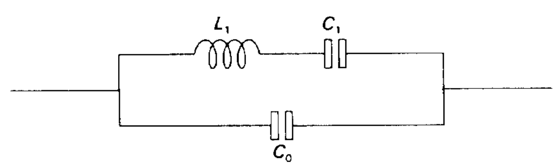

Fig. 6.2 Lossless equivalent circuit.

If Y is the admittance of the crystal unit, then Y can be written as

$$Y = jB_0 + jB_1$$

where $B_0 = \omega C_0$, ω is the angular frequency, and jB_1 is the admittance of the motional arm. jB_1 is given by the reciprocal of the motional arm impedance, ie, by

$$jB_1 = 1/(j\omega L_1 + 1/j\omega C_1) = j\omega C_1/(1 - \omega^2 L_1 C_1)$$

Therefore

$$Y = j(B_0 + B_1) = j\omega C_0 (1 + C_1/C_0 - \omega^2 L_1 C_1)/(1 - \omega^2 L_1 C_1)$$

and if Z is the crystal impedance,

$$Z = 1/Y = (1-\omega^2 L_1 C_1)/\{j\omega C_0(1+C_1/C_0-\omega^2 L_1 C_1)\}$$

The series resonance frequency f_r corresponds to the zero of the impedance Z, and the anti-resonance frequency f_a corresponds to the zero of Y. Consequently, if ω_r and ω_a are the corresponding angular frequencies, then

$$\omega_r^2 = 1/(L_1 C_1)$$

and

$$\omega_a^2 = (1+C_1/C_0)/(L_1 C_1)$$

or

$$\omega_a^2 = \omega_r^2(1+C_1/C_0)$$

Since $C_1/C_0 \ll 1$, to a good degree of approximation the last equation can be written

$$\omega_a = \omega_r(1+C_1/2C_0)$$

or

$$(\omega_a-\omega_r)/\omega_r = (f_a-f_r)/f_r = C_1/2C_0 \qquad (6.1)$$

which directly relates the separation between the resonance frequencies to the capacitance ratio.

6.2.2 Crystal with a load capacitance

Many practical oscillator circuits make use of a load capacitor C_L in series or parallel with the crystal, either in order to provide a means for final frequency adjustment, or perhaps for modulation or temperature compensation purposes. The presence of the load capacitor shifts the working frequency of the crystal by an amount depending upon the value of C_L and the values of C_0 and C_1. Figures 6.3 and 6.4 show the series and parallel connections respectively. Figures 6.5 and 6.6 show plots of the impedance and admittance

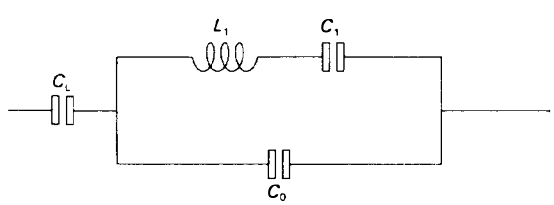

Fig. 6.3 Crystal with series load capacitor.

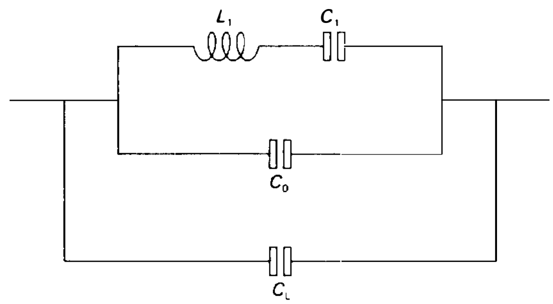

Fig. 6.4 Crystal with parallel load capacitor.

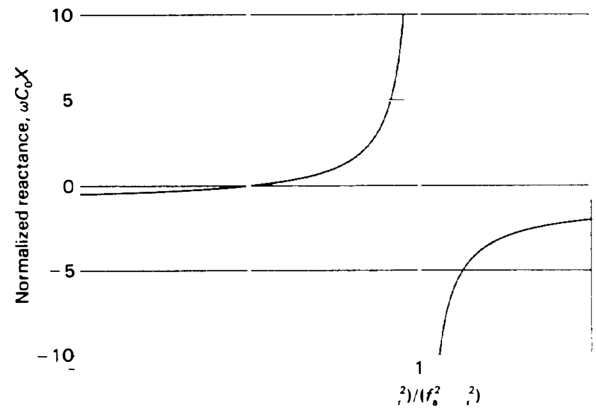

Fig. 6.5 Normaliz

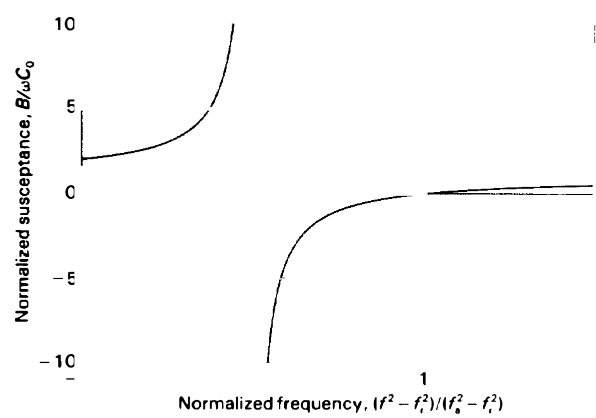

Fig. 6.6 Normalized crystal susceptance vs frequency.

of a crystal unit in the lossless approximation. From these plots it is clear that when C_L is connected in parallel with the crystal, the resonance frequency f_r is not affected, but that the anti-resonance frequency f_a is shifted down to a frequency f_L. Similarly, when C_L is connected in series with the crystal, f_a is unaffected but f_r is shifted up to a frequency f_L. It follows from the physical interpretation of f_r and f_a as short- and open-circuit resonances, respectively, that f_L has the same value in both the above cases, since an open-circuit parallel combination is the same as a short-circuit series combination. The value of f_L can therefore be very easily obtained from the expressions for f_a already given by substituting f_L for f_a and $(C_0 + C_L)$ for C_0. Therefore

$$f_L{}^2 = f_r^2(1 + C_1/(C_0 + C_L))$$

or

$$(f_L - f_r)/f_r = C_1/2(C_0 + C_L) \tag{6.2}$$

This is the *load resonance frequency offset*.

The sensitivity of the working frequency to small changes in C_L, ie, the 'pulling sensitivity', is given by

$$\mathrm{d}f_L/\mathrm{d}C_L = -C_1/2(C_0 + C_L)^2$$

For high overtone orders, the sensitivity is orders of magnitude lower than for fundamental mode units, because of the much lower typical C_1 values and consequently high ratio of C_0 to C_1.

From Eqns (6.1) and (6.2) it follows that

$$(f_L - f_r)/(f_a - f_r) = C_0/(C_0 + C_L) = k$$

For a purely capacitive load where C_L varies from zero upwards, the parameter k ranges from 1 to 0. This corresponds to the fact that the working frequency f_L is restricted to the range between f_r and f_a. Thus the maximum load resonance frequency offset is determined through Eqn (6.1) by the capacitance ratio C_0/C_1, and the location of f_L within this range is determined by k, ie, the value of C_L as compared to C_0. For example, if $C_L = 4C_0$, then $k = 0.2$ and the working frequency is shifted relative to f_r by 20% of the pole-zero spacing $(f_a - f_r)$.

6.3 LOSSY CIRCUIT ANALYSIS

The simple analysis of the characteristic frequencies of a crystal resonator based on the lossless equivalent circuit glosses over the subtle differences between the several 'resonance' frequencies that may be defined when losses are present. The existence of these several frequencies can easily lead to confusion in comparing the results of measurements made by different methods, especially at high frequencies and overtones so that an analysis of the equivalent circuit including losses is necessary.

6.3.1 Motional impedance and admittance

At an angular frequency ω the motional impedance Z_1 can be written

$$Z_1 = R_1 + j\omega L_1 + 1/j\omega C_1$$

or

$$Z_1 = R_1 + jL_1(\omega^2 - \omega_s^2)/\omega$$

where $\omega_s^2 L_1 C_1 = 1$ and ω_s is the angular frequency corresponding to the motional resonance frequency f_s where Z_1 is purely resistive.

If now the normalized frequency variable x is introduced by the relation

$$x = (f^2 - f_s^2)/(f_p^2 - f_s^2) = (\omega^2 - \omega_s^2)/(\omega_p^2 - \omega_s^2)$$

where $f_p^2 = f_s^2(1 + C_1/C_0)$ is the frequency corresponding to the anti-resonance frequency in the lossless case, then

$$Z_1 = R_1 + jX_1$$

where

$$X_1 = x(\omega_p^2 - \omega_s^2)L_1/\omega$$

From the definition of ω_p and ω_s, it follows that

$$(\omega_p^2 - \omega_s^2)L_1 = \omega_s^2 L_1 C_1/C_0 = 1/C_0$$

Therefore $X_1 = x/\omega C_0 = xX_0$ where $X_0 = 1/\omega C_0$. To a good approximation, X_0 can be regarded as a constant with a value equal to the magnitude of the reactance of C_0 at frequency ω_s, ie, $X_0 \sim 1/\omega_s C_0$.

It is convenient at this point to introduce the parameters Q, r, and M. Q is defined as the *quality factor* of the motional arm, ie, $Q = \omega_s L_1/R_1$, r is the capacitance ratio C_0/C_1, and M is the *figure of merit* Q/r. From the definition, it follows that $M = X_0/R_1$. Then the expression for Z_1 can be written in normalized form as

$$Z_1/R_1 = 1 + jxM$$

The motional admittance Y_1 is the reciprocal of the motional impedance. Thus if y_1 is the normalized admittance $R_1 Y_1$, it follows immediately that

$$y_1 = 1/(1 + jxM) = (1 - jxM)/(1 + x^2M^2)$$

If $y_1 = g_1 + jb_1$, then

$$g_1 = 1/(1 + x^2M^2)$$

$$b_1 = -xM/(1 + x^2M^2)$$

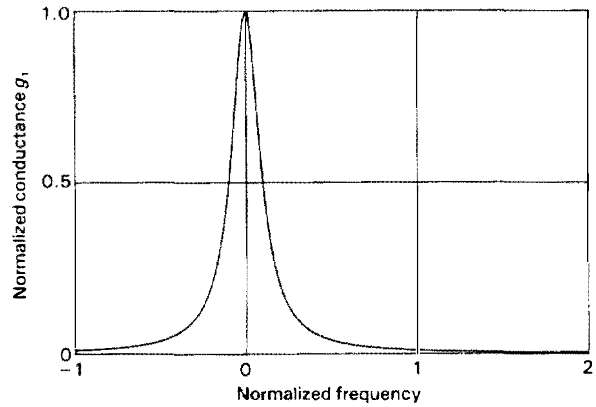

Fig. 6.7 Normalized crystal conductance vs frequency ($M = 10$).

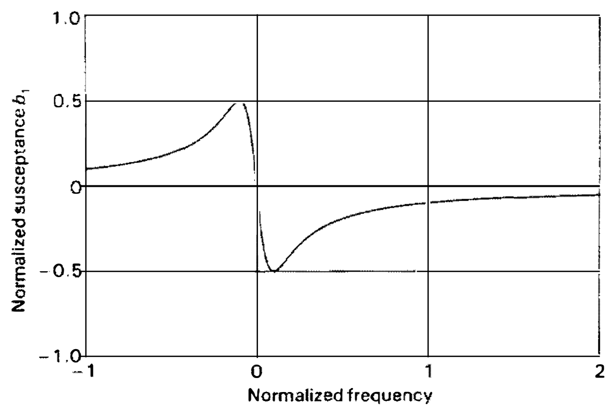

Fig. 6.8 Normalized motional susceptance ($M = 10$).

Figures 6.7 and 6.8 show plots of g_l and b_l as functions of the normalized frequency x. g_l has a single maximum of value 1 at $x = 0$ and is always > 0. b_l is > 0 for $x < 0$ and < 0 for $x > 0$, and has extrema at the points $x = \pm M^{-1}$. The extreme values are $+\frac{1}{2}$ at $x = -M^{-1}$ and $-\frac{1}{2}$ at $x = M^{-1}$.

The magnitude squared of the normalized admittance is $g_l^2 + b_l^2$ and is given by

$$g_l^2 + b_l^2 = 1/(1 + x^2M^2)$$

and again has a maximum of 1 at $x = 0$.

6.3.2 Crystal admittance

The crystal admittance Y is the sum of the motional admittance and the admittance of the shunt capacitance C_0. In the present degree of approximation, the admittance of C_0 is treated as a constant of value jB_0 with $B_0 = \omega_s C_0 = 1/X_0$. Its normalized value is thus $jR_1/X_0 = jM^{-1}$. Hence if the normalized crystal admittance is $y = g + jb$,

$$g = g_1 = 1/(1 + x^2M^2)$$
$$b = M^{-1} - xM/(1 + x^2M^2)$$

Clearly, the real part of the crystal admittance is just the real part of the motional admittance, being always >0 and having a single maximum of normalized value 1 at $x = 0$. The imaginary part is shifted by the additional contribution of C_0. As a consequence, the susceptance b may have two, one, or no zeros rather than the single zero at $x = 0$ possessed by the motional susceptance. The positions of the zeros on the x axis are given by the roots of the quadratic equation

$$x^2 - x + M^{-2} = 0$$

ie,

$$x = \{1 - (1 - 4M^{-2})^{1/2}\}/2$$

or

$$x = \{1 + (1 - 4M^{-2})^{1/2}\}/2$$

If $1 > 4M^{-2}$ or alternatively if $M > 2$, then two zeros exist. If $M = 2$ then the zeros coincide, and if $M < 2$ then there are no zeros.

For the case $M \gg 2$, the two zero locations are accurately given by the approximate formulae

$$x = M^{-2}$$
$$x = 1 - M^{-2}$$

On the other hand, when $M = 2$, the zeros coincide at

$$x = 0.5$$

Figure 6.9 shows a normalized plot of the crystal susceptance as a function of the normalized frequency x. In the present degree of approximation where C_0 is regarded as having a constant admittance in the frequency range of interest, the extrema of b occur at the same frequencies as those of b_1, ie, at $x = \pm M^{-1}$.

NOTE: the assumption that C_0 can be regarded as having a constant reactance or admittance in the frequency range of interest is well justified in the case of quartz resonators where the resonance frequencies are very closely grouped. However, for resonators made from materials with large

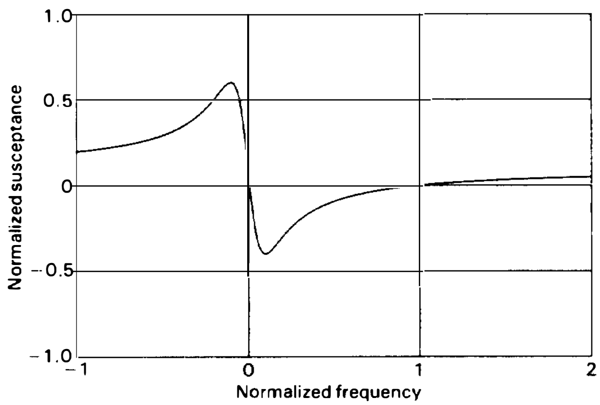

Fig. 6.9 Normalized crystal susceptance ($M = 10$).

electromechanical coupling and hence widely spaced resonance frequencies, the assumption may not be justified. Then an exact analysis such as that of Ballato (1970) must be used.

6.3.3 Phase characteristics

The phase angle ϕ of the crystal admittance is given by

$$\tan(\phi) = b/g$$
$$= (1 + x^2M^2 - xM^2)/M$$

Of particular importance in the zero phase measurement technique (Chapter 8), and in frequency stability considerations (Chapter 9), is the slope of the phase-frequency characteristic in the region of zero phase. Differentiating with respect to the normalized frequency x leads at once to

$$\mathrm{d}(\tan(\phi))/\mathrm{d}x = M(2x - 1)$$

Differentiating the defining equation for x with respect to the angular frequency ω gives

$$\mathrm{d}x/\mathrm{d}\omega = 2\omega Q R_1 C_0/\omega_s = 2Q/(\omega_s M)$$

where the last expression is valid to a high degree of accuracy in the vicinity of the resonance frequencies. Combining these expressions, it follows that

$$\mathrm{d}(\tan(\phi))/\mathrm{d}\omega = 2(Q/\omega_s)(2x - 1)$$

and also that

$$\mathrm{d}\phi/\mathrm{d}\omega = 2(Q/\omega_s)(2x - 1)\cos^2(\phi)$$

Clearly then in the neighbourhood of the series resonance frequency where x and ϕ are close to zero, the phase slope is primarily determined by the resonator Q.

6.3.4 The admittance circle

It can easily be verified from the expressions for g and b given previously, that

$$(g-\tfrac{1}{2})^2 + (b-M^{-1})^2 = (\tfrac{1}{2})^2$$

This is the equation of a circle in the $g+jb$ plane, with radius 1/2 and centre at the point $(\frac{1}{2}, M^{-1})$. Figure 6.10 illustrates the case for $M=10$. Each point on the circle corresponds to a value of frequency, with the frequency increasing as the circle is traversed in a clockwise direction.

The zeros of b, calculated in the preceding section, appear in Fig. 6.10 as the intersections of the admittance circle with the g axis. Clearly, as M decreases, the circle moves upwards along the b axis. The points of intersection gradually move closer until at $M=2$ they coincide at the point $(\frac{1}{2}, 0)$. For M smaller than 2, the circle does not intersect the g axis.

The point P_2 in Fig. 6.10 is the point where the magnitude of the crystal admittance reaches its maximum. It lies on the line from the origin of the g, b plane through the centre of the admittance circle. It thus follows that at the frequency of maximum admittance

$$b/g = M^{-1}/0.5 = 2/M$$

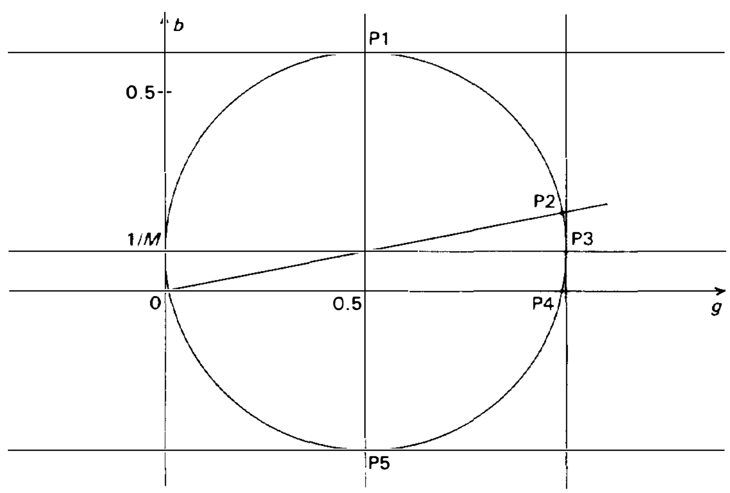

Fig. 6.10 Admittance circle ($M=10$).

The frequency of maximum admittance is therefore a root of the quadratic equation

$$x^2 - x - M^{-2} = 0$$

The roots are

$$x = \{1 - (1+4M^{-2})^{1/2}\}/2$$
$$x = \{1 + (1+4M^{-2})^{1/2}\}/2$$

For $M \gg 1$, the roots are accurately given by

$$x = -M^{-2}$$
$$x = 1 + M^{-2}$$

The root $x = -M^{-2}$ is the frequency of maximum admittance, and the other root is the frequency of minimum admittance.

6.3.5 The resonance frequencies

The admittance circle in Fig. 6.10 shows very closely the distinction between the three characteristic frequencies which may be loosely termed resonance frequencies of the crystal. The frequencies in question are (a) the frequency of maximum admittance, f_m, (b) the frequency of maximum conductance, f_s, and (c) the frequency of zero susceptance or zero phase, f_r. These correspond to the points P_2, P_3, and P_4, respectively, in Fig. 6.10. In addition, there is another pair of characteristic frequencies close to the normalized frequency $x=1$. These are the frequency of minimum admittance, f_n, and a second frequency of zero susceptance or phase, f_a. However, the first group is of most practical importance:

(a) The frequency of maximum admittance f_m is that frequency that would be measured in a simple transmission test circuit, and corresponds to the frequency at which maximum current flows through the crystal.
(b) The frequency of maximum conductance f_s is the resonance frequency of the motional arm of the equivalent circuit. It is the frequency as measured by admittance bridge methods when the capacitance C_0 is balanced out, or by transmission methods when C_0 is either physically tuned out or compensated for mathematically.
(c) The frequency of zero phase f_r, if it exists, is the lower of the two frequencies at which the crystal presents a purely resistive impedance. It has been adopted by the IEC in Publication 444 as the standard parameter for the characterization of crystal frequency.

6.3.6 Numerical estimates for the differences between f_m, f_s, and f_r

If the normalized frequencies corresponding to f_m, f_s, and f_r are x_m, x_s, and x_r, respectively, then from the preceding analysis

$$x_m = \{1 - (1+4M^{-2})^{1/2}\}/2$$
$$x_s = 0$$
$$x_r = \{1 - (1-4M^{-2})^{1/2}\}/2$$

Figure 6.11 shows x_m and x_r as a function of the figure of merit M for the range $M=2$ to $M=100$. Clearly for $M > 10$ both x_m and x_r are both close to zero, allowing the use of the approximate expressions

$$x_m = -1/M^{-2}$$
$$x_r = +1/M^{-2}$$

Removing the frequency normalization leads to the following expressions for the fractional frequency differences

$$(f_m - f_s)/f_s = -r/(2Q^2)$$
$$(f_r - f_s)/f_s = +r/(2Q^2)$$

For AT fundamental mode crystals with $r \sim 200$ and $Q \sim 50\ 000$ the fractional frequency error is 4×10^{-8} or 0.04 ppm, which is practically negligible.

For third overtone resonators with $r \sim 2000$, and $Q \sim 100\ 000$ the error increases to 1×10^{-7} or 0.1 ppm, still negligible for most purposes.

However, for fifth and higher overtones the frequency differences become significant. For a fifth overtone unit with $r \sim 7000$, and $Q \sim 70\ 000$, the error is 0.7 ppm, whereas for seventh and ninth overtone units the figure of merit will generally be sufficiently low to warrant the use of the exact

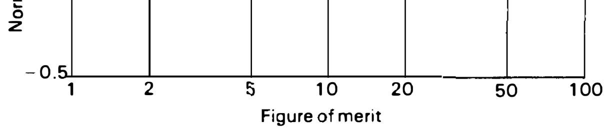

Fig. 6.11 Normalized frequencies vs figure of merit.

formulae for the frequency error. Taking an $r \sim 20\,000$ and $Q \sim 60\,000$ as typical of a seventh overtone crystal, then $M = 3$ and

$$x_m = -0.1$$
$$x_r = +0.127$$

Therefore the fractional frequency errors are -2.5 ppm and $+3.2$ ppm, respectively.

For a ninth overtone, the figure of merit M will typically be <2, so that f_r will not exist. Assuming $M = 1.5$, x_m will be -0.33 and for an r value of 40 000, the fractional frequency error of f_m will be -4.2 ppm.

6.3.7 Crystal impedance at resonance and anti-resonance

For a crystal with a figure of merit $M > 10$, it has been demonstrated that the resonance frequencies are closely grouped around the normalized frequency $x = 0$, and the anti-resonance frequencies are correspondingly grouped around $x = 1$. At these frequencies, the crystal susceptance and therefore the reactance are close to zero, so that the normalized crystal impedance z is almost purely resistive and is given by $z = 1/g$.

If r_r and r_a are the impedance values at resonance and anti-resonance, ie, at $x = 0$ and $x = 1$, then from the expression

$$g = 1/(1 + x^2M^2)$$

it follows that

$$r_r = 1$$
$$r_a = 1 + M^2 \sim M^2$$

NOTE: in the older literature, the term EPR or 'equivalent parallel resistance' is often used for the resistance of a crystal unit when operated at parallel or anti-resonance with a parallel load capacitor C_L. Removing the normalization in the expression for r_a above and writing $C_0 + C_L$ for C_0 gives the following relation between the EPR and the motional resistance R_1:

$$\text{EPR} = 1/[R_1(\omega_s(C_0 + C_L))^2]$$

6.4 EFFECTIVE CRYSTAL PARAMETERS WITH A SERIES LOAD CAPACITOR

The shift in the resonance frequencies of a crystal unit caused by a load capacitor has already been discussed in Section 6.2.1 on the basis of the lossless equivalent circuit. Analysis including losses demonstrates that a crystal unit with a series load capacitor is equivalent to a crystal unit without a load capacitor but with modified parameter values. The equivalence can be used to provide a means of obtaining parameter values that would otherwise be impracticable. This is particularly useful in filter design where a wide range of motional inductance values is required.

6.4.1 Circuit equivalence

Figure 6.12 shows the equivalent circuit of a resonator with a series load capacitor C_L. Figure 6.13 shows the equivalent circuit of a resonator with motional parameters L_1', C_1', and R_1', and static capacitance C_0'. The circuits of Figs. 6.12 and 6.13 are equivalent provided that the element values of Fig. 6.13 are suitably defined in terms of those of Fig. 6.12.

First, consider the behaviour of the circuits at high frequencies. The inductive arms may then be regarded as open-circuit, when it becomes clear that a necessary condition for equivalence is that

$$1/C_0' = 1/C_0 + 1/C_L$$

ie

$$C_0' = C_0 C_L/(C_0 + C_L)$$

For an arbitrary angular frequency ω, it then follows that

$$Z_0' = Z_0 + Z_L$$

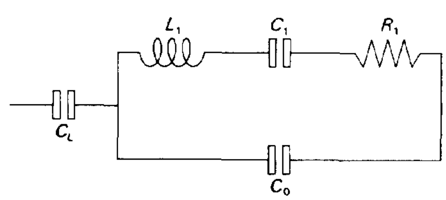

Fig. 6.12 Cry

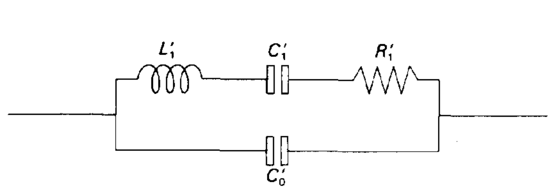

Fig. 6.13 Equivalent circuit for Fig. 6.12.

where Z_0', Z_0, and Z_L are the impedances of C_0', C_0, and C_L at frequency ω. Defining Z_m and Z_m' to be the impedances of the motional arms in Figs. 6.12 and 6.13, respectively, then by inspection the total admittance Y of crystal and load capacitor in Fig. 6.12 can be written

$$Y = (Z_0 + Z_m)/(Z_0 Z_L + Z_L Z_m + Z_m Z_0)$$

If now the admittance $j\omega C_0'$ is subtracted from Y, the equivalence is proved if the resulting impedance $1/(Y - j\omega C_0')$ can be put into the form Z_m'.

$$\begin{aligned} Y - j\omega C_0' &= (Z_0 + Z_m)/(Z_0 Z_L + Z_L Z_m + Z_m Z_0) - 1/(Z_0 + Z_L) \\ &= Z_0^2/(Z_0 + Z_L)(Z_0 Z_L + Z_L Z_m + Z_m Z_0) \end{aligned}$$

Therefore

$$1/(Y - j\omega C_0') = (1 + Z_L/Z_0)\{Z_L + (1 + Z_L/Z_0)Z_m\}$$

Since $Z_L/Z_0 = C_0/C_L$, a real constant K can be defined by

$$K = 1 + Z_L/Z_0 = 1 + C_0/C_L$$

Clearly, K is always > 1, since C_0 and C_L are positive. Then

$$1/(Y - j\omega C_0') = K(Z_L + KZ_m)$$

which has the form of Z_m' provided that

$$L_1'/L_1 = R_1'/R_1 = K^2$$
$$1/C_1' = K/C_L + K^2/C_1$$

Both the motional resistance and inductance are multiplied by the factor K^2. Since $K > 1$ for all values of C_L, the effective parameters are always increased by the addition of a series load capacitor. Since both resistance and inductance are increased by the same factor, the Q factor remains unchanged. The effective motional capacitance is however always decreased.

The effective capacitance ratio C_0'/C_1' is easily calculated to be

$$C_0'/C_1' = (C_0/C_1)\{(C_1 + C_0 + C_L)/C_L\}$$

or to a close degree of approximation

$$C_0'/C_1' = K(C_0/C_1)$$

Thus the effective capacitance ratio is always increased. Since the figure of merit M is defined as the quotient Q/r, where r is the capacitance ratio, and since the effective Q is not changed by the addition of a load capacitance, it follows that the presence of a load capacitance always degrades the figure of merit.

From the formulae given above for L_1' and C_1', it easily follows that if ω_s and ω_s' are defined by $\omega_s'^2 L_1' C_1' = \omega_s^2 L_1 C_1 = 1$ then

$$\omega_s'^2 = \omega_s^2 \{(1 + C_1/(C_0 + C_L))\}$$

To a good degree of approximation

$$(\omega_s' - \omega_s)/\omega_s = C_1/2(C_0 + C_L)$$

In the lossless case dealt with in Section 6.2.2, these formulae give directly the shift of the load resonance frequency relative to the resonance frequency. In the general case, for most situations of practical interest, ie, where the figure of merit is >10, the resonance frequency is still very close to the frequency ω_s, so that the formulae can still be applied with little error.

In the lossless case, the anti-resonance frequency ω_a is equal to the frequency ω_p defined by

$$\omega_p^2 = \omega_s^2(1 + C_1/C_0)$$

It is easily shown that if the frequency ω_p' is defined by

$$\omega_p'^2 = \omega_s'^2(1 + C_1'/C_0')$$

then $\omega_p' = \omega_p$. Thus the lossless anti-resonance frequency is not shifted by a series load capacitor. In the lossy case, provided the figure of merit is sufficiently large, the anti-resonance frequency remains close to ω_p, so that for practical purposes the shift in the anti-resonance frequency can be ignored.

It is sometimes the case that a crystal filter design requires a wider range of crystal inductances than can easily be achieved directly. The equivalence of the circuits in Figs. 6.12 and 6.13 provides one solution to the solution by allowing the use of a low inductance crystal with a series capacitor to simulate a higher inductance crystal.

To state the problem precisely, assume that the filter design requires a crystal of frequency f_s' and motional inductance L_1', while the available crystals have a motional inductance of L_1 and a static capacitance C_0. The problem is to determine a load capacitance C_L and a frequency f_s such that a crystal with parameters f_s, L_1 and C_0, in series with C_L, has the effective parameters f_s' and L_1'. The step-by-step procedure is

(a) Calculate $K = 1 + C_0/C_L$ from $K^2 = L_1'/L_1$.
(b) Calculate C_L from K.
(c) Calculate f_s from $(f_s' - f_s)/f_s' \sim C_1/2(C_0 + C_L)$ where C_1 is obtained from $\omega_s'^2 L_1 C_1 \sim 1$.