Appendix B

Derivation of Y-Parameter Equations for the Pierce Oscillator

The Pierce oscillator can be represented by the schematic diagram of Figure B-1.

Let the circuit be redrawn and the transistor replaced by a four-terminal network described by its Y-parameters. The diagram of Figure B-2 then results.

If the transistor is represented by its Y-parameters, then the feedback network must be represented by its Z-parameters if equation (A-12) is to be used. For this analysis, it is convenient to include the load resistance R_L in the output admittance of the transistor. This is accomplished by using a value of y_{oe} which is equal to y_{oe} (transistor) + $1/R_L$. Also, the input capacity of the transistor will be lumped in parallel with C_1, thus making $y_{ie} = g_{ie}$ purely resistive in the analysis. In like manner, the output capacity of the transistor will be

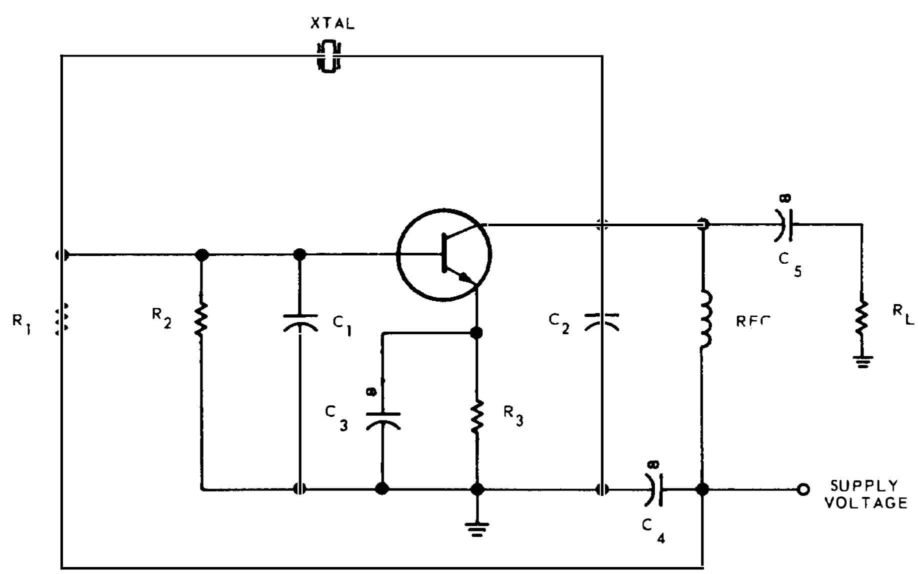

Figure B-1. Pierce oscillator: schematic diagram.

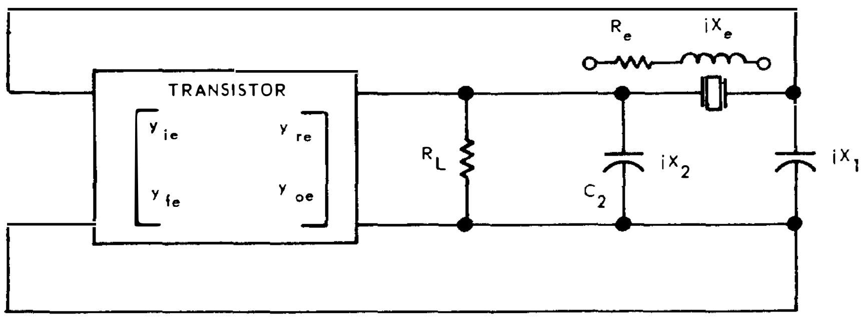

Figure B-2. Pierce oscillator: ac circuit diagram.

lumped in parallel with C_2, making $y_{oe} = g_{oe}$ purely resistive. In the analysis it will be assumed that the reverse transfer admittance of the transistor is purely imaginary: $y_{re} = jb_{re}$.

The remaining π-network is shown in Figure B-3.

From this circuit the Z-parameters can be calculated using the following definitions:

Input impeda $\quad \frac{V_1}{\;}\Big|$ (B-1)

Output impec — (B-2)

Forward tran: impedance $\quad Z_f = \frac{V_2}{I_1}\Big|_{I_2=0}$ (B-3)

Reverse trans impedance — | (B-4)

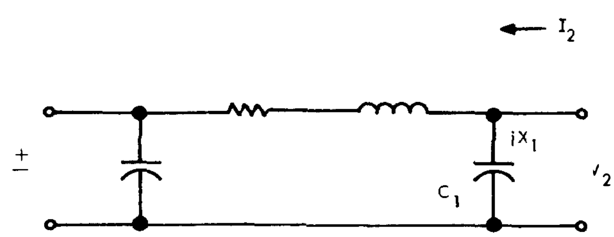

Figure B-3. Pierce oscillator: π-network: simplified diagram.

From equation (B-1) it can be seen that Z_i is merely the input impedance of the network with the output open-circuited and can be written by inspection as

$$Z_i = \frac{jx_2(R_e + jx_e + jx_1)}{jx_2 + (R_e + jx_e + jx_1)}. \tag{B-5}$$

If we let

$$Z = R_e + j(x_1 + x_2 + x_e) \tag{B-6}$$

we have

$$Z_i = \frac{1}{Z}\, jx_2(R_e + jx_e + jx_1) \tag{B-7}$$

$$Z_i = -\frac{1}{Z}\,(x_1x_2 + x_2x_e - jR_ex_2). \tag{B-8}$$

By symmetry, we may write:

$$Z_o = -\frac{1}{Z}\,[x_1x_2 + x_1x_e - jR_ex_1]. \tag{B-9}$$

The forward transfer impedance is given by the ratio of V_2 to I_1 with the output open-circuited. In order to calculate this, it is convenient to first determine I_3 in terms of I_1, which is given by

$$I_3 = \frac{I_1(jx_2)}{(R_e + jx_e + jx_1) + jx_2} \tag{B-10}$$

$$I_3 = \frac{I_1(jx_2)}{Z} \tag{B-11}$$

and

$$V_2 = I_3(jx_1); \tag{B-12}$$

therefore,

$$V_2 = \frac{I_1(jx_2)}{Z}\, jx_1 \tag{B-13}$$

and

$$\frac{V_2}{I_1} = \frac{-x_1x_2}{Z} = Z_f. \tag{B-14}$$

Since the π-network is composed entirely of reciprocal elements, the forward and reverse transfer impedances are equal, so that

$$Z_f = Z_r = \frac{-x_1 x_2}{Z}. \tag{B-15}$$

In using equation (A-12) the quantity ΔZ must be used. It is given by

$$\Delta Z = Z_i Z_o - Z_f Z_r \tag{B-16}$$

$$\Delta Z = -\frac{1}{Z}(x_1 x_2 + x_2 x_e - jR_e x_2)\left(-\frac{1}{Z}\right)(x_1 x_2 + x_1 x_e - jR_e x_1) - \left(-\frac{x_1 x_2}{Z}\right)\left(-\frac{x_1 x_2}{Z}\right) \tag{B-17}$$

$$\Delta Z = \frac{1}{Z^2}[(x_1 x_2 + x_2 x_e)(x_1 x_2 + x_1 x_e) - j(x_1 x_2 + x_2 x_e)R_e x_1 - j(x_1 x_2 + x_1 x_e)R_e x_2 - R_e^2 x_1 x_2] - \frac{x_1^2 x_2^2}{Z^2} \tag{B-18}$$

$$\Delta Z = \frac{1}{Z^2}(x_1^2 x_2^2 + x_1^2 x_2 x_e + x_2^2 x_1 x_e + x_1 x_2 x_e^2 - jR_e x_1^2 x_2 - jR_e x_1 x_2 x_e - jR_e x_1 x_2^2 - jR_e x_1 x_2 x_e - R_e^2 x_1 x_2 - x_1^2 x_2^2) \tag{B-19}$$

$$\Delta Z = \frac{1}{Z^2}\{R_e x_1 x_2 [-R_e - j(x_2 + x_e + x_1)] + jx_1 x_2 x_e [-R_e - jx_1 - jx_2 - jx_e]\} \tag{B-20}$$

$$\Delta Z = -\frac{1}{Z}(R_e x_1 x_2 + jx_1 x_2 x_e) \tag{B-21}$$

$$\Delta Z = -\frac{x_1 x_2}{Z}(R_e + jx_e). \tag{B-22}$$

Summarizing these results, we have

$$Z_i = -\frac{1}{Z}(x_1 x_2 + x_2 x_e - jR_e x_2) \tag{B-23}$$

$$Z_f = Z_r = -\frac{x_1 x_2}{Z} \tag{B-24}$$

$$Z_o = -\frac{1}{Z}(x_1x_2 + x_1x_e - jR_ex_1) \tag{B-25}$$

$$\Delta Z = -\frac{x_1x_2}{Z}(R_e + jx_e), \tag{B-26}$$

where

$$Z = R_e + j(x_1 + x_2 + x_e). \tag{B-27}$$

These results can be substituted in the general equation for oscillation as developed in appendix A.

$$y_{fe}Z_f + y_{ie}Z_o + y_{oe}Z_i + y_{re}Z_r + \Delta y\Delta Z + 1 = 0 \tag{B-28}$$

$$-y_{fe}\frac{(x_1x_2)}{Z} - y_{ie}\frac{(x_1x_2 + x_1x_e - jR_ex_1)}{Z} - y_{oe}\frac{(x_1x_2 + x_2x_e - jR_ex_2)}{Z}$$
$$-y_{re}\frac{(x_1x_2)}{Z} - \Delta y\frac{(x_1x_2)(R_e + jx_e)}{Z} + 1 = 0. \tag{B-29}$$

Multiplying by $-Z$ gives

$$y_{fe}x_1x_2 + y_{ie}(x_1x_2 + x_1x_e - jR_ex_1) + y_{oe}(x_1x_2 + x_2x_e - jR_ex_2)$$
$$+ y_{re}x_1x_2 + \Delta yx_1x_2(R_e + jx_e) - Z = 0. \tag{B-30}$$

Making the substitutions:

$$y_{fe} = g_{fe} + jb_{fe}, \quad y_{ie} = g_{ie}, \quad y_{oe} = g_{oe}, \quad y_{re} = jb_{re},$$

and

$$\Delta y = (y_{ie}y_{oe} - y_{fe}y_{re}) = g_{ie}g_{oe} - jb_{re}(g_{fe} + jb_{fe}),$$

and for Z results in the equation

$$\begin{aligned}(g_{fe} + jb_{fe})x_1x_2 &+ g_{ie}(x_1x_2 + x_1x_e - jR_ex_1)\\ &+ g_{oe}(x_1x_2 + x_2x_e - jR_ex_2) + jb_{re}x_1x_2\\ &+ [g_{ie}g_{oe} - jb_{re}(g_{fe} + jb_{fe})]\, x_1x_2(R_e + jx_e)\\ &-R_e - j(x_1 + x_2 + x_e) = 0,\end{aligned} \tag{B-31}$$

or

$$\begin{aligned}g_{fe}x_1x_2 &+ jb_{fe}x_1x_2 + g_{ie}x_1x_2 + g_{ie}x_1x_e - jg_{ie}R_ex_1\\ &+ g_{oe}x_1x_2 + g_{oe}x_2x_e - jg_{oe}R_ex_2 + jb_{re}x_1x_2\\ &+ g_{ie}g_{oe}x_1x_2R_e + jg_{ie}g_{oe}x_1x_2x_e - jb_{re}g_{fe}x_1x_2R_e\end{aligned}$$

$$+ b_{re}g_{fe}x_1x_2x_e + b_{re}b_{fe}x_1x_2R_e + jb_{re}b_{fe}x_1x_2x_e$$
$$- R_e - j(x_1 + x_2 + x_e) = 0. \tag{B-32}$$

The equation can be separated into real and imaginary components. The real parts of the equation are

$$g_{fe}x_1x_2 + g_{ie}x_1(x_2 + x_e) + g_{oe}x_2(x_1 + x_e) + R_ex_1x_2(g_{ie}g_{oe} + b_{fe}b_{re})$$
$$+ g_{fe}b_{re}x_1x_2x_e - R_e = 0. \tag{B-33}$$

The imaginary parts are

$$-x_1 - x_2 - x_e + (b_{fe} + b_{re})x_1x_2 + g_{ie}g_{oe}x_1x_2x_e - R_e(g_{ie}x_1 + g_{oe}x_2)$$
$$+ b_{re}b_{fe}x_1x_2x_e - g_{fe}b_{re}x_1x_2R_e = 0. \tag{B-34}$$

Rewriting these equations and separating the primary and secondary effects gives

$$g_{fe}x_1x_2 = R_e + K_1 \tag{B-35}$$

and

$$x_1 + x_2 + x_e = 0 + K_2, \tag{B-36}$$

where

$$\begin{aligned} K_1 = &-g_{ie}x_1(x_2 + x_e) - g_{oe}x_2(x_1 + x_e) \\ &-R_ex_1x_2(g_{ie}g_{oe} + b_{fe}b_{re}) \\ &- g_{fe}b_{re}x_1x_2x_e, \end{aligned}$$

and

$$\begin{aligned} K_2 = &(b_{fe} + b_{re})x_1x_2 + g_{ie}g_{oe}x_1x_2x_e \\ &- R_e(g_{ie}x_1 + g_{oe}x_2) + b_{re}b_{fe}x_1x_2x_e \\ &- g_{fe}b_{re}x_1x_2R_e. \end{aligned}$$

If we assume that K_2 is zero, then

$$x_e = -(x_1 + x_2) \tag{B-37}$$

or

$$x_e = \frac{1}{\omega C_1} + \frac{1}{\omega C_2}. \tag{B-38}$$

Let T be some parameter which causes C_1 to vary with a rate $\partial C_1/\partial T$, and C_2 with a rate $\partial C_2/\partial T$. It is assumed here that C_1 and C_2 include the transistor input and output capacitances which are primarily responsible for the changes in C_1 and C_2. Then

$$\frac{\partial C_1}{\partial T} \doteq \frac{\partial C_{\text{in}}}{\partial T} \quad \text{and} \quad \frac{\partial C_2}{\partial T} \doteq \frac{\partial C_{\text{out}}}{\partial T}.$$

Differentiating equation (B-38) gives

$$\frac{\partial x_e}{\partial T} = -\frac{1}{\omega C_1^2}\left(\frac{\partial C_1}{\partial T}\right) - \frac{1}{\omega C_2^2}\left(\frac{\partial C_2}{\partial T}\right). \tag{B-39}$$

Also, from equation (B-35), if we assume that K_1 is zero, $C_1 = g_{fe}/R_e C_2 \omega^2$. Putting this in equation B-39 gives

$$\frac{\partial x_e}{\partial T} = -\left[\frac{R_e^2 C_2^2 \omega^3}{g_{fe}^2}\left(\frac{\partial C_1}{\partial T}\right)\right] - \left[\frac{1}{\omega C_2^2}\left(\frac{\partial C_2}{\partial T}\right)\right]. \tag{B-40}$$

If this expression is minimized for C_2, then

$$\frac{\partial(\partial x_e/\partial T)}{\partial C_2} = -\frac{2R_e^2 C_2 \omega^3}{g_{fe}^2}\left(\frac{\partial C_1}{\partial T}\right) + \frac{2}{\omega C_2^3}\left(\frac{\partial C_2}{\partial T}\right) = 0. \tag{B-41}$$

Solving this for C_2 gives

$$C_2^4 = \frac{(\partial C_2/\partial T)}{(\partial C_1/\partial T)} \times \frac{gf_e^2}{\omega^4 R_e^2}. \tag{B-42}$$

In a similar manner, it can be shown that

$$C_1^4 = \frac{(\partial C_1/\partial T)}{(\partial C_2/\partial T)} \times \frac{gf_e^2}{\omega^4 R_e^2}. \tag{B-43}$$

Dividing equation (B-42) by (B-43) gives

$$\left(\frac{C_2}{C_1}\right)^4 = \frac{(\partial C_2/\partial T)^2}{(\partial C_1/\partial T)^2} \tag{B-44}$$

or

$$\frac{C_2}{C_1} = \left(\frac{\partial C_2/\partial T}{\partial C_1/\partial T}\right)^{1/2}. \tag{B-45}$$

This minimizes the frequency change with respect to some parameter T provided the assumption $K_1 = K_2 = 0$ is valid.

A close examination of equation (B-41) shows that the condition

$$\frac{\partial(\partial x_e/\partial T)}{\partial C_2} = 0$$

does not necessarily assure that a minimum occurs. It does, however, assure that a minimum value of $|\partial x_e/\partial T|$ occurs when $\partial C_1/\partial T$ and $\partial C_2/\partial T$ are of like sign. If they are opposite sign $\partial(\partial x_e/\partial T)/\partial C_2 \neq 0$, the quantity $\partial x_e/\partial T$ then may be set equal to zero, and solving equation (B-39) for this condition gives the result

$$\frac{C_2}{C_1} = \left(-\frac{\partial C_2/\partial T}{\partial C_1/\partial T}\right)^{1/2}. \tag{B-46}$$