3 Energy trapping

Within the linear theory of piezoelectric materials, simple and exact solutions are available for the idealized problem of thickness modes in thin plates. These solutions are invaluable for the basic understanding they give of the physical situation, but are not by themselves sufficient to give either a full appreciation of observed resonator behaviour, or to predict precise numerical results. Extensions to the simple theory are required in two directions. Firstly, non-linear effects are explicitly excluded from consideration in what is by definition a linear theory. Secondly, and most immediately important, the restriction to pure thickness modes has to be removed to give an understanding of such key phenomena as *energy trapping* and the coupling of the thickness modes to other modes dependent on the finite lateral dimensions of the plate.

3.1 WAVE PROPAGATION IN THIN PLATES

Pure thickness modes have a straightforward physical interpretation in terms of standing wave patterns set up by the reflection at the major surfaces of the plate of plane waves travelling in the thickness direction. The assumption of plane waves implies the assumption that the lateral dimensions of the plate, that is the length and width of a rectangular plate or the diameter of a circular plate, can be regarded as infinite. In practice, the ratio of the lateral dimensions to the thickness can vary from several hundred to one for very high frequency plates, to less than ten to one for low frequencies. Consequently, the thickness mode analysis cannot be expected to be anything more than a rough guide to the behaviour of plates in the low frequency region.

To take account of the finite lateral dimensions of an actual resonator, the most natural approach is to extend the interpretation in terms of standing wave patterns. That is, as well as considering waves propagating in the thickness direction and being reflected from the major surfaces, waves should also be considered that propagate in the plane of the plate and are reflected at the plate edges. An analysis of this sort would then be expected to cover not only the influence of the finite lateral dimensions on the thickness modes, but also as special cases the length extensional, flexural and face shear modes discussed in Chapter 1.

A prerequisite of such a program is a clear understanding of the propagation of waves in unbounded plates. As shown in the previous chapter, in an unbounded material, for any given direction of propagation, there are just three possible types of plane wave solution, making analysis straightforward. Additionally, the phase velocities of the waves are constants, independent of wave number, so that if the frequency ω is plotted against wavenumber k, the result is the set of three straight lines shown in Fig. 3.1. The slopes of the lines are just the three characteristic phase velocities V_k (Section 2.1). In this case, the group velocity $d\omega/dk$ for each mode is also constant and equal to the phase velocity. In general, a plot of ω versus k for a given type of wave or mode is termed the dispersion relation for that mode. The mode is termed dispersive when the phase velocity depends on the wavenumber, that is when the group velocity $d\omega/dk$ is not constant. In such case, the phase and group velocities are no longer equal.

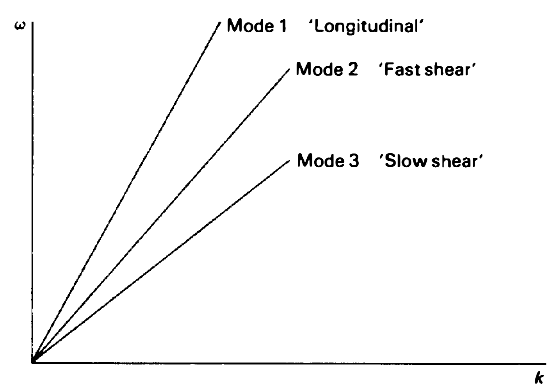

Fig. 3.1 Dispersion relations for plane waves.

In the case of waves in the plane of an unbounded plate, say with major surfaces at $y = x_2 = \pm h$, the situation is vastly more complicated as compared to the thickness mode case. The essential physical model of plate waves as being made up of combinations of plane waves successively reflected from the major surfaces remains correct, but it turns out that the number of possible combinations gives rise to many different types of plate wave. These can be broadly classified according to the nature of the particle displacements involved as *flexural* (F), *extensional* (E), *face shear* (FS), *thickness shear* (TS) and *thickness twist* (TT). Figure 3.2 illustrates the displacements for each type of wave.

The essential analysis required to obtain the dispersion curves for thickness

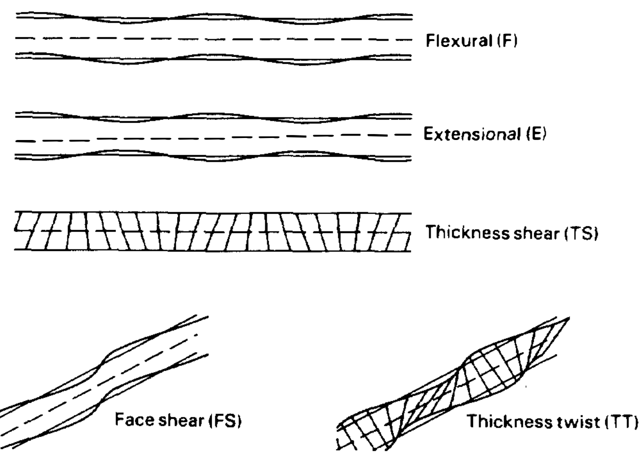

Fig. 3.2 Principal displacements for plate waves.

shear, flexural and extensional waves in *isotropic* plates was first given by Rayleigh (1889). As shown in Appendix 6, the satisfaction of the boundary conditions for stress-free major surfaces results in a transcendental equation involving the wavenumbers in both the thickness direction and the direction of propagation. The solutions of this equation determine the various branches of the dispersion relation for the isotropic plate. The dispersion curves for thickness twist and face shear modes are also obtained in Appendix 6 for the isotropic case, the analysis for these modes being very much simpler than for the thickness shear group. Figures 3.3 and 3.4 show very much simplified sketches of the normalized dispersion curves for those modes of most relevance in resonator theory, including only face shear, flexural,

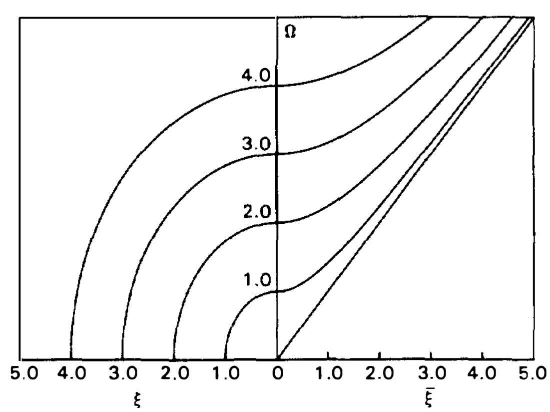

Fig. 3.3 Dispersion relations for TT and FS waves.

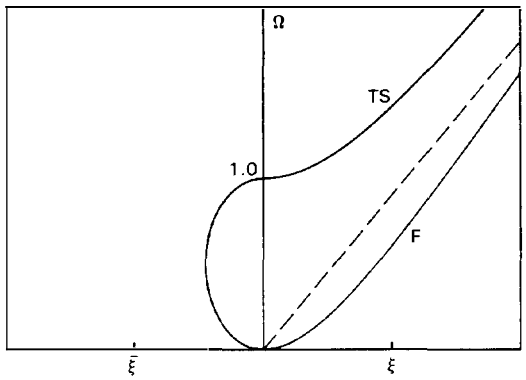

Fig. 3.4 Dispersion relations for TS and F waves.

thickness shear and thickness twist modes. The frequency Ω is normalized with respect to the lowest thickness mode frequency, whereas the wave-number along the plate ξ is expressed in terms of the number of half wave-lengths that would be contained in the thickness of the plate. In the Figures, the right-hand portion of the curves refers to real wavenumbers ξ, and the left-hand portion refers to imaginary ξ. The physical significance of this is that for imaginary wavenumbers, the particular mode in question is no longer a travelling wave, but an *evanescent* wave whose amplitude decreases exponentially with distance, and which is incapable of transmitting energy.

The important feature of the curves in Figs. 3.3 and 3.4 is that for both TT and TS modes, cut-off frequencies exist such that for frequencies below the cut-off for a particular mode, that mode exists only as an evanescent wave. As shown in Appendix 6, the cut-off frequencies correspond to the thickness mode frequencies. A simple physical explanation can be given of the cut-off phenomenon in terms of the model of the plate wave based on successive reflections of plane waves from the plate surfaces. In this model, there are clearly two distinct limiting situations. In the first, the component plane waves are actually travelling along the plate, parallel to the major surfaces, so that no reflections occur. Then the plate wave velocity coincides with the bulk wave velocity. As the angle of incidence of the component plane waves decreases from 90° towards normal incidence, then it is physically clear that the rate of transfer of energy *along* the plate, that is the *group velocity*, must decrease because of the increased path length due to an increased number of reflections. Clearly, at normal incidence, the second limiting case is reached, which is just that of pure thickness vibrations, with no energy being transmitted along the plate. This is the cut-off condition, in which the group velocity of the plate wave is zero.

3.2 RESONANCE FREQUENCIES FOR FINITE PLATES

Given a knowledge of the characteristics of waves in unbounded plates, the problem of determining the normal modes of vibration of a finite plate can in principle be regarded as equivalent to the problem of determining a linear combination of plate waves that will satisfy the boundary conditions on the edges of the plate. In this approach, since all the plate waves individually satisfy the boundary conditions on the major surfaces, it is only the edge conditions that need to be taken into account. Unfortunately, in all but the simplest limiting cases, solution in terms of a finite number of wave functions do not exist. This is so even in the case of isotropic plates; the additional complications of anisotropy and piezoelectricity only serve to complicate matters.

In the absence of exact solutions to the general problem, a great deal of work has been done in determining approximate solutions. There have been essentially two distinct approaches. The first uses the full set of field equations and boundary conditions and seeks to determine approximate solutions by such methods as perturbation theory or variational techniques. This approach is treated in detail by Holland and EerNisse (1969). The second approach is to use whatever special features of the problem exist to reduce the full set of equations to a more manageable, approximate set that can be solved exactly. In the context of resonator theory, the pioneering work in this area has been done by Mindlin and his co-workers in a series of papers beginning in 1951. The main features of this work are described by Tiersten (1969).

For present purposes, the details of both approaches can be passed over. The essential physical principles can be understood in simple terms by adopting as a general rule, applicable to any type of resonating system where a particular type of wave motion is predominant, the assumption that the condition for resonance is that an integral number of half wavelengths be contained in the dimension along the direction of propagation.

For thickness modes in rotated Y-cut quartz plates, it has been shown in Chapter 2 that the only piezoelectrically excited mode has $u = u_1$ as the only non-zero displacement. Consequently, in considering the modes of vibration of finite plates it is reasonable to assume that u will continue to be the major displacement component. From the discussion of isotropic plates in Appendix 6, the plate waves of immediate interest will then be thickness twist and face shear modes propagating in the $z = x_3$ direction, and thickness shear and flexural modes propagating in the x direction. In other words, the model of the rotated Y-cut resonator in terms of pure thickness modes is to be generalized by considering these specific types of wave motion in the plate.

For example, consider a plate with major faces at $y = \pm h$ as before, unbounded in the z direction, but with edges at $x = \pm L$. Then for those resonances associated with flexural waves propagating along x, the above

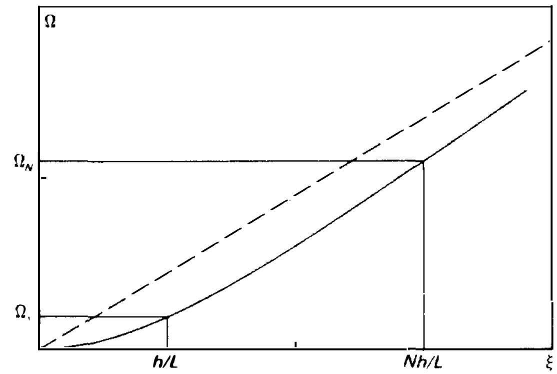

Fig. 3.5 Resonance frequencies for flexural modes.

rule, in conjunction with the dispersion relation for flexural waves, allows estimates to be made of the resonance frequencies as follows. First, from the definition of the normalized wavenumber ξ, the ξ values corresponding to the resonance condition are just

$$\xi = Nh/L \tag{3.1}$$

Figure 3.5 shows the dispersion relation for flexural modes, with the points Nh/L marked off along the ξ axis. The normalized resonance frequencies Ω_N are then determined as the frequency values corresponding to these ξ values. Notice that for Ω_N to be comparable with $\Omega = 1$, the lowest thickness shear frequency, N must be of the order of L/h, the length-to-thickness ratio of the plate.

The same argument may be applied to the estimation of the thickness shear frequencies of the plate. Figure 3.6 shows the dispersion relation for the TS

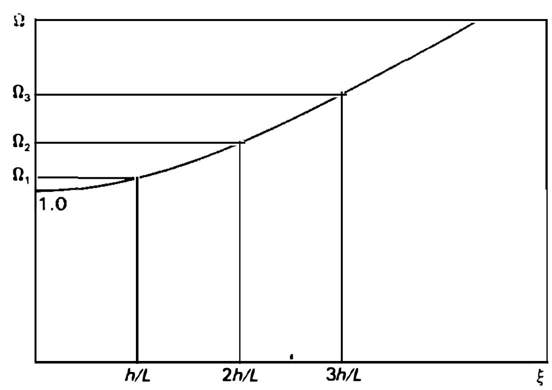

Fig. 3.6 Resonance frequencies for TS modes.

branch, again with the points Nh/L marked off on the ξ axis. Once again there are a whole family of resonances associated with the TS waves, but because of the shape of the dispersion curves the resonance frequencies are clustered just above the cut-off frequency. The lowest frequency mode is the main resonance associated with that particular branch of the dispersion relation, whereas the remaining modes are termed the *inharmonic modes* for that branch. For large L/h values, the frequency of the lowest or main resonance tends towards the cut-off frequency, that is the pure thickness mode frequency, just as would be expected.

For a strip resonator of infinite extent in the x direction but bounded by edges at $z = \pm W$, an exactly analogous discussion applies with face shear and thickness twist waves playing the role of flexural and thickness shear waves. Thus there will be a family of inharmonic thickness twist modes and a family of face shear resonances. In the more general case of a rectangular plate bounded in both directions, then the expectation must be that the inharmonic responses will consist of both thickness twist and thickness shear modes, and that both flexural and face shear resonances will be found. A more precise analysis reveals that not only is this the case, but that more complicated resonances exist, such as thickness twist overtones of the inharmonic thickness shear modes (Mindlin and Spencer, 1967). Nevertheless, the essential physical picture of the finite plate remains as one where the pure thickness mode suffers a slight upward displacement in frequency and is accompanied by a cluster of closely spaced inharmonics at higher frequencies, together with separate families of flexural and face shear modes.

At this point it is convenient to note that the inharmonics, as resonances associated with the same branch of the dispersion relation as the main thickness response, are governed by the same set of material constants. Consequently, their frequency–temperature characteristics can be expected to track the characteristic of the main response. However, the flexural and face shear modes, being associated with different branches of the dispersion relation and thus with a different set of material constants, can generally be expected to show quite different temperature characteristics. In practice, for the AT- and BT-cuts, these modes generally have large negative temperature coefficients compared to an essentially zero coefficient for the thickness mode and its inharmonics.

3.3 COUPLED MODES IN FINITE RESONATORS

Figure 3.7 shows again the dispersion relations for the flexural and thickness shear modes in a plate with edges at $x = \pm L$. Following the previous discussion the main thickness mode will have the resonance frequency Ω determined by the TS branch at the wavenumber $\xi = h/L$. At this frequency,

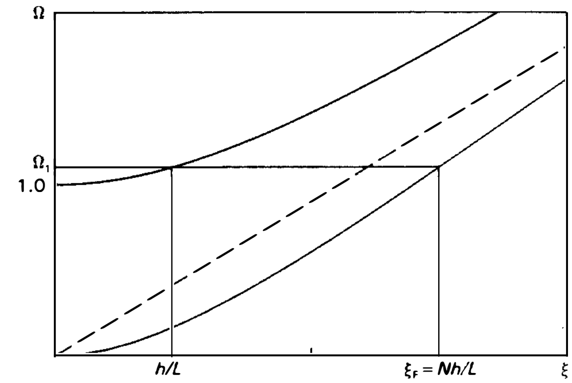

Fig. 3.7 Condition for coupling between TS and F modes.

the flexural mode will have the wavenumber ξ_F as shown in Fig. 3.7. If it should happen that $\xi_F = Nh/L$ for some integer N, then clearly the resonance frequencies of the main thickness mode and the Nth flexural mode coincide. In such a case, the basic assumption underlying the previous discussion is no longer valid, since it is no longer true that the resonance is primarily associated with a single type of wave motion.

Figure 3.8 shows in schematic form how the thickness modes of a strip resonator with edges at $x = \pm L$ vary with the length to thickness ratio L/h. As shown already, as L/h increases the frequencies tend in the limit to the frequencies of the pure thickness modes, and for reasonably large L/h values the variation is small. Also shown in the figure is the dependence of the frequency of a flexural mode with large N on the parameter L/h. Examination of the dispersion curves for flexural modes shows that for large wavenumbers, the dispersion relation approximates a straight line. This in

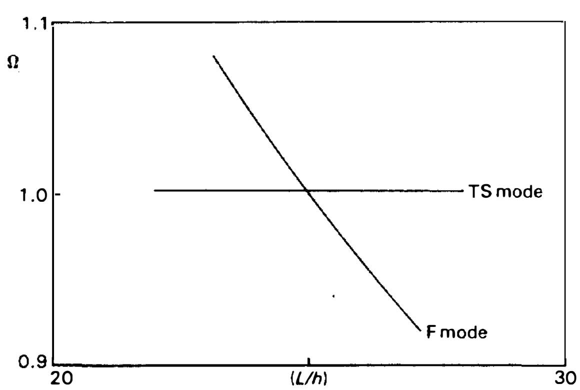

Fig. 3.8 TS and F mode frequencies vs (L/h).

turn implies that in this region the propagation of flexural waves is non-dispersive and that the dispersion curve can be approximated by a linear expression

$$\Omega = V\xi / V_s \tag{3.2}$$

where V is the phase velocity. Since for the Nth flexural mode in a strip, $\xi = Nh/L$, it follows that

$$\Omega = N(V/V_s)/(L/h) \tag{3.3}$$

Hence the curve shown in Fig. 3.8 for the flexural mode is part of a rectangular hyperbola.

Figure 3.8 shows the two curves for the thickness and flexural modes intersecting at a specific value of L/h. As already indicated, the basic assumption of the previous discussion is invalid at this point. A more exact analysis, supported by experiment, shows that if the relevant mode frequencies of a strip resonator are plotted as a function of L/h, with L being gradually reduced by, for example, grinding the edges of the resonator, the actual behaviour is as shown in Fig. 3.9. Here at the point A on the thickness mode curve, the flexural mode is too far away in frequency to have any effect. The particle displacements are predominantly thickness shear, as in Fig. 3.2. Moving down the thickness mode curve to point B, the flexural mode begins to influence the character of the displacements, and the frequency is perturbed upwards. Moving still further along the same curve, the character of the displacements changes to predominantly flexural, until finally a complete transition is made from thickness mode to flexural mode behaviour at point C. In the same way, starting at the point A' on the flexural mode curve and proceeding through the points B' and C', the mode character changes from flexural at A', through a combination of flexure and thickness shear at B', to purely thickness shear at C'. Both curves thus show a

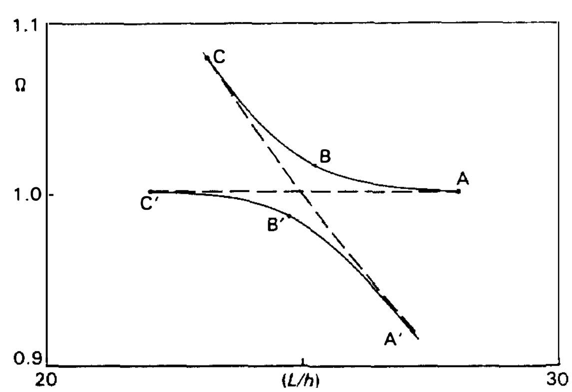

Fig. 3.9 Effect of coupling on TS and F modes.

continuous change in character, and do not intersect at the critical L/h value as predicted by the simple arguments used previously. The two modes in question are said to be *coupled*. The question of the exact mechanism of the coupling is begged by the discussion so far, but it suffices for present purposes to recall the statement made in Section 3.2 that satisfaction of the edge conditions requires a superposition of plate waves; if the dimensions of the resonator are such that only one mode satisfies the resonance conditions, then it is physically reasonable to suppose that this mode will predominate in the solution. However, if two or more modes simultaneously satisfy the resonance conditions, then clearly both modes must be expected to have significant amplitudes. Thus it can generally be supposed that where two modes can be simultaneously excited into resonance, they will be coupled together through the boundary conditions. The obvious exception to this rule would be the case where one or both modes could satisfy the edge conditions independently of the other, but as already indicated this is a rare situation.

The previous paragraphs apply equally well to both piezoelectric and non-piezoelectric resonators. In the former case, some further conclusions can be drawn based on the general character of the displacements in the thickness and flexural modes. Firstly, in the thickness mode case, when no coupled modes are present, the displacements are such that to a first approximation all points on each major face are moving in phase. Consequently, supposing the vibrations to be excited by electrodes on the major surfaces, the piezoelectric charges produced on the electrodes will likewise be in phase. On the other hand, in the case of the flexural modes at frequencies near the thickness mode, the wavenumber will be large (approximately L/h times the thickness mode wavenumber), and so there will be a large number of phase reversals from one edge of the plate to the other. The piezoelectric charges will therefore tend to cancel out, resulting in a relatively weak excitation of the flexural mode. In equivalent circuit terms, this implies that the effective resistance of the flexural mode is much higher than that of the thickness mode. Now if the resonator is used in a practical oscillator circuit, the preferred mode of vibration will be the one with the lower effective resistance. Hence if the output frequency of the oscillator is plotted as a function of the L/h ratio, the typical behaviour would be as shown in Fig. 3.10. Here at point A, corresponding to point A of Fig. 3.9, the resonator is working in its relatively low resistance thickness mode. As L/h is reduced and the coupling to the flexural mode increases, the oscillator frequency is 'pulled' upwards and at the same time the effective resistance increases. As L/h is further decreased, the effective resistance continues to increase, until at point B say, the condition is reached where the resistance at B' is less than at B. Then the oscillator 'jumps' in frequency from B to B', and with further reduction in L/h, continues along the curve through the point C'.

Although specific mention has been made only of coupling between the thickness shear and flexural modes, similar coupling exists between the

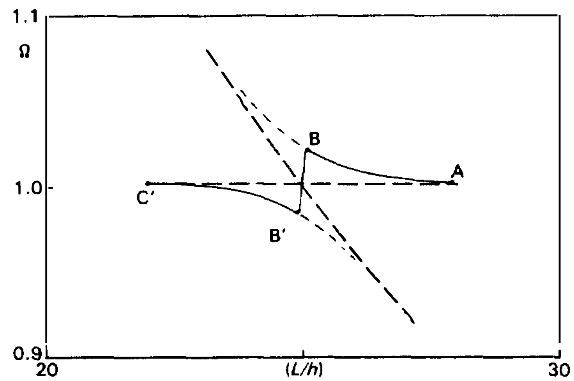

Fig. 3.10 Effect of coupling on oscillator frequency.

desired mode and other modes such as face shear and extensional. In all cases, the effects of such coupling are detrimental. At the end of the previous section it was pointed out that the temperature characteristics of the flexural and face shear modes are quite different to those of the thickness modes. Consequently, for resonators operating over wide temperature range, it can easily happen that although a particular flexural or other unwanted mode may be far removed from the desired thickness mode at one temperature, it may nevertheless coincide with the wanted mode at some other temperature in the range. When this occurs, coupling of the modes will produce frequency jumps at the particular temperature in question, usually accompanied by sharp increases in the crystal resistance. These effects of temperature-dependent coupling are variously known as 'activity dips' or 'bandbreaks'.

In the early days of the manufacture of AT- and BT-cuts, the need to minimize such coupling resulted in the introduction of elaborate dimensioning rules governing the allowed ranges of the length-to-thickness and width-to-thickness ratios. These resulted in plates of only slightly different frequencies having to be ground to different lengths and widths, considerably complicating the manufacturing process as compared to present-day techniques.

3.4 ENERGY TRAPPING

The key concept of the cut-off frequency for thickness twist and thickness shear waves in plates has already been introduced (Section 3.1 and Appendix 6). Consider for definiteness thickness twist waves in an isotropic plate, with thickness $2h$ along the y direction, particle displacement u along the x direction, and propagating along the z direction. From Appendix 6, the Nth odd overtone mode has the displacement

$$u = A\sin(N\pi y/2h)\exp(j\omega t - jk_z z) \tag{3.4}$$

For real k_z this represents a travelling wave, whereas for an imaginary k_z, say with $k_z = j\bar{k}_z$, the expression becomes

$$u = A\sin(N\pi y/2h)\exp(\bar{k}_z z)\exp(j\omega t) \tag{3.5}$$

representing an evanescent wave.

Now consider a plate as above except that in the region $|z| < W$ the thickness is $2H$, with $H > h$ (Fig. 3.11). Then in the central region the cut-off frequencies are lowered in the ratio h/H. Thus if the dispersion curves for both the central and outer regions of the plate are plotted on the same axes, the result will be as in Fig. 3.12, where only the Nth TT mode is shown for

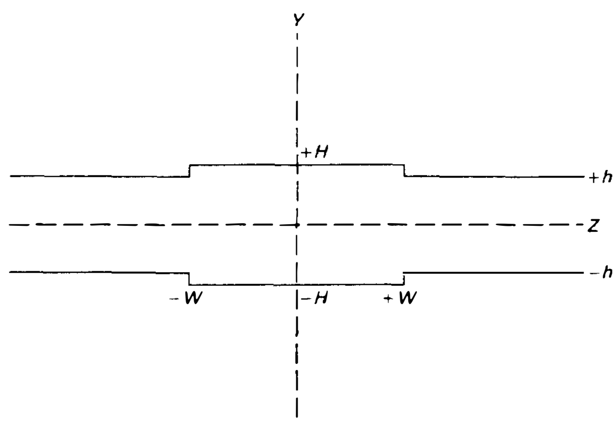

Fig. 3.11 Geometry of energy trapped resonator.

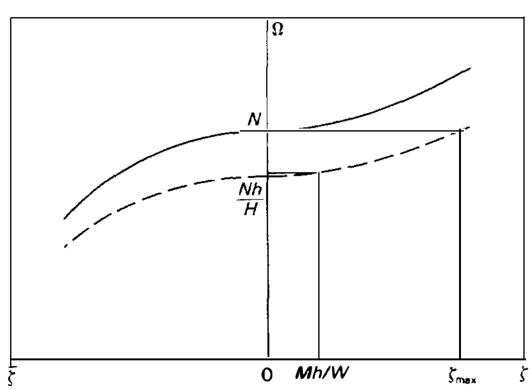

Fig. 3.12 Dispersion relations for energy trapping.

clarity. For normalized frequencies $\Omega > N$, that is above the cut-off for both regions, TT waves can propagate in both central and outer areas. Similarly, for $\Omega < Nh/H$, only evanescent waves can exist. However, in the frequency range $Nh/H < \Omega < N$, propagating waves can exist in the central region but not in the outer regions. This immediately gives rise to the possibility of producing a resonator by utilizing the reflections of the waves at the edges of the central region to set up a standing wave system. Such a resonator is termed a ***trapped energy resonator***, since the vibrational energy is 'trapped' in the central region of the device.

The width $2W$ is the critical factor in determining the resonance frequencies of the trapped modes, via the general assumption that resonance will occur when an integral number of half wavelengths are contained in the critical dimension. In terms of the normalized wavenumber $\zeta = 2hk_z/\pi$, this means that the relevant ζ values are

$$\zeta = Mh/W \tag{3.6}$$

where M is an integer. Eqn (3.6) is exactly analogous to Eqn (3.1) in Section 3.2, except for the fact that although the integer N in Eqn (3.1) can take on all values, the corresponding integer M in Eqn (3.6) can only take on a finite number of values. The reason for this can be seen from Fig. 3.12, where the points Mh/W are marked off on the ζ axis. Clearly for all M greater than some value M_0, the mode frequency will be above the upper limit N, so that those modes will no longer be trapped.

The number of trapped modes depends both on the width $2W$ and the difference between the cut-off frequencies in the two regions. If Δ is the fractional difference between these frequencies

$$\Delta = (H-h)/h \tag{3.7}$$

Also the dispersion relation in the central region is

$$\Omega^2 = (Nh/H)^2 + \zeta^2 \tag{3.8}$$

The maximum value of ζ for a trapped wave is obtained when $\Omega = N$, the cut-off frequency for the outer region, so that

$$\zeta_{max}^2 = N^2[1 - (h/H)^2]$$

or, in the usual case where $(H-h)/h \ll 1$,

$$\zeta_{max}^2 = 2N^2\Delta \tag{3.9}$$

The condition for there to be just M trapped modes is that the maximum wavenumber $\zeta_{max} = Mh/W$, that is

$$(Mh/W)^2 = 2N^2\Delta \tag{3.10}$$

Writing B_M for the ratio W/h, Eqn (3.10) can be rewritten as

$$B_M = M/(2N^2\Delta)^{1/2} \tag{3.11}$$

and for a given Δ, determines the width-to-thickness ratio required to selectively trap the first M thickness twist modes at the Nth overtone.

3.5 MASS LOADING

Although the preceding discussion has been based on the specific case of a plate with a thick central region and a thinner outer region, the phenomenon of energy trapping only requires that there be a difference in cut-off frequencies between the two regions. The precise mechanism by which this difference is achieved is immaterial. In practice, the example considered of a step change in the plate thickness is not used, although such a change could be achieved by selective etching of the crystal blank. The two commonly used techniques for achieving different cut-off frequencies in different regions of the resonator employ in the one case the frequency lowering due to the mass of the electrodes, and in the other a gradual change in thickness obtained by *contouring* or *bevelling* the crystal blank.

The *mass loading* effect results from the modification of the boundary conditions on the major surfaces of the resonator due to the presence of the electrodes. For major surfaces at $y = \pm h$, the stress free boundary conditions used in Chapter 2 and Appendix 6 were simply the vanishing of the surface tractions t_{2k}. In the presence of electrodes of mass density σ per unit area, the inertia of the electrodes results in the boundary conditions.

$$t_{2k} = -\sigma \ddot{u}_k \tag{3.12}$$

The relative magnitude of the mass loading is expressed in terms of the ratio R of the electrode mass per unit area to the mass per unit area of the resonator, $R = \sigma/\rho h$. For large R, the use of Eqn (3.12) rather than $t_{2k} = 0$ in the analysis of thickness modes in Chapter 2 leads to frequency equations best solved graphically or numerically (Ballato, 1977). However, it is usually the case with quartz resonators that the mass loading is small, with $R \ll 1$, and then the modified boundary conditions simply lead to the result that the resonance frequencies for thickness modes are lowered by the fractional amount R. That is, if f_m and f are the frequencies with and without mass loading

$$(f - f_m)/f = R \tag{3.13}$$

Again in the case of small R, this result can be carried over to the dispersion relations for wave propagation in mass loaded plates, insofar as the thickness shear and thickness twist branches, at least in the neighbourhood of zero wavenumber, can be regarded as having the same shape as for unloaded plates but shifted down on the frequency axis by the fractional amount R.

Consequently, if a resonator in the form of a thin plate or disc with a central region coated with electrodes is considered, then the mass loading of the electrodes will result in a finite number of TT and TS modes being trapped under the electrodes. The number of such modes will be given by an analogue of Eqn (3.11) with Δ replaced by R. The precise values of the numerical constants involved will of course differ from those in Eqn (3.11), since the analysis leading up to this equation only considered the simplest case of TT waves in an isotropic plate. Nevertheless, the essential points remain that the number of trapped modes increases both with the frequency lowering, in this case the mass loading R, and the size of the electrode. In particular, the possibility exists to limit the number of trapped modes to one only by judicious control of the mass loading and the electrode dimensions. This was first recognized in practice by Bechmann and the critical ratio of electrode dimension to plate thickness required to trap only one mode is known as *Bechmann's number* (for references, see Meeker, 1985).

The practical importance of this is that the energy of the trapped mode, being concentrated under the electrodes, gives rise to a strong resonator response, whereas the modes that are not trapped can propagate over the whole of the resonator. Thus not only is the proportion of the total energy in the mode that is available to the driving circuit restricted, but also these untrapped modes are subject to the usually large damping that occurs at the edges of the resonator plate, for example at the mounting points. The net effect of these two factors is that a properly designed resonator using the energy trapping criteria will have a much 'cleaner' response than other resonators, that is, it will be essentially free from 'unwanted' or 'spurious' responses.

3.6 CONTOURING AND BEVELLING

The second commonly used technique for using the energy trapping phenomenon, that of contouring the crystal to give a gradually changing thickness from the centre to the edge of the blank, was a well established empirical practice (Tyler, 1960) long before its theoretical analysis. The effect of contouring, that is the practice of imparting a spherical curvature to one or both major faces of a resonator as shown in Fig. 3.13, was recognized to be the restriction of the vibrating area of the plate to its central region, with the accompanying advantages of ease of mounting and reduced coupling to unwanted modes at the edge of the blank. Also recognized, but not understood on a theoretical basis, was the pronounced change in the first-order frequency–temperature characteristic brought about by contouring. Empirical data, in the form of curves and tables of 'good' designs, was, as stated above, well established for contoured units by the early 1960s. Similar data for *partially contoured* or *bevelled* blanks, where the spherical curvature is only applied in the outer region leaving a flat central region or *plateau* as in

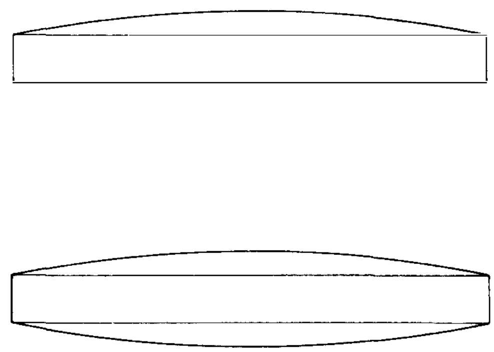

Fig. 3.13 Plano-convex and bi-convex contoured plates.

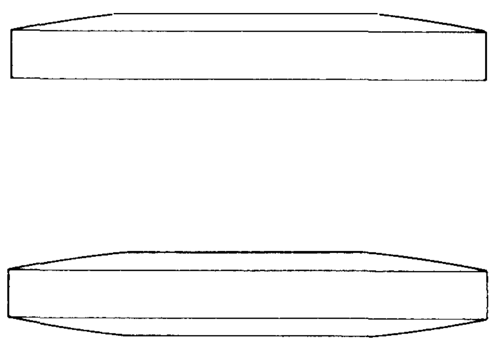

Fig. 3.14 Plano-convex and bi-convex bevelled plates.

Fig. 3.14, was not, however, readily available, and information on such designs is even now only available on a restricted basis.

The analysis of Section 3.4 is not directly applicable to contoured resonators, but nevertheless helps to give a physical understanding. A precise analysis (Tiersten and Smythe, 1979) shows that for the main response of a spherically contoured resonator, the amplitude of vibration falls away exponentially with the *square* of the distance from the centre, rather than as a simple exponential function of distance. This is to be expected on physical grounds since the cut-off frequency itself is a function of distance from the centre. Tiersten and Smythe give explicit solutions for both the main response and the inharmonic responses of a fully contoured resonator, from which the frequencies and the equivalent circuit parameters of each mode can be calculated. To the author's knowledge, no equivalent exact solution is available for the partially contoured or bevelled blank, except in the relatively simple and rarely used case of a *cylindrical* as opposed to a spherical contour. Thus to obtain solutions for the bevelled case, recourse has to be had to numerical methods.