Chapter 5
Electrical Characterization of Wideband Acoustic Devices

5.1 INTRODUCTION

In the development of the Christoffel equation, we assumed a homogeneous medium (the values of the stiffness components do not depend on position) in which there were no boundaries. For many situations (for example, an acousto-optic (A/O) device), this is a realistic and useful model; other devices utilize specific boundaries in their operations (acoustic resonators and transducers). In this chapter, we develop the tools needed to analyze and design acoustic structures with boundaries in one dimension. Configurations that involve the combination of the Christoffel equation with two- or three-dimensional boundaries are extremely difficult to solve precisely and require numerical techniques; they will not be considered here. We assume that the Christoffel equation has been solved and that the eigenvalues (velocities), polarization coupling constants (if the material is piezoelectric), and impedances of all propagating waves are known. We also assume that only one mode is propagating or that if two modes are propagating, they may be treated separately.

5.2 ONE-DIMENSIONAL EQUATIONS FOR A NONPIEZOELECTRIC SLAB

Our task is to develop a *one-dimensional* model that describes the electrical characteristics of an acoustic structure; the governing equations will be either programmed onto a computer or solved in closed form. We make the following analogues:

force → voltage

particle velocity → current

The use of force instead of tension is permissible here because we are dealing with a one-dimensional geometry, and thus shear and longitudinal waves are treated identically. Recall the definition of impedance (1.12):

$$Z = \frac{-T}{v} = \sqrt{\rho c} = \rho v_a = \frac{ck}{\omega} \tag{5.1}$$

where ρ is the material density and c is the relevant stiffness constant of the propagating mode (longitudinal or shear). As usual, we find c by solving the Christoffel equation; however, for a piezoelectric crystal c is determined either at constant **E** or **D**, depending on the external conditions. The definition of strain in one dimension is

$$S = \frac{\partial u}{\partial z} \tag{5.2}$$

where we assume propagation in the z direction. The particle displacement u may be either z-directed (longitudinal) or normal to the z-axis (shear).

Now, $F = AT$ where A is the area, so from (5.1) and (5.2):

$$F = AZv = \frac{Ackv}{\omega} \tag{5.3}$$

and

$$F = AcS = Ac\frac{\partial u}{\partial z} \tag{5.4}$$

These equations are valid for longitudinal or shear waves in an infinite homogeneous medium (the stiffness constant c characterizes the particular mode); if the medium is piezoelectrically active, we simply use the "stiffened" c component.

Next we consider a finite slab bounded by the planes $z = z_1$ and $z = z_2$ as shown in Figure 5.1. Because there is an acoustic reflection from each boundary, there must be two waves in the slab (one traveling to the left, the other to the right). The particle displacement is then (we suppress the time variation)

$$u = a\,e^{-jkz} + b e^{jkz} \tag{5.5}$$

$\uparrow$ traveling to right $\uparrow$ traveling to left

The interaction of these waves causes the resonance conditions that dramatically alter the electrical characteristics of acoustic devices. The coefficients a and b depend on the acoustic impedance mismatch at the boundaries. At the left boundary ($z = z_1$), the particle velocity is

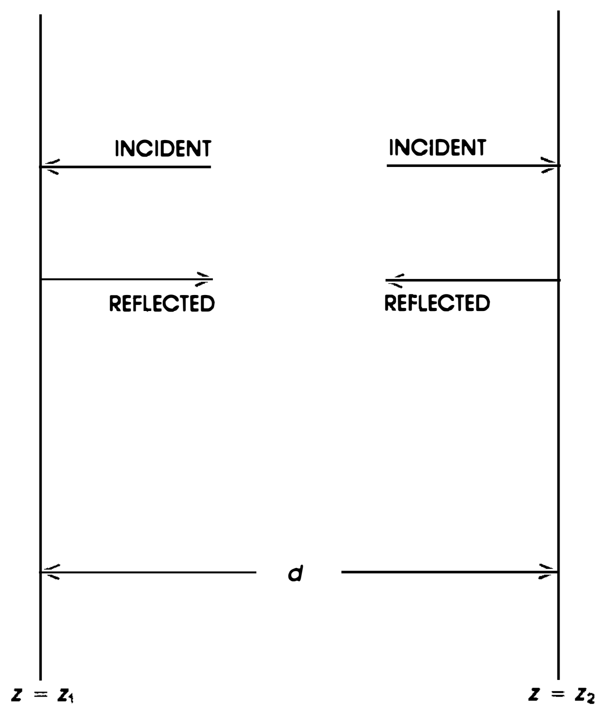


Figure 5.1 Finite thickness acoustic medium: at all points in the medium, there are right and left propagating plane waves.

$$\dot{u}_1 = v_1 = j\omega(a\,e^{-jkz_1} + b\,e^{jkz_1}) \tag{5.6}$$

At the right boundary ($z = z_2$), the particle velocity is

$$\dot{u}_2 = v_2 = j\omega(a\,e^{-jkz_2} + b\,e^{jkz_2}) \tag{5.7}$$

Our task is to solve for the coefficients a and b in terms of v_1 and v_2 (the particle velocities). Multiply (5.6) by e^{jkz_2}:

$$j\omega a\,e^{-jkz_1}\,e^{jkz_2} = v_1\,e^{jkz_2} - j\omega b\,e^{jkz_1}\,e^{jkz_2}$$

or (because $d = z_2 - z_1$)

$$j\omega a\,e^{jkd} = v_1\,e^{jkz_2} - j\omega b\,e^{jkz_1}\,e^{jkz_2} \tag{5.8}$$

Similarly, we multiply (5.7) by e^{-jkz_2}:

$$v_2\,e^{-jkz_2} = j\omega(a\,e^{-2jkz_2} + b)$$

or

$$j\omega b = v_2\,e^{-jkz_2} - j\omega a\,e^{-2jkz_2} \tag{5.9}$$

Substituting (5.9) into (5.8) gives

$$\begin{aligned} j\omega a\,e^{+jkd} &= v_1\,e^{jkz_2} - e^{jkz_1}\,e^{jkz_2}(v_2\,e^{-jkz_2} - j\omega a\,e^{-2jkz_2}) \\ &= v_1\,e^{jkz_2} - v_2\,e^{jkz_1} + j\omega a\,e^{jk(z_1 - z_2)} \end{aligned} \tag{5.10}$$

or

$$j\omega a(e^{jkd} - e^{-jkd}) = v_1\,e^{jkz_2} - v_2^{jkz_1} \tag{5.11}$$

Finally,

$$j\omega a = \frac{v_1\,e^{jkz_2} - v_2\,e^{jkz_1}}{2j\,\sin(kd)} \tag{5.12}$$

In an identical manner, we can solve for the coefficient b:

$$j\omega b = \frac{v_2\,e^{-jkz_1} - v_1\,e^{-jkz_2}}{2j\,\sin(kd)} \tag{5.13}$$

but from (5.5) we know that

$$u = a\,e^{-jkz} + b\,e^{jkz}$$

and, in a nonpiezoelectric medium:

$$T = c\frac{\partial u}{\partial z} \rightarrow F = Ac\frac{\partial u}{\partial z}$$

or

$$\begin{aligned} F &= jkcA(a\,e^{-jkz} - be^{jkz}) \\ &= Z(j\omega a\,e^{-jkz} - j\omega b\,e^{jkz}) \end{aligned} \tag{5.14}$$

because $Z = ckA/\omega$

We next substitute for $j\omega a$ and $j\omega b$ from (5.12) and (5.13); at $z = z_1$,

$$\begin{aligned} F_1 = Z\Bigg(& e^{-jkz_1}\left(\frac{v_1\,e^{jkz_2} - v_2\,e^{jkz}}{2j\sin(kd)}\right) \\ & - e^{jkz_1}\left(\frac{v_2\,e^{-jkz_1} - v_1\,e^{-jkz_2}}{2_j\sin(kd)}\right)\Bigg) \end{aligned} \tag{5.15}$$

where F_1 and F_2 are the forces *on* the slab at the left and right boundaries, respectively. Equation (5.15) can be simplified as follows:

$$\begin{aligned} F_1 &= Z\left(\frac{v_1\,e^{jkd} - v_2 - v_2 + v_1\,e^{jkd}}{2j\sin(kd)}\right) \\ &= Z\left(\frac{v_1 2\cos(kd) - 2v_2}{2j\sin(kd)}\right) \\ &= Z\left(\frac{v_1}{j\tan(kd)} - \frac{v_2}{j\sin(kd)}\right) \end{aligned} \tag{5.16}$$

Similarly, for the boundary at $z = z_2$,

$$F_2 = Z\left(\frac{v_1}{j\sin(kd)} - \frac{v_2}{j\tan(kd)}\right) \tag{5.17}$$

Now we use the trigonometric identity:

$$\frac{1}{\tan(kd)} = \frac{1}{\sin(kd)} + \tan\left(\frac{kd}{2}\right) \tag{5.18}$$

Substituting (5.18) into (5.16) and (5.17), we finally have

$$F_1 = \frac{Z}{j\sin(kd)}(v_1 - v_2) + jZ\tan\left(\frac{kd}{2}\right) v_1 \tag{5.19}$$

$$F_2 = \frac{Z}{j\sin(kd)}(v_1 - v_2) - jZ\tan\left(\frac{kd}{2}\right) v_2 \tag{5.20}$$

Next we consider the acoustic circuit shown in Figure 5.2. The current is represented by the particle velocity v, and the voltage by the force F. Applying Kirchhoff's voltage law to loop 1, we have

$$\underbrace{v_1 jZ\tan\left(\frac{kd}{2}\right)}_{\substack{\text{"voltage" drop} \\ \text{across } jZ\tan\left(\frac{kd}{2}\right)}} \quad - \quad \underbrace{\frac{jZ}{\sin(kd)}(v_1 - v_2)}_{\substack{\text{"voltage" drop} \\ \text{across } \frac{-jZ}{\sin(kd)}}} \quad = \quad \underbrace{F_1}_{\text{"source voltage"}}$$

In this equation, v_1 is the "acoustic" current and Z is the acoustic impedance. Writing Kirchhoff's voltage equation for loop 2 reproduces (5.20), and thus the circuit of Figure 5.2 is a valid representation for the acoustic equations in a nonpiezoelectric medium. It is a lumped element representation of an acoustic transmission or delay line of "acoustic" length kd.

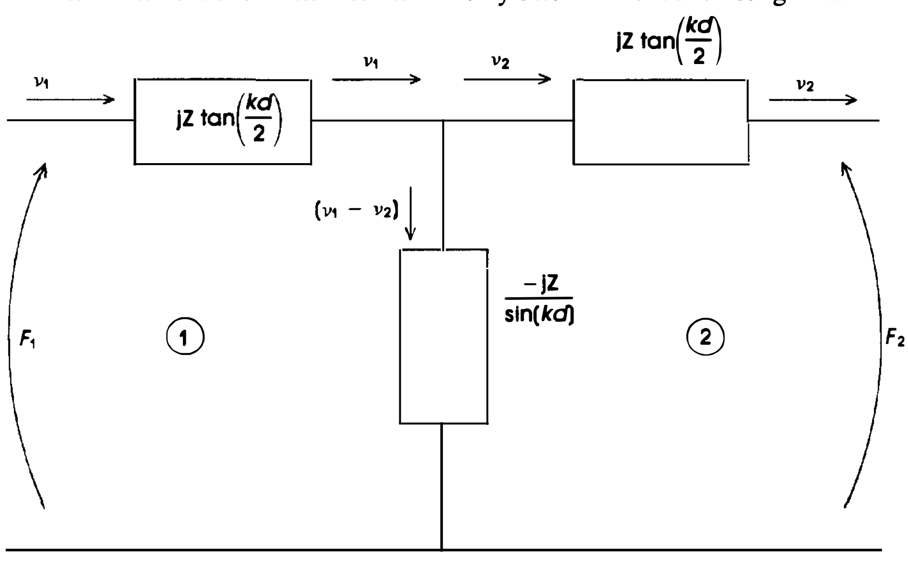


Figure 5.2 Equivalent circuit of a finite thickness acoustic delay line.

5.3 ONE-DIMENSIONAL EQUATIONS FOR A PIEZOELECTRIC SLAB

For a piezoelectric slab of thickness d (Figure 5.3), there is an extra term relating the coupling between the electrical and acoustic fields; in place of

$$T = cS = c\,\frac{\partial u}{\partial z}$$

we write

$$T = c^{\mathbf{E}}\,\frac{\partial u}{\partial z} - eE \tag{5.21}$$

where e, E, and c are the relevant components of the piezoelectric matrix, the local electric field, and the stiffness matrix measured at constant $\mathbf{E}$. Recall that (5.21) implies that $\mathbf{E}$ is held constant. In a static measurement, the body is usually electrically grounded, so $\mathbf{E} = 0$. The electrical conditions shown in Figure 5.3(a) imply that $\mathbf{D}$, and not $\mathbf{E}$, is constant (of course, in a nonpiezoelectric the distinction is irrelevant because $\mathbf{D}$ and $\mathbf{E}$ are proportional). The displacement vector is constant because (inside the piezolayer)

$$\nabla \cdot \mathbf{D} = \rho_e = 0$$

where ρ_e is the free charge density in the crystal. In one dimension, we have simply that D is constant in z. The case in which $\mathbf{E}$ is constant in z is shown in Figure 5.3(b). We will have occasion to study this case later. Equation (5.21) must be modified to reflect the fact that $\mathbf{D}$, and not $\mathbf{E}$, is the independent variable. Recall (4.25) (in one dimension):

$$D = \epsilon^{\mathbf{S}}E + eS \tag{5.22}$$

After we substitute for E, (5.21) becomes

$$\begin{aligned} T &= c^{\mathbf{E}}S - e\left[\frac{D - eS}{\epsilon^{\mathbf{S}}}\right] \\ &= \left[c^{\mathbf{E}} + \frac{e^2}{\epsilon^{\mathbf{S}}}\right]S - \frac{eD}{\epsilon^{\mathbf{S}}} \\ &= c^{\mathbf{D}}\,S - \frac{eD}{\epsilon^{\mathbf{S}}} \end{aligned} \tag{5.23}$$

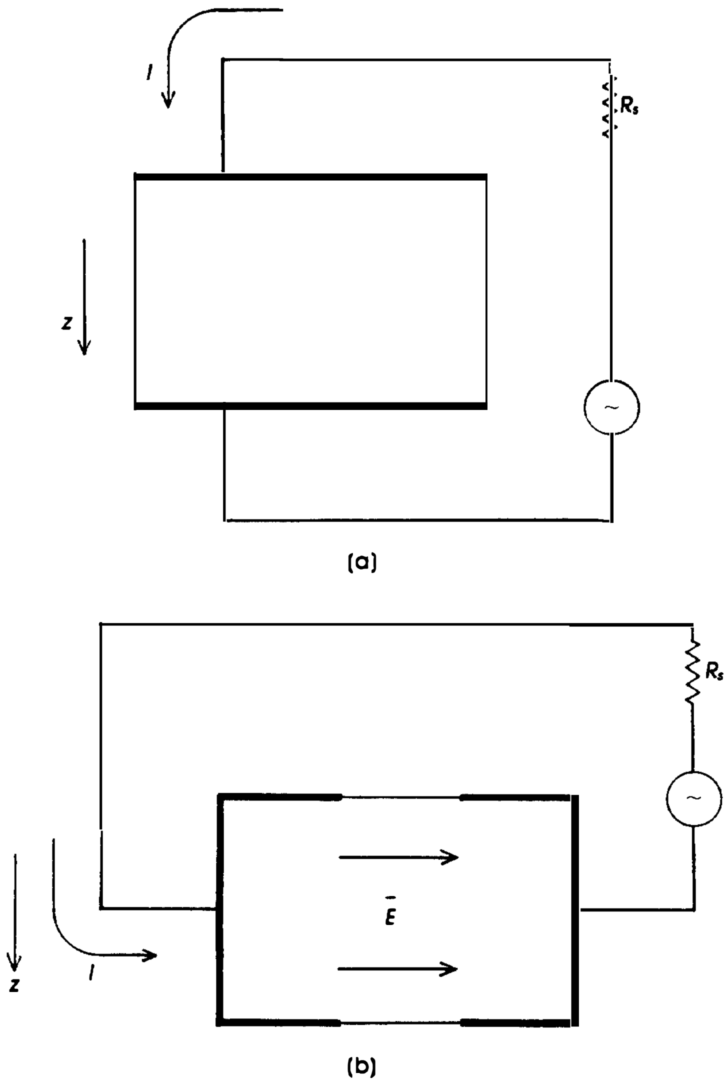


Figure 5.3 Electrical excitation of a finite thickness piezoelectric medium: (a) thickness excitation; (b) lateral field excitation. In both cases, acoustic propagation is in the z direction.

The presence of c^D (instead of c^E) in (5.23) results in the requirement of k_t^2 rather than K^2 in the description of coupling in a piezoelectric slab with thickness excitation (as in Figure 5.3a).

We are interested in the "current-voltage" (I-V) characteristic; the current is purely a displacement current if the piezolayer is dielectric (this is not true in piezoelectric semiconductors such as zinc oxide and cadmium sulfide), so

$$J = \frac{\partial \mathbf{D}}{\partial t} = j\omega D \tag{5.24}$$

and

$$I = j\omega DA$$

where J is the displacement current density with units of $[J] = \mathrm{A/m^2}$. To find the voltage, we integrate the electric field. From (5.22),

$$E = \underset{\substack{\uparrow \\ \text{external} \\ \mathbf{E}\text{ field}}}{\frac{D}{\epsilon^S}} - \underset{\substack{\uparrow \\ \text{internally generated} \\ \text{by the acoustic wave}}}{\frac{e}{\epsilon^S}\frac{\partial u}{\partial z}} \tag{5.25}$$

The voltage is thus

$$V = \int_{z_1}^{z_2} E \, \mathrm{d}z = \frac{Dd}{\epsilon^S} - \frac{\mathrm{e}}{\epsilon^S}(u(z_2) - u(z_1)) \tag{5.26}$$

where $d = z_2 - z_1$.

Substituting for D from (5.24) and using $\dot{u} = v = j\omega u$, we have

$$V = \frac{d}{\epsilon^S}\frac{I}{j\omega A} + \frac{h}{j\omega}(v_1 - v_2) \tag{5.27}$$

where

$$h = \frac{e}{\epsilon^S}$$

Finally, solving (5.27) for I, we get

$$I = j\omega C_0 V + hC_0(v_1 - v_2) \tag{5.28}$$

where

$$C_0 = \frac{\epsilon^S A}{d}$$

is the static or "clamped" capacitance, because it includes the permittivity at zero strain (determined from (4.23)). The current in the piezoelectric is composed of two terms:

1. The displacement current through a capacitance, $j\omega C_0 V$.
2. The current due to the conversion of mechanical energy (the piezoelectric effect), $hC_0(v_1 - v_2)$.

Equation (5.4) is modified to (at the left boundary)

$$F_1 = TA = c^D S\,A - \frac{eD}{\epsilon^S}A \tag{5.29}$$

where the first term is the mechanical term (as in (5.4)), and the second term is the piezoelectric contribution. The assumption is that there is an incident wave and a reflected wave at each boundary, and the operation leads to an expression similar to (5.19) with an additional term representing the electrical component. Substituting in (5.29) for the displacement vector from (5.24), we get

$$F_1 = \frac{Z}{j\sin(kd)}(v_1 - v_2) + jZ\tan\left(\frac{kd}{2}\right)v_1 + \frac{h}{j\omega}I \tag{5.30}$$

Likewise, we modify (5.20) for a piezoelectric medium by adding the electrical term $hI/j\omega$. The equivalent circuit of a piezoelectric slab is shown in Figure 5.4. The transformer represents the conversion of electrical energy to acoustic energy (or *vice versa*). An acoustic current (v_2-v_1) is present on the right side (terminals c-d) while an electrical current is present on the left side (terminals a-b). Writing Kirchhoff's voltage law for loop 1, we see that (from (5.30))

$$V_{cd} = \frac{h}{j\omega}I \tag{5.31}$$

From (5.28), which relates the electrical current to the acoustic current, the turns ratio of the transformer (ϕ) is just hC_0. Thus, the voltage V_{ab} is (which is a real voltage because it is on the electrical side of the transformer)

$$V_{ab} = \frac{I}{j\omega C_0} \tag{5.32}$$

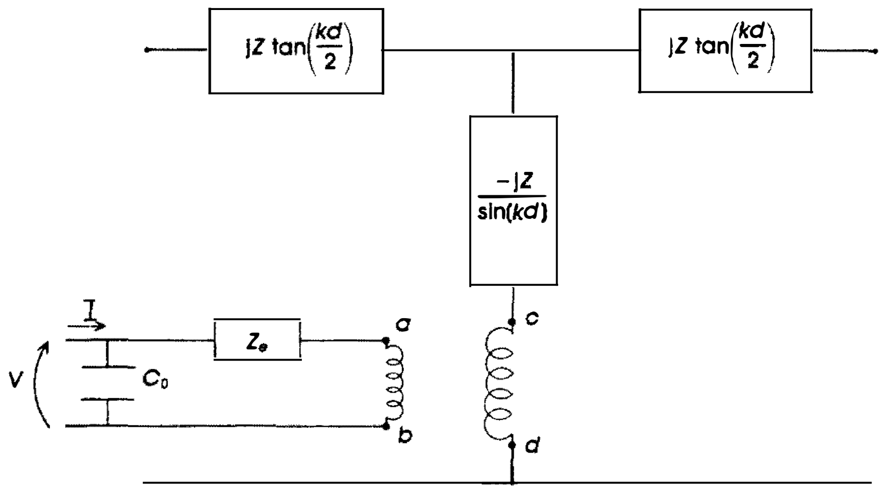


Figure 5.4 Mason model equivalent circuit of a finite thickness piezoelectric layer possesses two mechanical ports and one electrical port.

The electrical part of the piezoslab is shown in Figure 5.5; referring to Figure 5.5a, we write

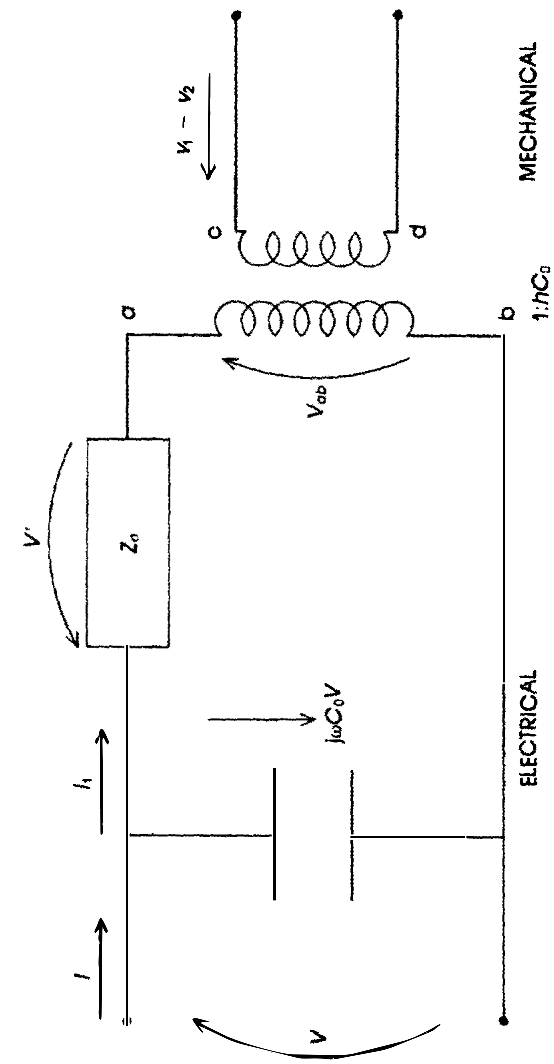
$$
\begin{aligned}
V' &= V - V_{ab} \\
&= \frac{I_2}{j\omega C_0} - \frac{I}{j\omega C_0}
\end{aligned}
\tag{5.33}
$$

But $I_2 = I + I_1$, so

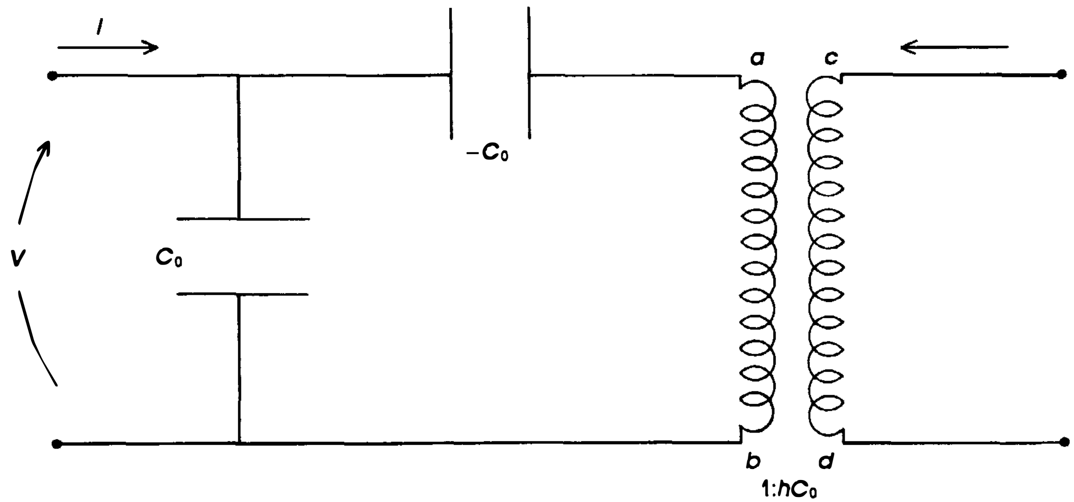
$$
V = -\frac{I_1}{j\omega C_0} \tag{5.34}
$$

The minus sign in (5.34) means that if I_1 is directed away from the node, then the voltage drop must be negative. The implication is that the series capacitor on the electrical side is negative. It behaves like a capacitor (i.e., its reactance varies inversely as the frequency), but the magnitude of its reactance is positive (like an inductor).

The circuit is redrawn in its final form in Figure 5.5(b). If $h \to 0$, then the secondary of the transformer is shortcircuited and $V_{ab} \to 0$. There is thus no current in the primary (and in $-C_0$), and the circuit reduces to



(a)



(b)

Figure 5.5 Electrical energy to mechanical energy conversion of Mason model contains a negative (nonrealizable) capacitor.

a simple capacitor with impedance given by the familiar expression:

$$X_C = \frac{1}{j\omega C_0}$$

In this case, the electrical and acoustic properties are decoupled, and the slab reverts to a simple acoustic delay line.

We are now in a position to attach the piezoelectric and delay line portions of the circuit, which results in a complete model for a piezoelectric slab, as shown in Figure 5.6. This circuit is often referred to as the Mason model, after Warren Mason, who performed much of the pioneering work in crystal acoustics. As we have already noted, for a nonpiezoelectric slab the electrical portion would not be present, and the acoustic-to-electric transformer would be short-circuited. The complete circuit has three ports and, in general, it is rather cumbersome to work with. A practical acoustic device is shown in Figure 5.7 and the Mason model schematic in Figure 5.8. The layer thicknesses are typically

$$d_1 \rightarrow .1\ \mu\text{m}, \qquad d_2 \rightarrow 1 \text{ to } 25\ \mu\text{m}$$

$$d_3 \rightarrow .1 \text{ to } 1\ \mu\text{m}, \quad d_4 \rightarrow .5 \text{ mm to } 5 \text{ cm}$$

Figure 5.9 illustrates device structures that contain piezo- and nonpiezoelectric materials. The first device is the two-port ultrasonic delay line. One of the transducers acts as a converter of electrical energy to acoustic energy. An acoustic wave (either longitudinal or shear mode) propagates in the delay line medium (which may or may not be piezoelectric) and is converted back into electrical energy at the second transducer. This device is characterized by its bandwidth (which should be as large as possible) and insertion loss. The latter is minimized by using high coupling transducer materials and ensuring a reasonable acoustic match between transducer and substrate. The delay medium is chosen to be consistent with the required time delay. Unless the coupling is very high, appreciable acoustic energy may be reflected and appear at the output after making three round trips. We can suppress this triple transit response by choosing a substrate of high enough attenuation constant α so that the wave is effectively attenuated after making more than one transit. The one-port delay line (Figure 5.9(b)) is a simple variation of this structure. It is significantly easier to fabricate, but requires a circulator or switch to operate effectively in a system.

The structure of Figure 5.9(c) is similar to the one-port delay line, but the acoustic wave interacts with an optic beam instead of returning to

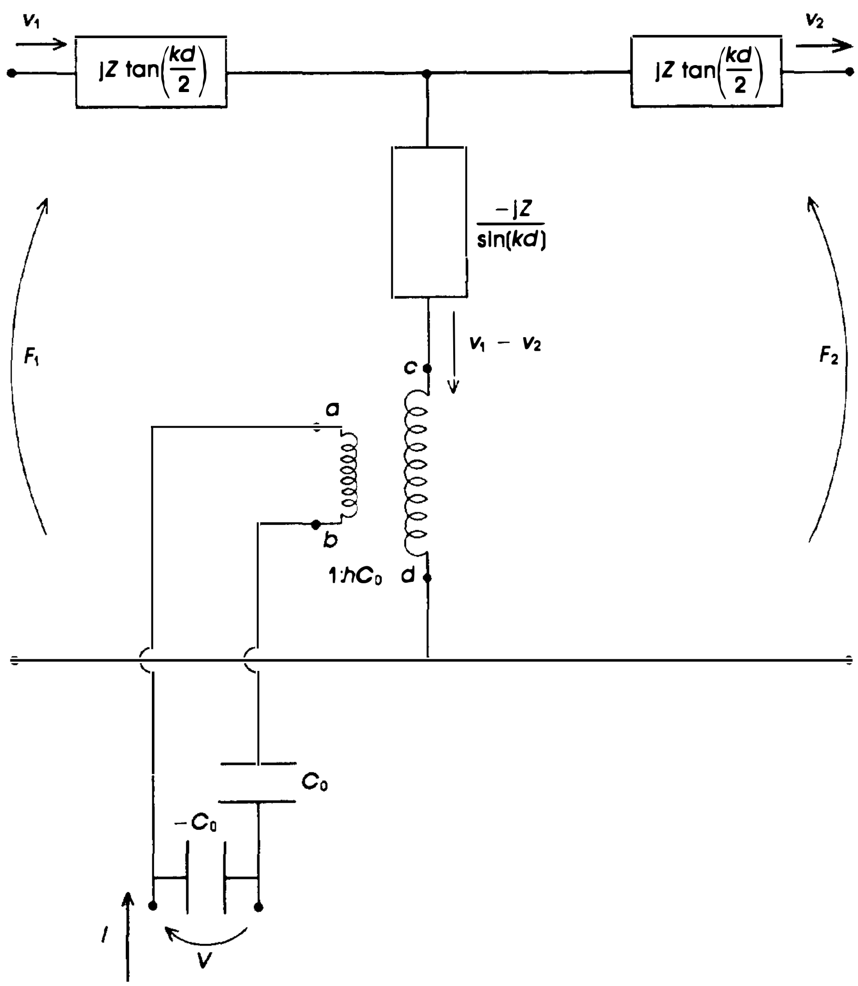


Figure 5.6 Final form of the three-port Mason model equivalent circuit.

the transducer. The return of the wave is prevented by an absorbing material placed at the end of the substrate, which is cut at an angle so as to deflect the wave from the transducer. Like the delay line, this device requires wide bandwidth and low conversion (of electrical to acoustic) loss. The substrate is simply a "dead" layer as in the delay line; its material properties determine the magnitude of the A/O interaction. We will study this configuration in Chapters 8 and 9.

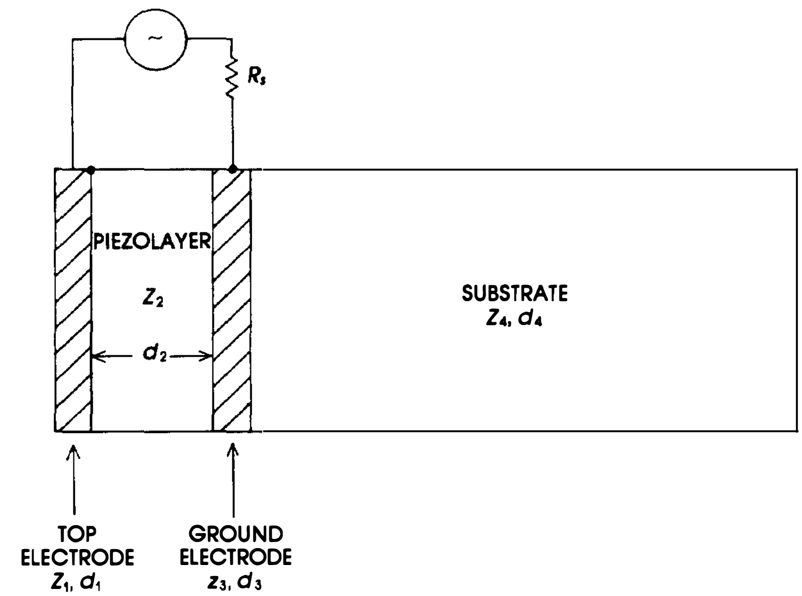


Figure 5.7 Schematic diagram of practical acoustic device containing four layers.

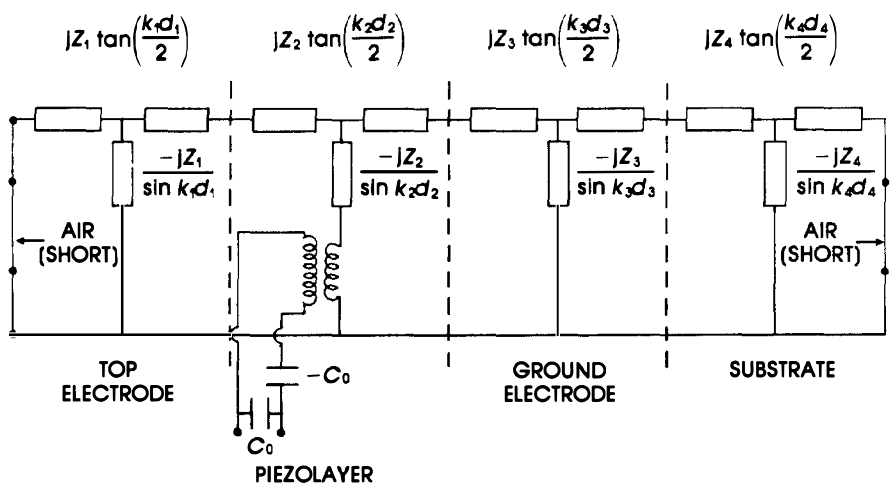


Figure 5.8 Equivalent circuit diagram of the four-layer acoustic device.

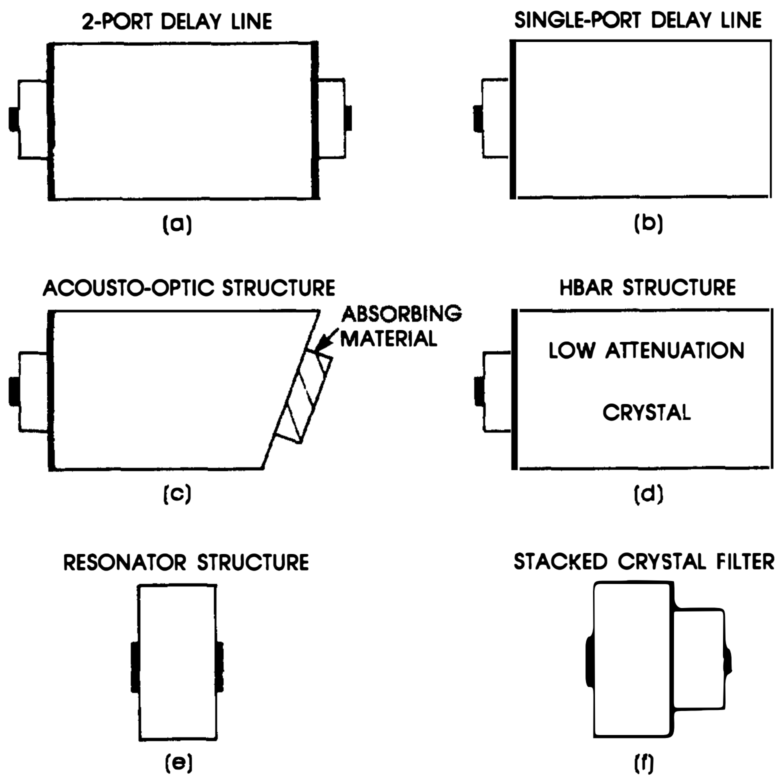


Figure 5.9 Representative devices that can be analyzed by acoustic modeling. We will consider configurations (b), (c), (d), and (e) in more detail.

If the attenuation in the substrate is substantially reduced, the wave is reflected back and forth many times before attenuating. This structure, shown in Figure 5.9(d), is a member of the class of devices called *composite resonators,* which we will investigate in Chapter 11. Because the device is used as a high-Q resonator, the substrate attenuation is the most critical factor. Use of high coupling transducers may actually degrade the performance. If the substrate is removed completely so that only the piezoelectric layer remains, the device (Figure 5.9(e)), is called a *thickness excited resonator.* Such devices are extremely important in many areas of electronics, from missiles to computers. This device configuration is the subject of Chapter 10. Finally, if two piezolayers are joined together as in Figure 5.9(f), the device is called a *stacked crystal filter.*

5.4 CLOSED-FORM EXPRESSION FOR THE INPUT IMPEDANCE

In general, it is not possible to obtain a closed-form solution for the electrical impedance, due to the complexity of the various layers in the Mason model. Each nonpiezoelectric layer is represented by a circuit of the form of Figure 5.2, whereas the piezolayer requires the three-port circuit of Figure 5.6. In two cases, however, it is possible to obtain closed-form solutions. One case is a piezoelectric resonator that is stress free on either side (no metallic or substrate layers). The second, somewhat more complex, case is the A/O structure of Figure 5.9(c). This is an example of a structure consisting of a combination of a piezoelectric body (the transducer) catenated with a nonpiezoelectric body. The metallic layers are assumed to be acoustically thin enough to neglect, and the substrate is infinitely long. The latter assumption implies that there is no reflected wave and thus no possibility of acoustic standing waves in the substrate (thus, only the transducer supports standing waves).

Our task is to solve for the input impedance as a function of frequency. The piezoelectric equations for the transducer are

$$F_1 = Z_T\left(\frac{v_1}{j\tan(kd)} - \frac{v_2}{j\sin(kd)}\right) + \frac{h}{j\omega}I \tag{5.35}$$

$$F_2 = Z_T\left(\frac{v_1}{j\sin(kd)} - \frac{v_2}{j\tan(kd)}\right) + \frac{h}{j\omega}I \tag{5.36}$$

where Z_T is the tranducer impedance. We also require (5.27):

$$V = \frac{h}{j\omega}(v_1 - v_2) + \frac{I}{j\omega C_0}$$

where

$$h = \frac{e}{\epsilon^S} \quad \text{and} \quad C_0 = \frac{\epsilon^S A}{d}$$

Equations (5.35), (5.36), and (5.27) are valid for a piezoelectric slab of arbitrary length. We now assume that the substrate is infinitely long and thus supports only one wave traveling to the right. Recall that F_1 is the force on the left face and F_2 is the force on the right side of the piezoelectric slab of length d. Referring to Figure 5.8, we note that the air interface on the left side of the piezolayer cannot exert a force, and thus $F_1 = 0$. The right side of the piezolayer "pushes" against the substrate, which pushes back with force:

$$F_2 = -Z_S v_2 \tag{5.37}$$

(because $T/v = Z$), where Z_S is the impedance of the substrate. Equation (5.37) is valid only if the substrate is either infinitely long or of finite length but very lossy (so that it possesses only a forward propagating wave). A wave that impinges on the piezolayer from its right (a left traveling wave) will interfere (either constructively or destructively) and change the force relationship. This point will be clarified later. With these substitutions (5.35), (5.36), and (5.27) become

$$0 = Z_T\left(\frac{v_1}{j\tan(kd)} - \frac{v_2}{j\sin(kd)}\right) + \frac{h}{j\omega}I \tag{5.38}$$

$$-Z_S v_2 = Z_T\left(\frac{v_1}{j\sin(kd)} - \frac{v_2}{j\tan(kd)}\right) + \frac{h}{j\omega}I \tag{5.39}$$

$$V = \frac{h}{j\omega}(v_1 - v_2) + \frac{I}{j\omega C_0}$$

These three equations contain the four unknowns v_1, v_2, V, and I. Recall that our task, however, is to solve for the input impedance (i.e., the ratio V/I). Dividing by I, we have three equations in three unknowns, namely the ratios V/I, v_1/I, and v_2/I.

We next make the substitutions:

$$x \rightarrow \frac{v_1}{I}, \quad y \rightarrow \frac{v_2}{I}, \quad c \rightarrow \frac{h}{j\omega}$$

Equations (5.38) and (5.39) become

$$0 = \frac{Z_T}{j\tan(kd)}x - \frac{Z_T}{j\sin(kd)}y + c \tag{5.40}$$

$$-Z_S v_2 = \frac{Z_T}{j\sin(kd)}x - \frac{Z_T}{j\tan(kd)}y + c \tag{5.41}$$

Equations (5.40) and (5.41) have the form

$$\begin{aligned} -c &= ax - by \\ -c &= bx - dy \end{aligned} \quad \text{or} \quad \begin{bmatrix} a & -b \\ b & -d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -c \\ -c \end{bmatrix}$$

with

$$a = \frac{Z_T}{j\tan(kd)}, \qquad b = \frac{Z_T}{j\sin(kd)}$$

$$d = \frac{Z_T}{j\tan(kd)} - Z_S$$

To solve for x and y, we simply invert the coefficient matrix:

$$\begin{bmatrix} x \\ y \end{bmatrix} = [\mathbf{A}]^{-1} \begin{bmatrix} -c \\ -c \end{bmatrix} \tag{5.42}$$

where

$$[\mathbf{A}]^{-1} = \begin{bmatrix} a & -b \\ b & -d \end{bmatrix}^{-1} = \frac{\begin{bmatrix} -d & -b \\ b & a \end{bmatrix}}{|\mathbf{A}|}$$

The determinant of the coefficient matrix is

$$\begin{aligned} |\mathbf{A}| &= -ad + b^2 \\ &= \frac{-Z_T^2\cos^2(kd) + Z_T^2 - jZ_SZ_T\sin(kd)\cos(kd)}{\sin^2(kd)} \end{aligned}$$

Solving for x (v_1/I), we have

$$\frac{v_1}{I} = \frac{(-Z_S\sin(kd) + j(Z_T\cos(kd) - Z_T))\sin(kd)}{-Z_T^2\cos^2(kd) + Z_T^2 - jZ_SZ_T\sin(kd)\cos(kd)} \cdot \left(\frac{h}{j\omega}\right) \tag{5.43}$$

Solving for y (v_2/I), we get

$$\frac{v_2}{I} = \frac{j(Z_T\cos(kd) - Z_T)\sin(kd)}{-Z_T^2\cos^2(kd) + Z_T^2 - jZ_SZ_T\sin(kd)\cos(kd)} \cdot \left(\frac{h}{j\omega}\right) \tag{5.44}$$

From (5.27), the input impedance is

$$\frac{V}{I} = \frac{h}{j\omega}(x - y) + \frac{1}{j\omega C_0} \tag{5.45}$$

Substituting for x and y from (5.43) and (5.44) (note that the denominators are equal); we obtain

$$\frac{V}{I} = \frac{h^2}{\omega^2 Z_T}\left(\frac{(Z_S/Z_T)\sin(kd) + 2j(1 - \cos(kd))}{\sin(kd) - j(Z_S/Z_T)\cos(kd)}\right) + \frac{1}{j\omega C_0} \tag{5.46}$$

The coefficient of (5.46) is written as

$$\frac{h^2}{\omega^2 Z_T} = \frac{e^2}{\epsilon^2\omega^2 Z_T} = \frac{e^2}{\omega^2 Z_T c\epsilon}\left(\frac{c}{\epsilon}\right) = k_t^2 \frac{c}{\epsilon Z_T \omega^2} \tag{5.47}$$

where

$$k_t^2 = \frac{e^2}{c^{\mathrm{D}}\epsilon^{\mathrm{S}}} \tag{5.48}$$

is the electromechanical coupling constant. The presence of c^{D} in (5.48) is a consequence of thickness excitation (Figure 5.3a). The relation between the electromechancial coupling constant k_t and the electromechancial coupling factor K follows from (4.62) and (5.48):

$$k_t^2 = \frac{e^2}{c^{\mathrm{D}}\epsilon^{\mathrm{S}}} = \frac{e^2}{(c^{\mathrm{E}} + e^2/\epsilon^{\mathrm{S}})\epsilon^{\mathrm{S}}} = \frac{K^2}{K^2 + 1} \tag{5.49}$$

in agreement with (4.63). Note the presence of k_t^2, which is due to c^{D} in (5.23).

Equation (5.47) can be further simplified to

$$\begin{aligned}\frac{k_t^2\, c}{\epsilon Z_T \omega^2} &= \frac{k_t^2\, cd}{\omega^2 \epsilon\sqrt{\rho c}\, Ad} = \frac{k_t^2 c}{\omega^2 C_0 d\sqrt{\rho c}} \\ &= \frac{k_t^2}{\omega^2 C_0 d}\sqrt{\frac{c}{\rho}} = \frac{k_t^2 v_a}{C_0 d\omega^2}\end{aligned} \tag{5.50}$$

Equation (5.46) takes the form:

$$Z_{\text{in}} = \underbrace{\frac{1}{j\omega C_0}}_{\substack{\uparrow \\ \text{capacitive} \\ \text{reactance of } C_0}} + \underbrace{P(\omega)\frac{R_0}{\omega^2 C_0^2}}_{\substack{\uparrow \\ \text{piezoelectric} \\ \text{contribution}}} \tag{5.51}$$

where

$$R_0 = \frac{k_t^2 v_a C_0}{d} \tag{5.52}$$

and

$$P(\omega) = \frac{(Z_S/Z_T)\sin(kd) + 2j(1 - \cos(kd))}{\sin(kd) - j(Z_S/Z_T)\cos(kd)}$$

If there is no piezoelectricity, $R_0 \rightarrow 0$ and the device looks like a "dead" capacitor.

We next consider the function $P(\omega)$ more closely. It is a complex function, so it can easily be written explicitly in terms of its real and imaginary parts.

$$P(\omega) = \frac{a + jb}{c - jd} = \frac{ac - bd + j(bc + ad)}{c^2 + d^2}$$

where

$$a = \frac{Z_S}{Z_T}\sin(kd), \quad b = 2(1 - \cos(kd),$$

$$c = \sin(kd), \qquad d = \frac{Z_S}{Z_T}\cos(kd)$$

The denominator is thus

$$c^2 + d^2 = \sin^2(kd) + \left(\frac{Z_S}{Z_T}\right)^2 \cos^2(kd) = \mathrm{D}$$

The real part is

$$ac - bd = -2(1 - \cos(kd)\left(\frac{Z_S}{Z_T}\cos(kd)\right) + \frac{Z_S}{Z_T}\sin^2(kd)$$
$$= \frac{Z_S}{Z_T}(\cos^2(kd) - 2\cos(kd) + 1)$$

Thus the real part of $P(\omega)$ is thus

$$P_r(\omega) = \frac{(Z_S/Z_T)\,(1 - \cos(kd))^2}{\mathrm{D}} \tag{5.53}$$

Similarly, for the imaginary part of $P(\omega)$, we write

$$bc + ad = 2(1 - \cos(kd))\sin(kd) + \left(\frac{Z_S}{Z_T}\right)^2 \sin(kd)\cos(kd)$$
$$= \sin(kd)\left(2 - 2\cos(kd) + \left(\frac{Z_S}{Z_T}\right)^2 \cos(kd)\right)$$

and thus

$$P_i(\omega) = 2\sin(kd)\,\frac{1 + \cos(kd)\,[1/2\,(Z_S/Z_T)^2 - 1]}{\mathrm{D}} \tag{5.54}$$

Resonance is *defined* as the frequencies at which $P_i(\omega) = 0$; from (5.54);

$$k_n d = (2n + 1)\pi, \qquad n = 0, 1, \ldots$$

or

$$\omega_n = \frac{(2n + 1)\pi v_a}{d} \tag{5.55}$$

At resonance, the resistive part of the complex impedance is near a maximum (which is precisely the opposite of our intuitive notion of resonance). It is perhaps more appropriate to refer to (5.55) as defining the antiresonances. For the fundamental,

$$\omega_0 = \frac{\pi v_a}{d} \tag{5.56}$$

Finally, we can write for the input impedance:

$$Z_{\text{in}} = \frac{1}{j\omega C_0} + Z_a$$

where Z_a is the contribution due to piezoelectric energy conversion:

$$Z_a = \hat{R}_a \left(\frac{\omega_0}{\omega}\right)^2 [P_r'(\omega) + jP_i'(\omega)] \tag{5.57}$$

where $P_r'(\omega)$ and $P_i'(\omega)$ are given by

$$\begin{aligned} P_r'(\omega) &= P_r(\omega)\left(\frac{Z_S}{4Z_T}\right) \\ P_i'(\omega) &= P_i(\omega)\left(\frac{Z_S}{4Z_T}\right) \end{aligned} \tag{5.58}$$

and

$$\hat{R}_a = \frac{4}{\pi}\frac{Z_T}{Z_S}\frac{k_t^2}{\omega_0 C_0} \tag{5.59}$$

Separating the factor $Z_S/4Z_T$ in (5.57) and (5.58) ensures the normalization of $P_r'(\omega)$ at resonance. At resonance,

$$kd = (2n + 1)\pi \tag{5.60}$$

and from (5.53), (5.54), and (5.58),

$$P_i'(\omega) = 0, \quad P_r'(\omega) = 1$$

The real part of the *acoustic* contribution $P(\omega)$ is a relatively complex even function of the resonance frequencies ω_n. The precise shape of this function depends on the impedance ratio Z_S/Z_T. Because the real part of Z_a determines the transfer of electrical energy to acoustic energy, it is clear that for frequencies far removed from resonances the energy conversion will decrease dramatically. The device "looks" reactive except near the resonance frequencies. The term that determines the real part of the input impedance is the *radiation resistance* $\hat{R}_a$, defined in (5.59). Indeed, $\hat{R}_a$ is equal to the real part of Z_{in} at the fundamental resonance ω_0. It is analogous to the radiation resistance of an antenna. At higher-order resonances, the radiation resistance decreases by n^2, where n is the harmonic number. This decrease is due to the $(\omega_0/\omega)^2$ term in (5.57). Alternatively, we may view the coupling constant as being decreased by n^2.

Wideband design requires that $\hat{R}_a$ be well matched electrically to the source resistance (usually 50 Ω). The factors that determine the radiation resistance are

1. the ratio of substrate to transducer impedance (Z_S/Z_T);
2. the clamped capacitance of the transducer C_0, which is determined by its dielectric constant (ϵ_r) and the active radiating area (A), at resonance,

$$C_0 = \frac{\epsilon_r^S \epsilon_0 A}{d} = \frac{\epsilon_r^S \epsilon_0 A \omega_0}{\pi v_a}$$

For a given choice of operating frequency and transducer, substrate materials, and orientations, the only factor that the designer can use to control $\hat{R}_a$ is the radiating area A of the transducer (the "hot" electrode). At high frequencies, the requirement that the radiation resistance be 50 Ω leads to extremely small active areas, which may degrade device performance (cause excessive acoustic diffraction or low deflection efficiency in A/O devices) and which are difficult to bond. In these cases, matching networks are required to allow efficient conversion from electrical energy to acoustic energy. This topic will be explored in Chapter 6.

Figure 5.10 shows the real part of Z_{in} for the fundamental and third harmonic frequencies for several impedance ratios (Z_S/Z_T). The radiation resistance at the fundamental frequency has been normalized to 1. We note that $\hat{R}_a$ at the third harmonic has been reduced by a factor of 9; the reduction in $\hat{R}_a$ is independent of the impedance ratio, and makes harmonic operation very difficult at high frequencies. We change not only the value of $\hat{R}_a$ by varying the impedance ratio (in these figures it has been normalized by varying A), but also the shape $\text{Re}(Z_{\text{in}})$. The curves are not even functions around ω_0 due to the $(\omega_0/\omega)^2$ term in (5.57)

The imaginary part of the input impedance is composed of the capacitive reactance of C_0 (the piezotransducer acting as a "dead" capacitor) in series with the imaginary part of the acoustic impedance, as given in (5.54). As for the imaginary part of the acoustic impedance, the function $P_i(\omega)$ is an odd function of frequency around resonances. At resonance, $P_i(\omega) = 0$, and the device looks like a resistor of resistance $\hat{R}_a$ in series

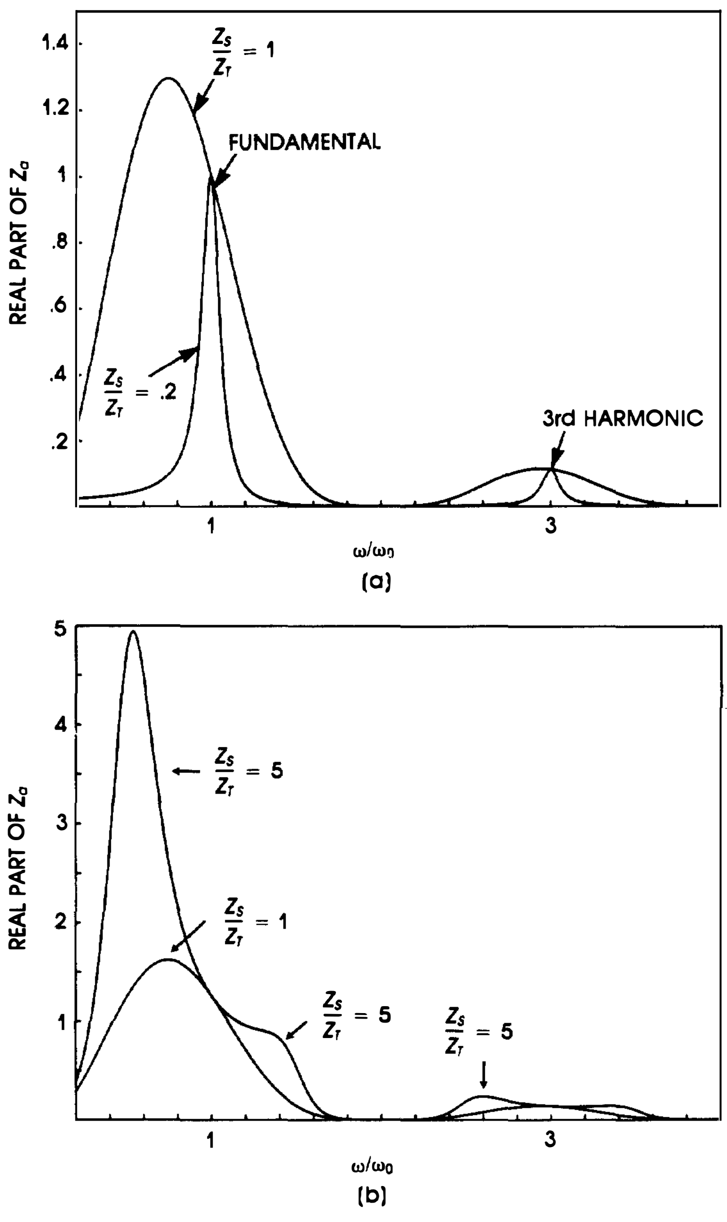


Figure 5.10 Real part of the acoustic contribution to the electrical input impedance. All curves are normalized (i.e., the radiating area is adjusted), so $\hat{R}_a = 1$ at resonance.

with C_0, as shown in Figure 5.11. We can make the impedance purely resistive by "resonating" C_0 with an inductor of value,

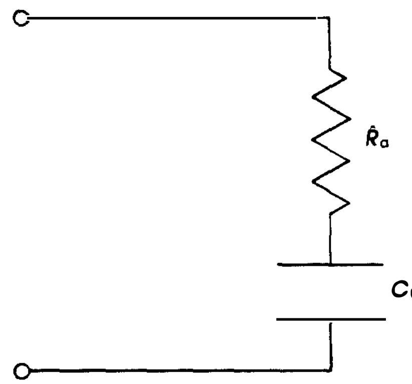


Figure 5.11 Electrical equivalent circuit of configuration 5.9(c) at resonance.

$$L_0 = \frac{1}{\omega_0^2 C_0} \tag{5.61}$$

For frequencies less than ω_0, $P_i(\omega)$ is positive (inductive); for $3\omega_0 > \omega > \omega_0$, $P(\omega)$ is capacitive. For acoustic impedance ratios greater than about .3, the inductive region is swamped by the capacitive reactance of C_0, so the device looks capacitive for all frequencies. This behavior changes dramatically for finite-dimensional substrates in which acoustic resonances occur in the substrate. The reflected wave interferes either constructively or destructively, causing the input impedance to appear inductive over appreciable frequency bands periodically as a function of the substrate length. These substrate standing waves add another dimension to an already complex problem. It is quite difficult to derive a closed-form expression for the impedance of such devices, but in Chapter 6 we will study their behavior by using computer models. Even for the case under consideration here, in which there are no acoustic reflections in the substrate (infinite substrate length), we cannot determine the reactive character of the device for all frequencies unless we know the acoustic impedance ratio. Thus, the device is represented in Figure 5.12 as a resistor in series with C_0 and an electrical reactance due to X_a, which may be capacitive or inductive.

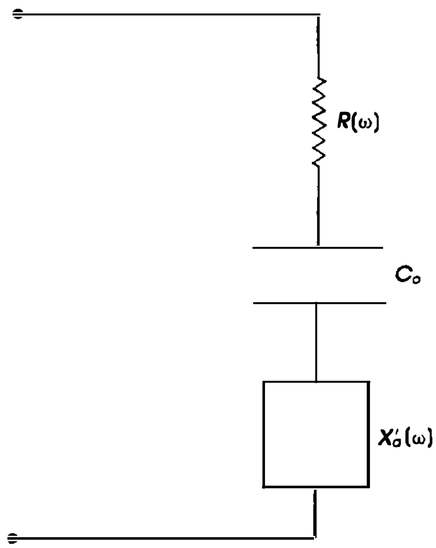


Figure 5.12 Electrical equivalent circuit of configuration 5.9(c) away from resonance. The reactance $X_a'(\omega)$ depends on the impedance ratio Z_S/Z_T.

At resonance, the device looks like a capacitor C_0 in series with a resistor $\hat{R}_a$, as shown in Figure 5.11; hence, it is natural to define the Q of this circuit in the ordinary way:

$$Q = \frac{1}{\omega_0 C_0 \hat{R}_a} \tag{5.62}$$

Substituting for $\hat{R}_a$ from (5.59) yields

$$Q = \frac{f_0}{\Delta f} = \frac{\pi}{4k_t^2}\frac{Z_T}{Z_S} \tag{5.63}$$

Wideband operation requires a close match between transducer and substrate and a large coupling constant. For given material properties, an obvious but important consequence of (5.63) is the linear dependence of the bandwidth on center frequency.

Figure 5.13 shows the sum of Z_a (the reactance due to acoustic conversion) and the capacitive reactance of C_0. If the ratio $Z_s/Z_T = 1$, the total reactance is capacitive for all frequencies. Also in Figure 5.13 is the

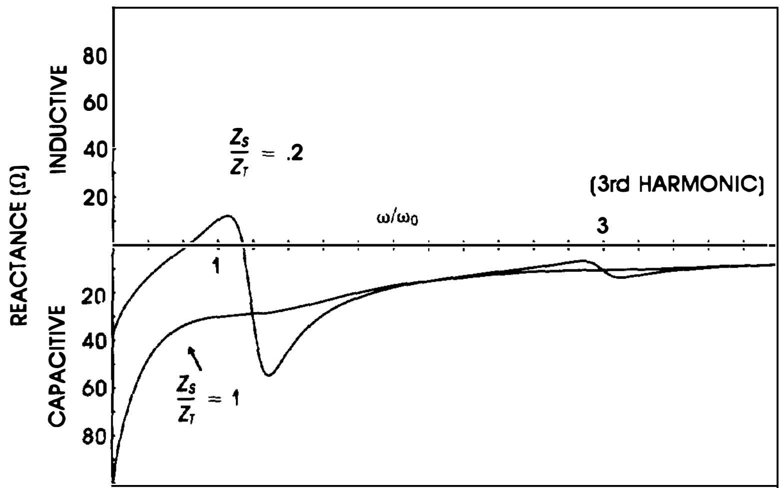


Figure 5.13 Imaginary part of the acoustic contribution to the input impedance. $\mathrm{Im}(Z_{i\bar{m}}) = Z_a + (1/j\omega C_0)$. If Z_S/Z_T is small enough, the reactance becomes inductive near resonance. As in Figure 5.10, the area is adjusted (normalized curves), so $\hat{R}_a = 1$ at resonance.

device reactance for an impedance ratio of.2. By making the ratio small enough, we can make the reactance inductive for a small band of frequencies. Decreasing the ratio further would cause an inductive region in the higher-order harmonics. Increasing the ratio sufficiently also allows the reactance to become inductive for frequencies just below resonance. In both of these cases, if the impedance mismatch between the transducer and substrate is large enough, most of the acoustic wave generated in the transducer is not transferred to the substrate. As $Z_S/Z_T \rightarrow 0$, we have the case of thickness excited crystal resonators, which we will study in Chapter 10. The case of $Z_S/Z_T > 5$ in which a low impedance transducer is connected to a high impedance (rigid) material, such as tungsten, is important in many low frequency applications. In this case, quarter-wave resonances are obtained. We leave the details as an exercise.

5.5 INPUT IMPEDANCE EXAMPLES

In this section, we consider practical examples that reflect impedance ratios considered previously. As in the previous section, the radiation resistances are normalized (by varying the active radiating area).

Example 5.1. Polyvinyl fluoride (PVF_2) on fused quartz. This example illustrates the case in which the substrate impedance is much larger than the impedance of acoustic transducer. Polyvinyl fluoride belongs to a new class of piezoelectric plastics characterized by relatively low dielectric constant, moderate piezoelectric coupling, low acoustic velocities, and low density. Typical values are

$$\left.\begin{array}{l}\rho = 1.8 \times 10^3 \text{ kg/m}^3 \\ v_a = 2.2 \times 10^3 \text{ m/s}\end{array}\right\} \rightarrow Z_T = 4 \times 10^6 A \text{ kg/s}$$

The permittivity is smaller by a factor of about 10 compared with single-crystal transducers, whereas the coupling constant is about .2 to .3.

For fused quartz we need only the acoustic parameters:

$$\left.\begin{array}{l}\rho = 2.65 \times 10^3 \text{ kg/m}^3 \\ v_a = 6 \times 10^3 \text{ m/s}\end{array}\right\} \rightarrow Z_S = 1.6 \times 10^7 A \text{ kg/s}$$

This example corresponds to the case in which the substrate has a much higher impedance than does the transducer. For low frequency applications, the radiation resistance may become quite large. Operation at 50 MHz requires that the transducer thickness be

$$d = \frac{v_a}{2f} = 22\ \mu m\ (\approx 1 \text{ mil})$$

Example 5.2. $\langle x \rangle$ quartz (longitudinal mode) on $\langle z \rangle$ rutile (TiO_2). For this case, the substrate impedance is moderately larger than the transducer impedance. For the quartz transducer,

$$\left.\begin{array}{l}\rho = 2.65 \times 10^3 \text{ kg/m}^3 \\ v_a = 5.7 \times 10^3 \text{ m/s}\end{array}\right\} \rightarrow Z_T = 15 \times 10^6 A \text{ kg/s}$$

and $\epsilon_r = 4.5$ and $k_t^2 = 8.5 \times 10^{-3}$. For the substrate,

$$\left.\begin{array}{l}\rho = 4.2 \text{ kg/m}^3 \\ v_a = 7.9 \times 10^3 \text{ m/s}\end{array}\right\} \rightarrow Z_S = 3.3 \times 10^7 A \text{ kg/s}$$

Operation at 100 MHz requires a transducer thickness of 28 μm. The transducer dimensions for a 50-Ω $\hat{R}_a$ are

$$C_0 = 1.4 \text{ pF} \rightarrow A = 10 \times 10^{-7} \text{m}^2 (1 \text{ mm} \times 1 \text{ mm})$$

The impedance ratio Z_S/Z_T in this case is about 2. Even though quartz has a very low coupling constant, a low dielectric constant, and a low acoustic impedance compared with most other single crystals, its low acoustic impedance allows wider bandwidths than might otherwise be expected with reasonable transducer areas. Even so, the low coupling constant limits the fractional acoustic bandwidth to at most 20%.

Example 5.3: 36° $\langle y \rangle$ $LiNbO_3$ (longitudinal mode) on $\langle x \rangle$ $LiNbO_3$. In this example, the transducer and substrate are well matched, a setup used in the fabrication of high frequency A/O devices. For the transducer,

$$\left.\begin{array}{l} \rho = 4.6 \times 10^3 \text{ kg/m}^3 \\ v_a = 7.4 \times 10^3 \text{ m/s} \end{array}\right\} \rightarrow Z_T = 3.4 \times 10^7 A \text{ kg/s}$$

and $\epsilon_r = 40$ and $k_t^2 = .25$. For the substrate,

$$\left.\begin{array}{l} \rho = 4.6 \times 10^3 \text{ kg/s} \\ v_a = 6.6 \times 10^3 \text{ m/s} \end{array}\right\} \rightarrow Z_S = 3 \times 10^7 A \text{ kg/s}$$

These values correspond to the components of input impedance in which the transducer and substrate are closely matched. For operation at 2 GHz, the transducer thickness is about 1.7 μm, and the transducer area for 50 Ω is

$$C_0 = 6 \times 10^{-13} \rightarrow A = 3 \times 10^{-9} \text{ (2 mil} \times \text{2 mil)}$$

For a more reasonable transducer area, the clamped capacitance will increase considerably, causing the radiation resistance to drop, probably to below 10 Ω. Because of the close acoustic match, it is possible to fabricate devices with bandwidths of nearly an octave by using broadband matching techniques, which we consider in Chapter 6.

Example 5.4: We conclude with an example in which $Z_S/Z_T \ll 1$. Consider $\langle x \rangle$ $LiNbO_3$ (shear) on $\langle 1, 1, 0 \rangle$ TeO_2 (slow shear). As in the previous example, this combination of transducer and substrate is quite important in practical A/O devices. For the transducer,

$$\left.\begin{array}{l} \rho = 4.6 \times 10^3 \text{ kg/m}^3 \\ v_a = 4.8 \times 10^3 \text{ m/s} \end{array}\right\} \rightarrow Z_T = 2.2 \times 10^7 A \text{ kg/s}$$

and $\epsilon_r = 44$ and $k_t^2 = .46$. The substrate values are

$$\left.\begin{array}{l}\rho = 6 \times 10^3 \text{ kg/m}^3 \\ v_a = .62 \times 10^3 \text{ m/s}\end{array}\right\} \rightarrow Z_S = 3.7 \times 10^6 A \text{ kg/s}$$

In this case, there is a severe mismatch because the impedance of the substrate is extremely low. The radiation resistance is increased at the expense of reduced bandwidth (5.63) when $Z_S/Z_T \leq .2$. In this case, the bandwidth is severely restricted by the large acoustic mismatch. This is analogous to the optic mismatch encountered at an interface between media of different indices of refraction. We will design an acoustic *anti-reflection* coating in Chapter 6 for this transducer-substrate combination.

Table 5.1 lists the properties of some common transducer-substrate combinations.

Table 5.1

Transducer/Substrate	Z_S/Z_T	k_t^2	*Application*
$\frac{LiNbO_3\ (36\ \langle y \rangle)}{GaP\ (\langle 1, 1, 0 \rangle)}$	.76	.25	Bragg cell
$\frac{LiNbO_3\ (\langle x \rangle)}{TeO_2\ (\langle 1, 1, 0 \rangle)}$	.18	.46	Bragg cell
$\frac{Ba_2NaNb_5O_{15}(\langle z \rangle)}{GaP\ (\langle 1, 1, 0 \rangle)}$	1.0	.26	Bragg cell
$\frac{ZnO\ (\langle z \rangle)}{YAG}$	1.1	.08	Delay line

PROBLEMS

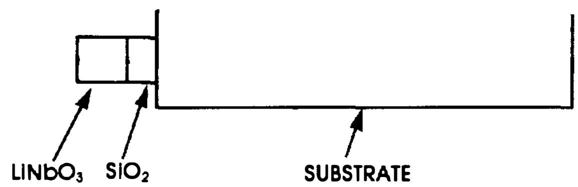
5.1 Find the required transducer area for a 36° $\langle y \rangle$ lithium niobate transducer on gallium phosphide $\langle 1, 1, 0 \rangle$ that is to operate in the third harmonic at 1 GHz. The properties of the substrate are $\rho = 4.13$ and $v_a = 6.3 \times 10^3$ m/s.

5.2 Let $Z_T/Z_S \rightarrow 0$. Write the acoustic component of the input impedance Z_a in closed form. What are the resonance conditions?

5.3 Repeat Problem 5.1 for the fifth harmonic.

5.4 Plot the functions $P_i(\omega)$ and $P_r(\omega)$ for the acoustic impedance ratios (Z_S/Z_T) .5, 1, and 2.

5.5 Discuss the performance of the following acoustic structure:



5.6 Draw the equivalent Mason model circuit of the stacked crystal filter (Figure 5.9(f)).

5.7 Write a computer program to find the electrical input impedance of the infinite delay line structure (see (5.51)).

REFERENCES

1. B. Auld, *Acoustic Fields and Waves in Solids,* John Wiley and Sons, New York, 1973, Chapter 9.
2. V. Ristic, *Principles of Acoustic Devices,* John Wiley and Sons, New York, 1983 Chapter 5.
3. T. Reeder and D. Winslow, "Characteristics of Microwave Acoustic Transducer for Volume Wave Excitation," *IEEE Trans. Microwave Theory and Tech.* **MTT-17** (11), 927 (1969).
4. N. Uchida, S. Fukunishi, and S. Saito, "Performance of Single Crystal $LiNbO_3$ Transducers Operating above 1 GHz," *IEEE Trans. Sonics and Ultrasonics* **SU-20** (3), 285 (1973).
5. I. Sato and A. Fukumoto, "Input Characteristic of Acousto-Optic Light Deflector," *IEEE Trans. Sonics and Ultrasonics* **SU-20** (3), 287 (1973).
6. D. Denburg, "Wide-Bandwidth High Coupling Sputtered ZnO Transducers on Sapphire," *IEEE Trans. Soncis and Ultrasonics* **SU-18,** 31 (1971).
7. A. Meitzler and E. Sittig, "Characterization of Piezoelectric Transducers Used in Ultrasonic Devices Operating above .1 GHz," *J. Appl. Phys.* **40,** 4341 (1969).