Appendix I

Nonlinear Analysis of the Colpitts Oscillator Based on the Principle of Harmonic Balance

A simplified schematic diagram of the Colpitts oscillator is shown below in Figure I-1. Here the crystal is represented by its series resistance R and an equivalent inductance L. For the analysis, it is assumed that R_1, R_2, and R_3 are large enough so that they produce only negligible effects on the RF signals. The transistor input and output capacitances are also neglected in this analysis. The effects of these reactances may be determined using the method given in sections 6.4.1 and 7.2.2.

From Appendix G, equation (G-7), we see that the emitter current of the transistor is given by

$$I_e = I_R \exp qV_b/KT \tag{I-1}$$

where I_R is a constant defined in equation (G-5) and V_b is the intrinsic base-to-emitter voltage. The base current is then given by $(\beta + 1)I_b = I_e$, where β is the common-emitter current gain. If we let $I_r = I_R/(\beta + 1)$, we may write

$$I_b = I_r \exp qV_b/KT \tag{I-2}$$

in the active region.*

The circuit of Figure I-1 is redrawn in Figure I-2 to show the base-to-emitter diode and the collector current generator. We note that $V_b = V_1 + V_0$, where V_0 is the base bias voltage.

The equations for the system can be written as follows:

$$V_2 - V_1 = Ri_3 + L\frac{di_3}{dt} \tag{I-3}$$

$$i_2 = C_2\frac{dV_2}{dt} \tag{I-4}$$

*Note that the "active region" here means any operating condition in which the transistor is not saturated ($V_{cb} > 0$) and is much greater than the linear region.

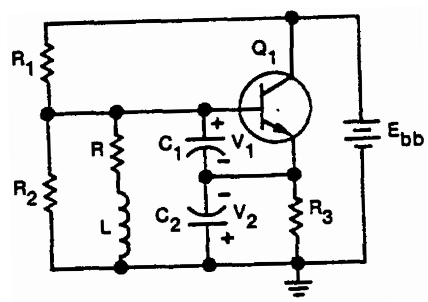

Figure I-1. Colpitts oscillator.

$$i_c + i_2 + i_3 = 0 \tag{I-5}$$

$$i_1 = C_1 \frac{dV_1}{dt} \tag{I-6}$$

$$i_3 = i_1 + I_b \tag{I-7}$$

$$I_b = I_r \exp qV_b/KT \tag{I-8}$$

$$I_c = \beta I_b \tag{I-9}$$

$$V_b = V_0 + V_1. \tag{I-10}$$

With considerable manipulation, these simultaneous equations can be used to solve for V_1. The resulting expression is

$$\frac{d^3 V_1}{dt^3} + \frac{d^2 V_1}{dt^2}\left\{\frac{R}{L} + \frac{1}{C_1}\left(\frac{qI_r}{KT}\right)\exp\left[\frac{q}{KT}(V_1 + V_0)\right]\right\}$$
$$+ \left(\frac{dV_1}{dt}\right)^2 \left\{\frac{I_r}{C_1}\left(\frac{q}{KT}\right)^2 \exp\left[\frac{q(V_1 + V_0)}{KT}\right]\right\}$$

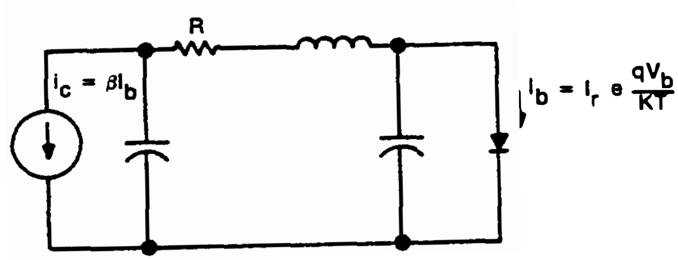

Figure I-2. Equivalent circuit of Colpitts oscillator (bias circuit not shown).

$$+\frac{dV_1}{dt}\left\{\frac{1}{L}\left(\frac{1}{C_1}+\frac{1}{C_2}\right)+\frac{R}{LC_1}\left(\frac{qI_r}{KT}\right)\exp\left[\frac{q}{KT}(V_1+V_0)\right]\right\}$$

$$+\frac{I_r(\beta+1)}{LC_1C_2}\exp\left[\frac{q}{KT}(V_1+V_0)\right]=0. \tag{I-11}$$

This equation is nonlinear not only because of the exponential terms but also because of the presence of the $(dV_1/dt)^2$ term. If this equation could be solved, it would exactly describe the behavior of the equivalent circuit from which it was derived. Unfortunately, an exact solution cannot be found. There are several techniques by which an approximate solution for this equation can be obtained. Perhaps the least difficult is the principle of harmonic balance. Using this technique, the voltage V_1 is assumed to have a solution of the form

$$V_1 = E\cos\omega t. \tag{I-12}$$

This expression is substituted for V_1 in the equation, and coefficients E and ω are adjusted so that the equation is exact insofar as terms of the fundamental frequency are concerned. Terms containing $\cos 2\omega t$, $\sin 2\omega t$, $\cos 3\omega t$, etc., are simply ignored. The justification for this rests in the theory that it is primarily the fundamental terms that determine the amplitude of oscillation and the frequency.

We see that if $V_1 = E\cos\omega t$,

$$\frac{dV_1}{dt} = -\omega E\sin\omega t \tag{I-13}$$

$$\frac{d^2V_1}{dt^2} = -\omega^2E\cos\omega t \tag{I-14}$$

$$\frac{d^3V_1}{dt^3} = \omega^3E\sin\omega t. \tag{I-15}$$

Making these substitutions, we obtain

$$\omega^3E\sin\omega t-\omega^2E\cos\omega t\left[\frac{R}{L}+\frac{1}{C_1}\left(\frac{qI_r}{KT}\right)\exp\left(\frac{qV_0}{KT}\right)\exp\left(\frac{qE\cos\omega t}{KT}\right)\right]$$

$$+\omega^2E^2\sin^2\omega t\left[\frac{I_r}{C_1}\left(\frac{q}{KT}\right)^2\exp\left(\frac{qV_0}{KT}\right)\exp\left(\frac{qE\cos\omega t}{KT}\right)\right]$$

$$-\omega E\sin\omega t\left[\frac{1}{L}\left(\frac{1}{C_1}+\frac{1}{C_2}\right)\right.$$

$$+\frac{R}{LC_1}\left(\frac{qI_r}{KT}\right)\exp\left(\frac{qV_0}{KT}\right)\exp\left(\frac{qE\cos\omega t}{KT}\right)\Bigg]$$

$$+\frac{I_r(\beta+1)}{LC_1C_2}\exp\left(\frac{qV_0}{KT}\right)\exp\left(\frac{qE\cos\omega t}{KT}\right)=0. \tag{I-16}$$

This equation can be expanded using the identity,

$$\exp Z\cos\theta = I_0(Z)+2\sum_{n=1}^{\infty} I_n(Z)\cos n\theta, \tag{I-17}$$

where $I_n(Z)$ represents modified Bessel function of the first kind and order n.

Retaining only the fundamental terms, after simplification, leads to the equation for oscillation shown below:

$$\omega^3 E\sin\omega t-\frac{R\omega^2 E}{L}\cos\omega t-\frac{\omega^2 VM}{C_1}\left[I_0(V)+I_2(V)\right]\cos\omega t \tag{I-18}$$

$$+\frac{\omega^2V^2M}{2C_1}\left[I_1(V)-I_3(V)\right]\cos\omega t-\omega E\left[\frac{1}{L}\left(\frac{1}{C_1}+\frac{1}{C_2}\right)\right]\sin\omega t$$

$$-\frac{\omega RVM}{LC_1}\left[I_0(V)-I_2(V)\right]\sin\omega t+\frac{2(\beta+1)MI_1(V)\cos\omega t}{LC_1C_2}=0.$$

Here use has been made of the trigonometric identities:

$$\sin^2\theta=\tfrac{1}{2}-\tfrac{1}{2}\cos 2\theta$$

$$\cos^2\theta=\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta$$

$$\cos x\cos y=\tfrac{1}{2}\cos(x+y)+\tfrac{1}{2}\cos(x-y)$$

$$\sin x\cos y=\tfrac{1}{2}\sin(x+y)+\tfrac{1}{2}\sin(x-y).$$

We have also defined $V=qE/KT$ and $M=I_r\,e^{qV_0/KT}$

In order for equation (I-18) to be satisfied, both the coefficients of the sine terms and the coefficients of the cosine terms must equate independently to zero. The equation resulting from the sine terms represents the frequency equation, and the equation from the cosine terms represents the amplitude equation.

We note also, using equations (I-2), (I-10), (I-12), and (I-17), that the emitter current is given by

$$I_e=(\beta+1)I_r\,e^{(q/KT)(V+V_0)}$$

$$=(\beta+1)M\left[I_0(V)+2\sum_{n=1}^{\infty}I_n(V)\cos n\omega t\right]. \tag{I-19}$$

The dc component is given by

$$I_e(\text{mean}) = (\beta + 1)MI_0(V) \tag{I-20}$$

or

$$M = \frac{I_e(\text{mean})}{(\beta + 1)I_0(V)}. \tag{I-21}$$

Turning now to the frequency equation resulting from the sine terms of equation (I-18), we have

$$\omega^2 = \frac{1}{L}\left(\frac{1}{C_1} + \frac{1}{C_2}\right) + \frac{RVM}{ELC_1}\,[I_0(V) - I_2(V)] = 0. \tag{I-22}$$

Substituting for M from equation (I-21) and noting that $V/E = q/KT$, we have

$$\omega^2 = \frac{1}{L}\left(\frac{1}{C_1} + \frac{1}{C_2}\right) + \frac{RqI_e(\text{mean})}{KTLC_1(\beta + 1)}\left[\frac{I_0(V) - I_2(V)}{I_0(V)}\right] = 0. \tag{I-23}$$

Using the identity $2I_1(V)/V = I_0(V) - I_2(V)$, and also noting from equation (G-27) that the base-to-emitter input resistance is given by

$$R_{\text{in}0} = \frac{(\beta + 1)KT}{qI_e(\text{mean})},$$

we have

$$\omega^2 = \frac{1}{L}\left[\frac{1}{C_1}\left(1 + \frac{2I_1(V)R}{VI_0(V)R_{\text{in}0}}\right) + \frac{1}{C_2}\right]. \tag{I-24}$$

If we now define R_{in} by the relationship,

$$\frac{R_{\text{in}}}{R_{\text{in}0}} = \frac{VI_0(V)}{2I_1(V)}, \tag{I-25}$$

equation (I-24) becomes

$$\omega^2 = \frac{1}{L}\left[\frac{1}{C_1}\left(1 + \frac{R}{R_{\text{in}}}\right) + \frac{1}{C_2}\right]. \tag{I-26}$$

It is interesting to observe that equation (I-25) is the same as that derived in Appendix G, equation (G-28A), and that consequently the graph of Figure 3-6 can be used to determine R_{in} after the amplitude V has been determined from the amplitude equation (to be discussed subsequently).

Equation (I-26) cannot be conveniently used to determine the change in frequency resulting from a change in the amplitude of oscillation V because L, as we have used it, is the equivalent steady-state inductance of the crystal, which is itself a function of frequency. We may rewrite equation (I-26), however, in the form

$$\omega L = \frac{1}{\omega C_1}\left(1 + \frac{R}{R_{\text{in}}}\right) + \frac{1}{\omega C_2}. \tag{I-27}$$

Since the Q of the crystal is very high (normally several hundred thousand), the frequency of oscillation is always very near the resonant frequency of the crystal and the reactances of C_1 and C_2 may be considered to be constants the values of which are calculated at the nominal frequency of the crystal. We may then rewrite equation (I-27) as follows:

$$X_e + X_1\left(1 + \frac{R}{R_{\text{in}}}\right) + X_2 = 0, \tag{I-28}$$

where X_e is the crystal reactance and X_1 and X_2 are the capacitor reactances.

It is interesting to observe that the effect on frequency caused by R_{in} is the same as that which would be determined by placing the value of R_{in} determined from the nonlinear analysis of Appendix G into the linear Y-parameter, equation (7-12). This results from the $R_e X_1 g_{ie}$ term of K_2. Equation I-28 then becomes equivalent to equation (7-12) if we neglect the reactances of the transistor and the output conductance.

Turning now to the amplitude equation resulting from the coefficients of the cosine terms in equation (I-18), we have

$$-\frac{R\omega^2 E}{L} - \frac{\omega^2 VM}{C_1}[I_0(V) + I_2(V)] + \frac{\omega^2 V^2 M}{2C_1}[I_1(V) - I_3(V)] + \frac{2(\beta + 1)MI_1(V)}{LC_1C_2} = 0. \tag{I-29}$$

By making the substitutions, $E = VKT/q$ (as defined earlier) and

$$M = \frac{I_e(\text{mean})}{(\beta + 1)I_0(V)}$$

[from equation (I-21)] and simplifying, we obtain the expression

$$R = \frac{qI_e(\text{mean})L}{2(\beta + 1)KTC_1}\left[\frac{VI_1(V) - VI_3(V) - 2I_0(V) - 2I_2(V)}{I_0(V)}\right] + \left(\frac{1}{\omega^2 C_1 C_2}\right)\left(\frac{qI_e(\text{mean})}{KT}\right)\frac{2I_1(V)}{VI_0(V)}. \tag{I-30}$$

Now using the identity

$$\frac{2n}{Z} I_n(Z) = I_{n-1}(Z) - I_{n+1}(Z), \tag{I-31}$$

with $n = 2$, we have

$$\frac{4I_2(Z)}{Z} = I_1(Z) - I_3(Z).$$

Making this substitution in (I-30) gives

$$R = \frac{-qI_e(\text{mean})L}{(\beta+1)KTC_1}\left[\frac{I_0(V) - I_2(V)}{I_0(V)}\right] + \left(\frac{1}{\omega^2 C_1 C_2}\right)\left[\frac{qI_e(\text{mean})}{KT}\right]\frac{2I_1(V)}{VI_0(V)}. \tag{I-32}$$

Again using (I-31) for $n = 1$, we have

$$\frac{2I_1(Z)}{Z} = I_0(Z) - I_2(Z).$$

Performing this substitution gives

$$R = \frac{-qI_e(\text{mean})L}{(\beta+1)KTC_1}\left[\frac{2I_1(V)}{VI_0(V)}\right] + \left(\frac{1}{\omega^2 C_1 C_2}\right)\left(\frac{qI_e(\text{mean})}{KT}\right)\frac{2I_1(V)}{VI_0(V)}. \tag{I-33}$$

Now let $R_{\text{in}0} = (\beta+1)KT/qI_e(\text{mean})$ from equation (G-27) and

$$g_{m0} = \frac{qI_e(\text{mean})}{KT}\,\frac{\beta}{\beta+1}$$

from equation (G-22). We have added the factor $\beta/(\beta+1)$ here to account for the fact that $I_c = I_e\beta/(\beta+1)$, since in Appendix G the assumption was made that $I_c \doteq I_e$. The resulting equation is

$$R = \frac{-L}{R_{\text{in}0}C_1}\left[\frac{2I_1(V)}{VI_0(V)}\right] + \left[\frac{1}{\omega^2 C_1 C_2}\right]\left[\left(1+\frac{1}{\beta}\right)g_{m0}\right]\left[\frac{2I_1(V)}{VI_0(V)}\right]. \tag{I-34}$$

Now observing that

$$\left(\frac{1}{\beta}\right) g_{m0} = \frac{qI_e(\text{mean})}{KT} \frac{\beta}{(\beta + 1)} \left(\frac{1}{\beta}\right) = \frac{qI_e(\text{mean})}{KT(\beta + 1)} = \frac{1}{R_{\text{in0}}},$$

and also substituting $X_1 = -1/\omega C_1$, $X_2 = -1/\omega C_2$, and $X_e = \omega L$, we have

$$X_1 X_2 g_{m0} \left[\frac{2I_1(V)}{VI_0(V)}\right] = R - X_1(X_e + X_2) \left[\frac{2I_1(V)}{VI_0(V)}\right] \frac{1}{R_{\text{in0}}}. \tag{I-35}$$

Now letting $g_m/g_{m0} = 2I_1(V)/VI_0(V)$ and $R_{\text{in}} = R_{\text{in0}}\ VI_0(V)/2I_1(V)$, as defined in Appendix G, so that Figure 3-6 can be used to determine the values, we have

$$X_1 X_2 g_m = R - \frac{X_1(X_e + X_2)}{R_{\text{in}}}. \tag{I-36}$$

It is interesting to note that this equation is consistent with linear equation (7-11) if we neglect the transistor reactances and the output admittance. Then K_1, equation (7-13), becomes $-X_1(X_2 + X_e)g_{ie}$, as we determined in equation (I-36) above.