Appendix K

Derivation of Crystal Equations

The equivalent circuit of a crystal is shown in Figure K-1. This circuit is well known, and the definitions of the components are as follows:

C_0 = holder capacitance,
L_1 = motional arm inductance,
C_1 = motional arm capacitance, and
R_1 = motion arm resistance.

If we define

$$Z_0 = \frac{-j}{2\pi f C_0} \tag{K-1}$$

and

$$Z_1 = R_1 + j\left(2\pi f L_1 - \frac{1}{2\pi f C_1}\right), \tag{K-2}$$

then the complex impedance of the crystal at any frequency is given by

$$Z_p = \frac{Z_0 Z_1}{Z_0 + Z_1} = \frac{\left[\frac{-j}{2\pi f C_0}\right]\left[R_1 + j\left(2\pi f L_1 - \frac{1}{2\pi f C_1}\right)\right]}{\left[R_1 + j\left(2\pi f L_1 - \frac{1}{2\pi f C_1} - \frac{1}{2\pi f C_0}\right)\right]} \tag{K-3}$$

$$Z_p = \frac{\left[\frac{2\pi f L_1 - (1/2\pi f C_1)}{2\pi f C_0} - \frac{jR_1}{2\pi f C_0}\right]}{\left[R_1 + j\left(2\pi f L_1 - \frac{1}{2\pi f C_1} - \frac{1}{2\pi f C_0}\right)\right]}. \tag{K-4}$$

For a resonance to occur, Z_p must be resistive and, therefore,

$$\frac{-R_1/2\pi f C_0}{\frac{2\pi f L_1 - (1/2\pi f C_1)}{2\pi f C_0}} = \frac{2\pi f L_1 - \frac{1}{2\pi f C_1} - \frac{1}{2\pi f C_0}}{R_1} \tag{K-5}$$

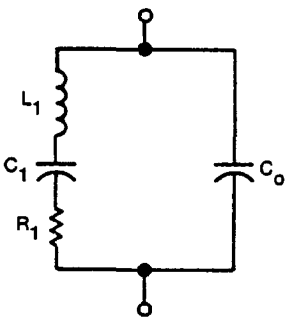

Figure K-1. Crystal equivalent circuit.

$$-R_1^2 = \left(2\pi f L_1 - \frac{1}{2\pi f C_1}\right)\left(2\pi f L_1 - \frac{1}{2\pi f C_1} - \frac{1}{2\pi f C_0}\right) \tag{K-6}$$

$$R_1^2 + 4\pi^2 f^2 L_1^2 - \frac{L_1}{C_1} - \frac{L_1}{C_0} - \frac{L_1}{C_1} + \frac{1}{4\pi^2 f^2 C_1^2} + \frac{1}{4\pi^2 f^2 C_1 C_0} = 0 \tag{K-7}$$

$$(2\pi f)^4 + (2\pi f)^2 \left(\frac{R_1^2}{L_1^2} - \frac{2}{L_1 C_1} - \frac{1}{L_1 C_0}\right) + \left(\frac{1}{L_1^2 C_1^2} + \frac{1}{L_1^2 C_1 C_0}\right) = 0. \tag{K-8}$$

Using the quadratic formula and solving for $(2\pi f)^2$ gives

$$(2\pi f)^2 = \frac{1}{2}\left\{\left(\frac{2}{L_1 C_1} + \frac{1}{L_1 C_0} - \frac{R_1^2}{L_1^2}\right) \pm \left[\left(\frac{2}{L_1 C_1} + \frac{1}{L_1 C_0} - \frac{R_1^2}{L_1^2}\right)^2 - 4\left(\frac{1}{L_1^2 C_1^2} + \frac{1}{L_1^2 C_1 C_0}\right)\right]^{1/2}\right\}, \tag{K-9}$$

$$f = \frac{1}{2\pi}\left\{\left(\frac{1}{L_1 C_1} + \frac{1}{2L_1 C_0} - \frac{R_1^2}{2L_1^2}\right) \pm \left[\left(\frac{1}{L_1 C_1} + \frac{1}{2L_1 C_0} - \frac{R_1^2}{2L_1^2}\right)^2 - \left(\frac{1}{L_1^2 C_1^2} + \frac{1}{L_1^2 C_1 C_0}\right)\right]^{1/2}\right\}^{1/2}. \tag{K-10}$$

We now consider the quantity,

$$\left[\left(\frac{1}{L_1 C_1} + \frac{1}{2L_1 C_0} - \frac{R_1^2}{2L_1^2}\right)^2 - \left(\frac{1}{L_1^2 C_1^2} + \frac{1}{L_1^2 C_1 C_0}\right)\right]^{1/2}$$

$$= \left(\frac{1}{L_1^2 C_1^2} + \frac{1}{L_1^2 C_1 C_0} - \frac{R_1^2}{L_1^3 C_1} + \frac{1}{4L_1^2 C_0^2} - \frac{R_1^2}{2L_1^3 C_0} + \frac{R_1^4}{4L_1^4} - \frac{1}{L_1^2 C_1^2} - \frac{1}{L_1^2 C_1 C_0}\right)^{1/2}$$

$$= \left[\left(\frac{1}{2L_1 C_0} - \frac{R_1^2}{2L_1^2}\right)^2 - \frac{R_1^2}{L_1^3 C_1}\right]^{1/2}. \tag{K-11}$$

For any practical crystal, however, it is normally true that

$$\left(\frac{1}{2L_1C_0}-\frac{R_1^2}{2L_1^2}\right)^2 \gg \frac{R_1^2}{L_1^3C_1}.$$

Then

$$\left[\left(\frac{1}{2L_1C_0}-\frac{R_1^2}{2L_1^2}\right)^2-\frac{R_1^2}{L_1^3C_1}\right]^{1/2} \doteq \left(\frac{1}{2L_1C_0}-\frac{R_1^2}{2L_1^2}\right). \tag{K-12}$$

Then

$$f \doteq \frac{1}{2\pi}\left[\left(\frac{1}{L_1C_1}+\frac{1}{2L_1C_0}-\frac{R_1^2}{2L_1^2}\right)\pm\left(\frac{1}{2L_1C_0}-\frac{R_1^2}{2L_1^2}\right)\right]^{1/2}. \tag{K-13}$$

This equation gives two resonant frequencies; the first, obtained using the minus sign, is series resonance and the other is parallel resonance.

$$f_s \doteq \frac{1}{2\pi}\left(\frac{1}{L_1C_1}\right)^{1/2} \tag{K-14}$$

$$f_a = \frac{1}{2\pi}\left(\frac{1}{L_1C_1}+\frac{1}{L_1C_0}-\frac{R_1^2}{L_1^2}\right)^{1/2}. \tag{K-15}$$

But

$$\frac{1}{L_1C_0} \gg \frac{R_1^2}{L_1^2}, \quad f_a \doteq \frac{1}{2\pi}\left(\frac{1}{L_1C_1}+\frac{1}{L_1C_0}\right)^{1/2},$$
$$f_a \doteq \frac{1}{2\pi}\left(\frac{1}{L_1C_1}\right)^{1/2}\left(1+\frac{C_1}{C_0}\right)^{1/2}. \tag{K-16}$$

But $C_1/C_0 \ll 1$. Therefore, the binomial approximation,

$$(1+x)^n \doteq 1+nx \quad \text{if} \quad x \ll 1,$$

may be used, and

$$f_a = \frac{1}{2\pi}\left(\frac{1}{L_1C_1}\right)^{1/2}\left(1+\frac{C_1}{2C_0}\right) = f_s\left(1+\frac{C_1}{2C_0}\right). \tag{K-17}$$

If $\Delta f = f_a - f_s$, then $\Delta f = (C_1/2C_0)f_s$ and the pullability $\Delta f/f_s = C_1/2C_0$.

The frequency at any load point C_L now can be calculated merely by making C_0 in the equation equal to the holder capacity plus the load capacity. Then

$$\frac{\Delta f}{f_s} = \frac{C_1}{2(C_0+C_L)}. \tag{K-18}$$

Substituting

$$\frac{C_0}{C_1} = r$$

gives

$$\frac{\Delta f}{f_s} = \frac{C_0}{2r(C_0 + C_L)}. \tag{K-19}$$

The resistance of the crystal at any load capacitance C_L can be found conveniently by redrawing the equivalent circuit of a crystal as given in Figure K-1. Here the resultant reactance of L_1 and C_1 is replaced by X, and the circuit of Figure K-2 results.

The impedance of the circuit may be written by inspection as

$$Z = \frac{(R_1 + jx)(jX_{C0})}{R_1 + j(X + X_{C0})}, \tag{K-20}$$

where $X_{C0} = -1/\omega C_0$.

Separating the real and imaginary parts gives, after some algebra,

$$Z = \frac{R_1 X_{C0}^2}{R_1^2 + (X + X_{C0})^2} + \frac{jX_{C0}\left[R_1^2 + X(X + X_{C0})\right]}{R_1^2 + (X + X_{C0})^2}. \tag{K-21}$$

The effective resistance is given by the real part and is

$$R_e = \frac{R_1 X_{C0}^2}{R_1^2 + (X + X_{C0})^2}. \tag{K-22}$$

To find the effective resistance at a specific load capacitance, it is necessary to determine the value of X at that load capacitance. This can be done with the aid of Figure K-3.

By definition, the crystal is operating at a load capacitance C_L when it is inductive and resonant with C_L. From Figure K-3 it can be seen that resonance of

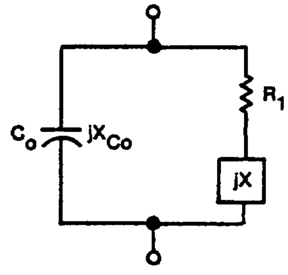

Figure K-2. Quartz crystal resonator.

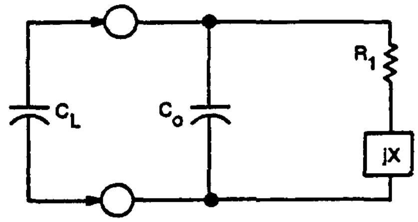

Figure K-3. Crystal and load capacitance.

the crystal with C_L occurs when resonance of X with $C_L + C_0$ occurs. Therefore when the crystal is operating into a load capacitance C_L the reactance X is given by

$$X = \frac{1}{\omega(C_L + C_0)}.$$

From equation (K-22), the effective resistance of the crystal is then given by

$$R_e = \frac{R_1 X_{C0}^2}{R_1^2 + \left[\frac{1}{\omega(C_L + C_0)} - \frac{1}{\omega C_0}\right]^2}. \qquad \text{(K-23)}$$

This can be simplified to the form:

$$R_e = \frac{R_1 X_{C0}^2}{R_1^2 + \frac{1}{\omega^2 C_0^2}\left(-\frac{C_L}{C_L + C_0}\right)^2}$$

$$R_e = \frac{R_1 X_{C0}^2}{R_1^2 + X_{C0}^2\left(\frac{C_L}{C_L + C_0}\right)^2}. \qquad \text{(K-24)}$$

If

$$\left|X_{C0}\left(\frac{C_L}{C_L + C_0}\right)\right| >> R_1,$$

then

$$R_e \doteq \frac{R_1 X_{C0}^2}{X_{C0}^2\left(\frac{C_L}{C_L + C_0}\right)^2}$$

$$R_e = R_1\left(\frac{C_L + C_0}{C_L}\right)^2. \qquad \text{(K-25)}$$