# Chapter 2
# Propagation of Acoustic Waves in Crystals

## 2.1 INTRODUCTION

When applied to a three-dimensional geometry, Hooke's law takes a relatively complicated form because the stiffness components couple two $3 \times 3$ matrices. The nature of the coupling is determined by the *symmetry* properties of the particular crystal medium, which determine a stiffness *matrix*. In this chapter, we develop the symmetry conditions and show how they reduce the number of stiffness components from a maximum of 21 in the most general class to only 2 for isotropic media.

The three-dimensional wave equation is generally referred to as the *Christoffel equation*. It admits three solutions, the properties of which are determined by the relation of the propagation direction to the stiffness matrix. We examine the propagation in several important directions for isotropic, cubic, tetragonal, and orthorhombic symmetries. The solutions for an arbitrary direction are carried out by computer in Chapter 3.

### 2.1.1 Hooke's Law in Three Dimensions

The application of an external force to a solid body produces internal stresses and distortions (strains) in the medium. Consider, for example, a stress in the $x$ direction (Figure 2.1) The relation between the stress and strain is given by Hooke's law:

$$\mathbf{T} = \mathbf{c} : \mathbf{S} \tag{2.1}$$

It is a fundamental law of physics that the existence of a stress in any direction is accompanied by strains not only in the same direction but in perpendicular directions as well. For example, if a solid is compressed in

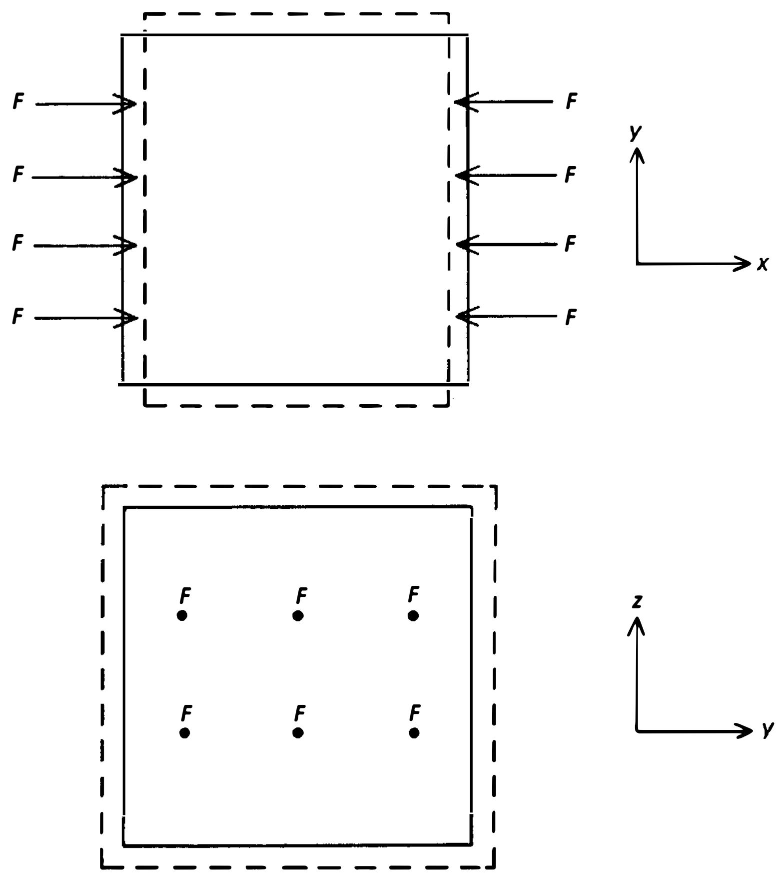


**Figure 2.1** Relation of orthogonal stresses and strains. An $x$-directed longitudinal stress $T_{xx}$ couples to $y$- and $z$-directed strains $S_{yy}$ and $S_{zz}$.

the $x$ direction, there will be an elongation in the $y$ and $z$ directions. This effect is easily observable in highly elastic materials such as rubber. We say that an $x$-directed stress *couples* to $y$- and $z$-directed strains; this coupling also obeys Hooke's law, with different proportionality constants. Furthermore, the coupling is generally not the same for different directions. In rubber, an isotropic material, an $x$-directed stress couples to equal $y$- and $z$-directed strains, whereas in wood (an anisotropic material) the perpendicular strains depend on the orientations of the particular directions with respect to the grain. The ratio of the perpendicular strain to the primary strain is called the *Poisson ratio*, which is a (dimensionless) constant of the material only. It is found experimentally that the Poisson ratio lies between 1/4 and 1/3 for most materials, with a limiting value of about 1/2.

Thus, our first task is to examine the constitutive relations between **T** and **S**. From the foregoing argument, we can write (for an isotropic body)

$$T_{xx} = \underbrace{c_{xxxx}S_{xx}}_{\text{primary term}} + \underbrace{c_{xxyy}S_{yy} + c_{xxzz}S_{zz}}_{\text{terms due to Poisson ratio}} \tag{2.2}$$

Similarly,

$$T_{yy} = c_{yyxx}S_{xx} + c_{yyyy}S_{yy} + c_{yyzz}S_{zz} \tag{2.3}$$

and

$$T_{zz} = c_{zzxx}S_{xx} + c_{zzyy}S_{yy} + c_{zzzz}S_{zz} \tag{2.4}$$

where, for isotropic materials:

$$c_{xxyy} = c_{xxzz} = c_{yyxx} = c_{yyzz} = c_{zzxx} = c_{zzyy}$$

and

$$c_{xxxx} = c_{yyyy} = c_{zzzz}$$

It is also clear that shear stress couples to shear strain with a different proportionality; for isotropic materials:

$$T_{xy} = c_{xyxy}S_{xy},\ T_{xz} = c_{xzxz}S_{xz},\ T_{yz} = c_{yzyz}S_{yz} \tag{2.5}$$

with $c_{xyxy} = c_{xzxz} = c_{yzyz}$. The shear stiffness components generally are significantly less than the longitudinal components. Note that there is no analogue to the Poisson ratio in the shear stress-strain relation for an isotropic material; in anisotropic crystals it is, however, possible for a given shear stress to couple to all three shear strains. Indeed, although not intuitively obvious, it is nevertheless true that, depending on the internal crystal structure, a given stress can couple to the six possible shear strains. In general, we write

$$\begin{aligned} T_{xx} = {} & \overset{\text{"normal" Hooke's law term}}{\overset{\downarrow}{c_{xxxx}S_{xx}}} + c_{xxxy}S_{xy} + c_{xxxz}S_{xz} \\ & + c_{xxyx}S_{yx} + \overset{\text{Poisson term}}{\overset{\downarrow}{c_{xxyy}S_{yy}}} + c_{xxyz}S_{yz} \\ & + c_{xxzx}S_{zx} + c_{xxzy}S_{zy} + \underset{\text{Poisson term}}{\underset{\uparrow}{c_{xxzz}S_{zz}}} \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} T_{yx} = {} & c_{yxxx}S_{xx} + c_{yxxy}S_{xy} + c_{yxxz}S_{xz} + c_{yxyx}S_{yx} + c_{yxyy}S_{yy} \\ & + c_{yxyz}S_{yz} + c_{yxzx}S_{zx} + c_{yxzy}S_{zy} + c_{yxzz}S_{zz} \end{aligned} \tag{2.7}$$

Equation (2.6) is a generalization of the isotropic relation (2.2). In (2.6) and (2.7) the three diagonal terms (in which the strains have repeated subscripts) represent the relation of a given stress (longitudinal in (2.6) and shear in (2.7)) to longitudinal strains, and the six off-diagonal terms represent the coupling of the stress to the shear strains. The first two subscripts on the $c$ values correspond to the stress terms, and the latter two correspond to the strains. Each equation of the form of (2.6) and (2.7) contains nine terms, and there are nine such equations (corresponding to the nine terms of the stress matrix). Thus, there are a total of 81 possible $c$ values. In (2.6) and (2.7), there are six "nonobvious" terms (which in most materials are not present); in (2.6) these terms represent the coupling of the longitudinal stress in the $x$ direction $T_{xx}$ to *shear* strains. In (2.7), the only intuitive term is $T_{yx} = c_{yxyx} S_{yx}$ and in most materials all other terms are not present.

Alternatively, we can write the stress as the independent variable:

$$\begin{aligned} S_{xx} = {} & s_{xxxx}T_{xx} + s_{xxxy}T_{xy} + s_{xxxz}T_{xz} + s_{xxyx}T_{yx} + s_{xxyy}T_{yy} \\ & + s_{xxyz}T_{yz} + s_{xxzx}T_{zx} + s_{xxzy}T_{zy} + s_{xxzz}T_{zz} \end{aligned} \tag{2.8}$$

Equations (2.6) and (2.7) can be put into a shorthand notation by writing

$$T_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} c_{ijkl}S_{kl} \tag{2.9}$$

where $i$, $j$, $k$, and $l$ represent $x$, $y$, and $z$.

In (2.6), $i = j = 1$, whereas in (2.7) $i = 2$ and $j = 1$. In (2.9), we note that the *repeated* letters are arbitrary (we could have substituted any other symbols for $k$ and $l$) and are called *dummy indices*. Following the accepted convention that a repeated index implies a summation over that index, we can write (2.8) and (2.9) as

$$T_{ij} = c_{ijkl}S_{kl} \tag{2.10}$$

$$S_{ij} = s_{ijkl}T_{kl} \tag{2.11}$$

where each stress ($T_{ij}$) represents one equation with nine terms. Equations (2.10) and (2.11) can also be written in a matrix format by writing the stress and strain matrices as $9 \times 1$ column vectors. The $c$ and $s$ values would be arranged as a matrix of $9 \times 9$ elements. The **c** matrix is called the *stiffness* matrix, and the **s** matrix is called the *compliance* matrix. While these definitions are logical (a stiff medium requires a large stress to couple to a given strain), the symbols are the reverse of what may be expected (i.e., **s** is compliance and **c** is stiffness).

Various basic physical principles can reduce the number and complexity of the stiffness and compliance matrices. From (1.48) and (1.51), we know that the stress and strain matrices are symmetric; i.e.,

$$T_{ij} = T_{ji}, \quad i \neq j$$

Therefore, of the nine equations of (2.10) only six are independent; this reduces the number of independent $c$ values by 27 (three equations each with nine terms) to 54. Furthermore,

$$S_{kl} = S_{lk}, \quad l \neq k$$

Because there are three of these off-diagonal terms for each of the remaining six equations, the number of independent $c$ values is further reduced by 18 for a total of 36. Thus the symmetry conditions for the stress and strain matrices (which are valid for all materials) result in the constraints:

$$c_{ijkl} = c_{jikl} \quad \text{(symmetry of } \mathbf{T}\text{)} \tag{2.12}$$

$$c_{ijkl} = c_{ijlk} \quad \text{(symmetry of } \mathbf{S}\text{)} \tag{2.13}$$

We can use the concept of the abbreviated subscripts (2.10) to write

$$\rho v \frac{\partial v}{\partial t} = \frac{\partial u_v}{\partial t}$$

$u_v$ is the kinetic energy density, and

$$T\frac{\partial S}{} \quad \frac{\partial u_s}{}$$

$u_s$ is the In three dimensions, the rate of change of strain e

$$\frac{\partial u}{\partial t} \quad -$$

where : ltiplication. In the summation notation, we have

$$\frac{\partial u}{\partial t} \quad T_I\frac{\partial S_I}{\partial t} = c_{IJ}S_J\frac{\partial S_I}{\partial t} \tag{2.20}$$

Eliminating the time derivative gives

$$\mathrm{d}u_s = c_{IJ}S_J\,\mathrm{d}S_I$$

or

$$\frac{\partial u_s}{\partial S_I} = c_{IJ}S_J$$

Differentiating with respect to $S_J$, we obtain

$$\frac{\partial^2 u_s}{\partial S_I \partial S_J} = c_{IJ} \tag{2.21}$$

Now the same procedure can be repeated with the order of differentiation reversed so that

$$c_{IJ} = \frac{\partial^2 u_s}{\partial S_I \partial S_J} = \frac{\partial^2 u_s}{\partial S_J \partial S_I} \equiv c_{JI}$$

The stiffness matrix is symmetric. Because there are 30 off-diagonal terms (36 minus the 6 diagonal terms) and half of them are dependent, we conclude that in the most general case there must be 21 (36 minus 15) independent terms in the stiffness matrix. The reduction process is shown in Table 2.2.

**Table 2.2**

| | | | |
|---|---|---|---|
| $T_{ij} = T_{ji}$ | $c_{ijkl} = c_{jikl}$ | $i \neq j$ | reduces from 81 to 54 |
| $S_{kl} = S_{lk}$ | $c_{ijkl} = c_{ijlk}$ | $k \neq l$ | reduces from 54 to 36 |
| symmetry of $\mathbf{c}$ | $c_{IJ} = c_{JI}$ | $I \neq J$ | reduces from 36 to 21 |

## 2.3 THE STIFFNESS AND COMPLIANCE MATRICES IN AN ISOTROPIC MEDIUM

As an example of the properties of the stiffness and compliance matrices, we consider an isotropic solid. As we have seen, a stress $T_1$ couples to strains $S_1$, $S_2$, and $S_3$. The ratio of $T_1$ to $S_1$ (or $T_2$ to $S_2$) is called Young's modulus, $Y_0$. If only $T_1$ is present, we write

$$S_1 = \frac{T_1}{Y_0}, \quad S_2 = \frac{\sigma T_1}{Y_0}, \quad S_3 = \frac{\sigma T_1}{Y_0} \tag{2.22}$$

and

$$\sigma \triangleq -\frac{S_2}{S_1} = -\frac{S_3}{S_1}$$

where $\sigma$ is the Poisson ratio, which is negative (a compressive stress in $x$ usually couples to elongations in the $y$ and $z$ directions). If $T_1$, $T_2$, and $T_3$ are present, we write

$$\begin{aligned} S_1 &= \underbrace{\frac{T_1}{Y_0}}_{\substack{\text{primary} \\ \text{strain}}} + \underbrace{\frac{\sigma(T_2 + T_3)}{Y_0}}_{\substack{\text{strain } S_1 \text{ due to } T_2 \\ \text{and } T_3 \text{ ("Poisson" term)}}} \\ S_2 &= \frac{T_2}{Y_0} + \frac{\sigma(T_1 + T_3)}{Y_0} \\ S_3 &= \frac{T_3}{Y_0} + \frac{\sigma(T_1 + T_2)}{Y_0} \end{aligned} \tag{2.23}$$

The first term represents the "primary" strain for a given stress, and the second term represents the contributions from Poisson's ratio. We will show that these are the only contributions (there is no coupling between longitudinal strains and shear stress). There are, however, shear strains that are coupled to shear stresses. These strains are given by

$$S_4 = \frac{T_4}{\mu}, \quad S_5 = \frac{T_5}{\mu}, \quad S_6 = \frac{T_6}{\mu} \tag{2.24}$$

The quantity $\mu$ is called the shear modulus. These six equations can be put into the form:

$$S_I = s_{IJ} T_J$$

or, explicitly,

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} = \begin{bmatrix} Y_0^{-1} & \frac{\sigma}{Y_0} & \frac{\sigma}{Y_0} & 0 & 0 & 0 \\ \frac{\sigma}{Y_0} & Y_0^{-1} & \frac{\sigma}{Y_0} & 0 & 0 & 0 \\ \frac{\sigma}{Y_0} & \frac{\sigma}{Y_0} & Y_0^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu^{-1} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} \tag{2.25}$$

Comparing (2.25) and (2.11), we see that

$$s_{11} = Y_0^{-1}, \quad s_{12} = \frac{\sigma}{Y_0}, \quad s_{44} = \mu^{-1}$$

Although there are three constants in the compliance matrix (and likewise in the stiffness matrix), only two are independent. To show this, consider the application of equal and opposite stresses $T_1$ and $-T_2$, in which $T_1$ is a tensile stress and $-T_2$ is a compressive stress:

$$S_1 = \frac{T_1}{Y_0} + \frac{\sigma T_2}{Y_0}, \quad S_2 = \frac{T_2}{Y_0} + \frac{\sigma T_1}{Y_0}, \quad S_3 = \frac{\sigma}{Y_0}(T_1 - T_2) = 0$$

(A formal proof using Mohr circles is given in [4].) There are no shear stresses, so $S_4 = S_5 = S_6 = 0$. Now we rotate the coordinate system by

$\pi/4$. The combination of the stresses $T_1$ and $-T_2$ is transformed to shear stresses $T_{xy}$. From Example 1.10, we know that a pure shear can be produced by a combination of orthogonal tensions and compressions along coordinated axes rotated by 45°. Hence, we write

$$T_1, \quad T_2 \rightarrow T_{xy} = T_6$$

This transformation is valid only if the $\pi/4$ rotation leaves the properties of the material unchanged (for an isotropic body a rotation at an arbitrary angle does not change any of its properties). The strains are also transformed to

$$S_1, \quad S_2 \rightarrow \frac{S_6}{2} = T_{xy}\frac{1 - \sigma}{Y_0} \tag{2.26}$$

but

$$S_6 = \frac{T_{xy}}{\mu} = \frac{T_6}{\mu} \tag{2.27}$$

Combining (2.26) and (2.27), we conclude that

$$\mu = \frac{Y_0}{2(1 - \sigma)} \tag{2.28}$$

It is useful to think of the internal structure of an isotropic body as an amorphous mass of randomly scattered particles; (2.28) states that the combination of longitudinal stresses $T_1$ and $T_2$ in such a medium couples equally well to a pure shear strain as to expansions and dilations of the body. The important result of (2.28) is that there are only two independent components to the compliance matrix for an isotropic material. It is important to stress that this is a consequence of the invariance of the material properties under the 45° rotation. From (2.22), it is clear that

$$\frac{s_{12}}{s_{11}} = \frac{s_{13}}{s_{11}} \tag{2.29}$$

Substituting (2.29) and the definitions of $\mu$ and Young's modulus $Y_0$ into (2.28), we have

$$s_{12} = s_{11} - \frac{s_{44}}{2} \tag{2.30}$$

There is an analogous relation between the *stiffness* constants in an isotropic medium. From (2.19), it is clear that

$$c_{11} \neq s_{11}^{-1}$$

To express $c_{11}$ in terms of the compliance constants, we must find the 11 components of the *inverse* of **s**. The form of (2.25) dramatically simplifies our task. Because there is no coupling between shear stress and longitudinal strain, we need only work with the upper left quadrant. Indeed, we can write immediately that

$$c_{44} = s_{44}^{-1} \tag{2.31}$$

From basic matrix theory, $c_{11}$ is given by

$$c_{11} = \frac{A_{11}}{|\mathbf{s}'|} \tag{2.32}$$

where $A_{11}$ is the cofactor of element 11 and $|\mathbf{s}'|$ is the determinant of the reduced, upper-left-quadrant (3 × 3) matrix. Performing the operations of (2.32), we have

$$c_{11} = \frac{s_{11} + s_{12}}{(s_{11} - s_{12})(s_{11} + 2s_{12})} \tag{2.33}$$

Because **c** and **s** are inverses, we conclude immediately that the relation between $s_{11}$ and the stiffness matrix components is identical to the form of (2.33) with all $s$ components replaced by $c$ components. Similarly, we have for $c_{12}$:

$$c_{12} = \frac{A_{12}}{|\mathbf{s}'|} = \frac{-c_{12}}{(c_{11} - c_{12})(c_{11} + 2c_{12})} \tag{2.34}$$

Substituting (2.28) and (2.30) into (2.34), we have

$$2c_{44} = c_{11} - c_{12} \tag{2.35}$$

Equation (2.35) will be referred to as the *isotropy condition*. The stiffness matrix is thus

$$\mathbf{c} = \begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{12} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{44} \end{bmatrix}$$

The two independent stiffness components are called the Lamé constants by many authors and are written as $\lambda$ and $\mu$, where $c_{44} = \mu$ and $c_{12} = \lambda$; thus $c_{11} = \lambda + 2\mu$. Finally, we can write Young's modulus in terms of the stiffness constant by using (2.33):

$$Y_0 = s_{11}^{-1} = \frac{(c_{11} - c_{12})(c_{11} + 2c_{12})}{c_{11} + c_{12}}$$

## 2.4 THE CHRISTOFFEL EQUATION

The Christoffel equation describes the propagation of mechanical waves in three dimensions. Consider the dynamical equations (1.91) and (1.86):

$$\nabla \cdot \mathbf{T} = \rho \frac{\partial^2 \mathbf{u}}{\partial t}$$

$$\nabla_s \mathbf{u} = \mathbf{S}$$

Our task is to manipulate these equations by using the constitutive relations to form a wave equation. We write (1.91) and (1.86) in terms of **T** and **v**; recall that **T** and **v** are in phase in the one-dimensional model (they remain in phase in three dimensions):

$$\nabla \cdot \mathbf{T} = \rho \frac{\partial \mathbf{v}}{\partial t} \tag{2.36}$$

$$\nabla_s \mathbf{v} = \frac{\partial \mathbf{S}}{\partial t} = \mathbf{s}{:}\frac{\partial \mathbf{T}}{\partial t} \tag{2.37}$$

Differentiating (2.36) with respect to $t$ and multiplying (2.37) on the left by **c** give

$$\nabla \cdot \frac{\partial \mathbf{T}}{\partial t} = \rho \frac{\partial^2 \mathbf{v}}{\partial t^2} \tag{2.38}$$

$$\mathbf{c}:\nabla_s \mathbf{v} = \mathbf{c}:\mathbf{s}:\frac{\partial \mathbf{T}}{\partial t} = \frac{\partial \mathbf{T}}{\partial t} \text{ (because } \mathbf{c}:\mathbf{s} = 1) \tag{2.39}$$

Now we substitute $\partial \mathbf{T}/\partial t$ from (2.39) into (2.38):

$$\nabla \cdot \frac{\partial \mathbf{T}}{\partial t} = \nabla \cdot \mathbf{c}:\nabla_s \mathbf{v} = \rho \frac{\partial^2 \mathbf{v}}{\partial t^2} \tag{2.40}$$

Equation (2.40) is the wave equation we seek in three dimensions. It is not difficult to derive an equation of identical form to (2.40) for the particle displacement $\mathbf{u}$. If $\mathbf{c}$ and $\mathbf{v}$ are scalars (with particle velocity in the $z$ direction), (2.40) reduces to

$$\nabla \cdot \mathbf{c}:\nabla_s \mathbf{v} \to c\nabla \cdot \nabla v = c\nabla^2 v = c\frac{\partial^2 v}{\partial z^2} = \rho \frac{\partial^2 v}{\partial t^2}$$

where $\nabla \cdot \nabla = \nabla^2$ is the Laplacian operator. We recognize the preceding equation as the one-dimensional wave equation. Equation (2.40) can be written in a more convenient matrix form:

$$\nabla_s \to \nabla_{Ij} \qquad I = 1 \text{ to } 6, j = 1 \text{ to } 3$$

is a $6 \times 3$ matrix. The expression $\mathbf{c}:\nabla_s \mathbf{v}$ is a $6 \times 6$ matrix, so $\nabla \cdot (\mathbf{c}:\nabla_s \mathbf{v})$ is a vector (recall that the divergence of a matrix is a vector (1.90)). Also recall that the divergence matrix operator $(\nabla\ \cdot)$ is the transpose of $\nabla_{Ij}$:

$$\nabla \cdot \to \nabla_{iJ} \quad i = 1 \text{ to } 3, J = 1 \text{ to } 6$$

Therefore the three-dimensional wave equation reduces to

$$\nabla \cdot \mathbf{c}:\nabla_s \mathbf{v} = \rho \frac{\partial^2 \mathbf{v}}{\partial t^2} \to \nabla_{iK} c_{KL} \nabla_{Lj} v_j = \rho \frac{\partial^2 v_i}{\partial t^2} \tag{2.41}$$

Equation (2.41) contains three summations over the dummy indices $j$, $K$, and $L$.

Finally, we consider a plane wave propagating in a direction $\hat{\mathbf{l}}$ (which is arbitrary) in the medium. We write

$$\hat{\mathbf{l}} = l_x\hat{\mathbf{i}} + l_y\hat{\mathbf{j}} + l_z\hat{\mathbf{k}} \tag{2.42}$$

where $l_x$, $l_y$, and $l_z$ are the projections of the *unit* vector $\hat{\mathbf{l}}$ on the three Cartesian axes. Now recall from Chapter 1 that for a one-dimensional wave propagating in the positive $z$ direction the fields have the time and space dependence $e^{j(\omega t - \beta z)}$. In three dimensions, the form is

$$v \propto e^{j(\omega t - k\hat{\mathbf{l}}\cdot\bar{\mathbf{r}})} \tag{2.43}$$

where $\hat{\mathbf{l}}$ is given by (2.42), $\bar{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, and $k = \omega/v_a$.

Performing the operation $\hat{\mathbf{l}} \cdot \bar{\mathbf{r}}$, we can write (2.43) as

$$e^{j(\omega t - k\hat{\mathbf{l}}\cdot\bar{\mathbf{r}})} = e^{j(\omega t - k(l_x x + l_y y + l_z z))} \tag{2.44}$$

In this form, it is a simple matter to differentiate to obtain

$$\begin{aligned} \frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial x}\left(e^{j(\omega t - k\hat{\mathbf{l}}\cdot\bar{\mathbf{r}})}\right) &= -jkl_x\, e^{j(\omega t - k\hat{\mathbf{l}}\cdot\bar{\mathbf{r}})} \\ &= -jkl_x\mathbf{v} \end{aligned} \tag{2.45}$$

The $y$ and $z$ components are written in a similar fashion. Now recall the gradient and divergence operator matrices. Both contain components of the form of (2.45). For the divergence,

$$\nabla_{Lj} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \rightarrow -jk \begin{bmatrix} l_x & & 0 \\ 0 & l_y & 0 \\ 0 & 0 & l_z \\ 0 & l_z & l_y \\ l_z & 0 & l_x \\ l_y & l_x & 0 \end{bmatrix} = -jkl_{Lj} \tag{2.46}$$

The $\mathbf{l}$ matrix has the same form as the gradient operator matrix; each of its components represents a propagation direction of the acoustic wave.

Likewise, the divergence matrix operator becomes

$$\nabla_{ik} \to -jk \begin{bmatrix} l_x & 0 & 0 & 0 & l_z & l_y \\ 0 & l_y & 0 & l_z & 0 & l_x \\ 0 & 0 & l_z & l_y & l_x & 0 \end{bmatrix} = -jkl_{iK} \tag{2.47}$$

Now we can rewrite (2.41) as

$$\nabla_{iK} c_{KL} \nabla_{Lj} v_j \to -(k^2 l_{iK} c_{KL} l_{Lj}) v_j = -k^2 \Gamma_{ij} v_j = -\rho\omega^2 v_i \tag{2.48}$$

where

$$\Gamma_{ij} = l_{iK} c_{KL} l_{Lj} \tag{2.49}$$

is called the *Christoffel matrix*. This matrix is $3 \times 3$ with elements that depend only on the propagation direction of the wave (through the $l$ values) and the stiffness constants of the crystal. For any crystal material and propagation direction, we can form the Christoffel matrix by using (2.49). Solving the Christoffel matrix involves solving an eigenvalue problem; the eigenvalues are three real positive numbers that are simply the three phase velocities of the possible propagating waves (one longitudinal and two shear waves), and the three corresponding eigenvectors are the particle velocity directions, which are defined as the *acoustic* polarization. We cannot solve for the magnitudes of the particle velocities because, as we showed in (1.35), they depend on the intensity of the acoustic wave. Furthermore, because **u** and **v** are parallel, solving for the direction of one automatically determines the direction of the other. Thus the acoustic polarization may be defined as the direction of either **u** or **v**. The three polarizations are mutually orthogonal (the eigenvectors of symmetric matrices are always orthogonal), but are not necessarily parallel to the propagation directions. Thus, it is not always true that the longitudinal polarization will be parallel to the propagation direction or that the shear polarizations will be normal to it; only certian (symmetry) directions will yield polarizations that are precisely parallel and normal to the propagation direction.

Now we consider an isotropic crystal in which there are two independent components of the stiffness matrix. We form the Christoffel matrix for an arbitrary propagation direction **l**:

$$\hat{\mathbf{l}} = l_x \hat{\mathbf{i}} + l_y \hat{\mathbf{j}} + l_z \hat{\mathbf{k}}$$

and

$$\hat{\mathbf{l}} \cdot \hat{\mathbf{l}} = l_x^2 + l_y^2 + l_z^2 = 1 \qquad (\hat{\mathbf{l}} \text{ is a unit vector}) \tag{2.50}$$

The Christoffel matrix is given by (2.49):

$$\Gamma_{ij} = l_{iK} c_{KL} l_{Lj}$$

$$= l_{iK}\colon \begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{12} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{44} \end{bmatrix} \begin{bmatrix} l_x & 0 & 0 \\ 0 & l_y & 0 \\ 0 & 0 & l_z \\ 0 & l_z & l_y \\ l_z & 0 & l_x \\ l_y & l_x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} l_x & 0 & 0 & 0 & l_z & l_y \\ 0 & l_y & 0 & l_z & 0 & l_x \\ 0 & 0 & l_z & l_y & l_x & 0 \end{bmatrix} \begin{bmatrix} c_{11}l_x & c_{12}l_y & c_{12}l_z \\ c_{12}l_x & c_{11}l_y & c_{12}l_z \\ c_{12}l_x & c_{12}l_y & c_{11}l_z \\ 0 & c_{44}l_z & c_{44}l_y \\ c_{44}l_z & 0 & c_{44}l_x \\ c_{44}l_y & c_{44}l_x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} c_{11}l_x^2 + c_{44}(l_z^2 + l_y^2) & (c_{12} + c_{44})l_x l_y & (c_{12} + c_{44})l_x l_z \\ (c_{12} + c_{44})l_x l_y & c_{11}l_y^2 + c_{44}(l_x^2 + l_z^2) & (c_{12} + c_{44})l_y l_z \\ (c_{12} + c_{44})l_z l_x & (c_{12} + c_{44})l_y l_z & c_{11}l_z^2 + c_{44}(l_x^2 + l_y^2) \end{bmatrix} \tag{2.51}$$

There are only two independent stiffness components, because

$$\frac{c_{11} - c_{12}}{2} = c_{44} \quad \text{(isotropy condition)}$$

There are also conditions on $\hat{\mathbf{l}}$ imposed from (2.50):

$$l_z^2 + l_y^2 = 1 - l_x^2, \quad l_z^2 + l_x^2 = 1 - l_y^2, \quad l_x^2 + l_y^2 = 1 - l_z^2$$

As we will show, the form of the Christoffel equation for cubic crystals is identical to (2.51), but does not have the isotropy condition, which requires identical propagation characteristics in all directions. We now investigate the propagation in the $xy$ plane:

$$l_z = 0 \text{ and } l_x^2 + l_y^2 = 1 \tag{2.52}$$

The Christoffel matrix reduces to

$$\Gamma_{ij} = \begin{bmatrix} c_{11}l_x^2 + c_{44}(1 - l_x^2) & (c_{12} + c_{44})l_x l_y & 0 \\ (c_{12} + c_{44})l_x l_y & c_{11}l_y^2 + c_{44}(1 - l_y^2) & 0 \\ 0 & 0 & c_{44} \end{bmatrix} \tag{2.53}$$

Now recall the Christoffel equation (2.48):

$$k^2\Gamma_{ij}v_j = \rho\omega^2 v_i$$

In the $xy$ plane of an isotropic material, we substitute (2.50) and (2.53) into (2.48) and arrive at

$$k^2 \begin{bmatrix} A & B & 0 \\ B & C & 0 \\ 0 & 0 & c_{44} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \rho\omega^2 \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \tag{2.54}$$

where

$$A = c_{11}l_x^2 + c_{44}(1 - l_x^2)$$

$$B = (c_{12} + c_{44})l_x l_y$$

$$C = c_{11}l_y^2 + c_{44}(1 - l_y^2)$$

and $l_x^2 + l_y^2 = 1$.

Note that our task is to solve for the three particle velocities (the acoustic polarizations). Equation (2.54) is a set of three equations, of which two are *coupled* equations. We first consider the simplest uncoupled equation (for $v_z$):

$$k^2 c_{44} v_z = \rho\omega^2 v_z$$

This equation can only be true if

$$\frac{\omega^2}{k^2} = \frac{c_{44}}{\rho} \quad \text{or} \quad v_a = \frac{\omega}{k} = \sqrt{\frac{c_{44}}{\rho}} \tag{2.55}$$

Equation (2.55) shows that for *any* direction in the $xy$ plane there can exist a wave with particle velocity or polarization (**v**) that is $z$-directed. Because the polarization is normal to the propagation direction of the wave, the wave is *pure* shear; the phase velocity is independent of the propagation direction.

Now consider the remaining equations in (2.54):

$$k^2 \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \rho\omega^2 \begin{bmatrix} v_x \\ v_y \end{bmatrix} \tag{2.56}$$

for an arbitrary propagation direction $B \neq 0$, so (2.54) represents two coupled equations. To diagonalize the matrix and uncouple the equations, we form the determinant:

$$\begin{vmatrix} A - \lambda & B \\ B & C - \lambda \end{vmatrix} = 0 \tag{2.57}$$

and solve for the two values of $\lambda$:

$$(A - \lambda)(C - \lambda) - B^2 = 0$$

or

$$\lambda_{1,2} = \frac{A + C \pm \sqrt{(A + C)^2 - 4AC + 4B^2}}{2} \tag{2.58}$$

We substitute for $A$, $B$, and $C$ under the square root and use (2.35) (the isotropy condition) to obtain

$$\lambda_{1,2} = \frac{(A + C) \pm (c_{11} - c_{44})}{2} \tag{2.59}$$

Using the isotropy condition (2.35) again, we find that

$$A + C = c_{11} + c_{44} \tag{2.60}$$

Thus, for an arbitrary propagation direction in the $xy$ plane, we have

$$\lambda_{1,2} = \frac{c_{11} + c_{44} \pm (c_{11} - c_{44})}{2} \qquad \text{for all } l_x, l_y; \quad l_z = 0$$

$$\lambda_1 = c_{11}, \qquad \lambda_2 = c_{44} \tag{2.61}$$

The $\lambda$ values are called the *eigenvalues* of the Christoffel matrix; to each eigenvalue there corresponds an *eigenfunction*. The eigenfunctions of a symmetric matrix (the Christoffel matrix is always symmetric) form an orthogonal set. From the eigenvalue, we find the phase velocity $v_a$; the *eigenvector* provides the particle velocity direction (which is identical to the particle displacement direction) and thus the acoustic polarization. We have already solved for one of the eigenfunctions $v_z$; for the remaining two, we return to the Christoffel matrix. We substitute each eigenvalue in the Christoffel equation; for $\lambda_1$:

$$\begin{bmatrix} A - \lambda_1 & B \\ B & C - \lambda_i \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$
$$= 0 \rightarrow \begin{bmatrix} (c_{11} - c_{44})(l_x^2 - 1) & (c_{11} - c_{44})l_x l_y \\ (c_{11} - c_{44})l_x l_y & (c_{11} - c_{44})(l_y^2 - 1) \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$
$$= 0 \tag{2.62}$$

Extracting the common factor $(c_{11} - c_{44})$ and expanding, we get

$$\begin{aligned} -l_y^2 v_x + l_x l_y v_y &= 0 \\ l_x l_y v_x - l_x^2 v_y &= 0 \end{aligned} \tag{2.63}$$

These equations are linearly dependent. From either equation, we find immediately that

$$v_x = \frac{l_x}{l_y} v_y \tag{2.64}$$

A suitable eigenvector corresponding to $\lambda_1$ is

$$\hat{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ \dfrac{l_y}{l_x} \end{bmatrix} = \begin{bmatrix} \dfrac{l_x}{l_y} \\ 1 \end{bmatrix} \tag{2.65}$$

We normalize $\hat{\mathbf{v}}_1$ by dividing by $1/l_x$

$$\hat{\mathbf{v}}_1 = \begin{bmatrix} l_x \\ l_y \end{bmatrix}$$

where $|\hat{\mathbf{v}}_1| = 1$ because $l_x^2 + l_y^2 = 1$. In three dimensions, we write

$$\hat{\mathbf{v}}_1 = \begin{bmatrix} l_x \\ l_y \\ 0 \end{bmatrix} \tag{2.66}$$

Note that we are not able to determine the *magnitude* of the particle velocity, which depends on the power in the acoustic wave. If we carry out the identical procedure for the eigenvalue $\lambda_2 = c_{44}$, we obtain the normalized particle velocity:

$$\lambda_2 = c_{44} \rightarrow \hat{\mathbf{v}}_2 = \begin{bmatrix} l_y \\ l_x \\ 0 \end{bmatrix} \tag{2.67}$$

Recall that the third eigenvalue and eigenvector are

$$\lambda_3 = c_{44} \rightarrow \hat{\mathbf{v}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{2.68}$$

Now consider the dot product of $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$:

$$\hat{\mathbf{v}}_1 \cdot \hat{\mathbf{v}}_2 = (l_x\hat{\mathbf{i}} + l_y\hat{\mathbf{j}}) \cdot (-l_y\hat{\mathbf{i}} + l_x\hat{\mathbf{j}}) = 0 \tag{2.69}$$

Likewise, it is easy to show that

$$\hat{\mathbf{v}}_1 \cdot \hat{\mathbf{v}}_3 = \hat{\mathbf{v}}_2 \cdot \hat{\mathbf{v}}_3 = 0 \tag{2.70}$$

The eigenvectors are always orthogonal for all symmetries.

Now we consider the orientations of the polarizations with respect to the propagation direction $\hat{\mathbf{l}}$:

$$\lambda_1 = c_{11}, \quad \hat{\mathbf{v}}_1 \rightarrow \hat{\mathbf{v}}_1 \cdot \hat{\mathbf{l}} = (l_x\hat{\mathbf{i}} + l_y\hat{\mathbf{j}}) \cdot (l_x\hat{\mathbf{i}} + l_y\hat{\mathbf{j}}) = 1 \tag{2.71}$$

Because the particle motion (polarization) is parallel to the propagation direction, this wave is clearly a longitudinal wave. For $\lambda_2 = c_{44}$:

$$\hat{\mathbf{v}}_2 \cdot \hat{\mathbf{l}} = (-l_x\hat{\mathbf{i}} + l_y\hat{\mathbf{j}}) \cdot (l_x\hat{\mathbf{i}} + l_y\hat{\mathbf{j}}) = 0 \tag{2.72}$$

The polarization $\hat{\mathbf{v}}_2$ is orthogonal to the propagation direction, so this wave is a shear wave. In (2.55), we saw that the third wave with $z$-directed polarization is also a shear wave. The full solution consists of one longitudinal wave and two identical shear waves. It is not difficult to show that this result is also valid not only in the $xz$ and $yz$ planes but for an arbitrary direction. For other symmetry classes, it will not be generally true that the polarizations will be parallel or normal to the propagation direction. Propagation in an isotropic medium is characterized by the following facts:

1. There are three velocities corresponding to three plane waves. This is true for all crystal symmetries.

2. The longitudinal mode corresponds to the eigenvalue $c_{11}$ and thus has the phase velocity:

$$v_a = \sqrt{\frac{c_{11}}{\rho}}$$

3. There are two shear modes with the same velocity for all directions of propagation; this situation is called a shear mode degeneracy. The shear mode phase velocity is

$$v_a = \sqrt{\frac{c_{44}}{\rho}}$$

4. The phase velocities of the three modes are the same for all directions of propagation; this is a consequence of the isotropy condition (2.35).

## 2.5 ACOUSTIC PROPAGATION IN ANISOTROPIC CRYSTALS

It is customary to speak of seven crystal systems (eight if isotropic media are included). Each system represents a general symmetry that is further divided into crystal classes, of which there are 32. The acoustically important systems are shown in Figure 2.2. The connecting lines represent springs; the stiffness of each spring is symbolized by its length. Other classes, namely the triclinic and monoclinic, have not been exploited widely in crystal acoustics and are not included in our discussion. The crystal systems (referring to Figure 2.2) are given below.

*1. Isotropic*: An arbitrary rotation of the stiffness matrix leaves the mechanical properties unchanged. The molecules in this system can be visualized as an amorphous jelly of randomly positioned particles; thus the propagation is independent of direction. The most important member of this class is fused (amorphous) quartz. Other isotropic materials, such as Lucite, and plastics do not support high frequency acoustic waves. The stiffness matrix contains two independent components.

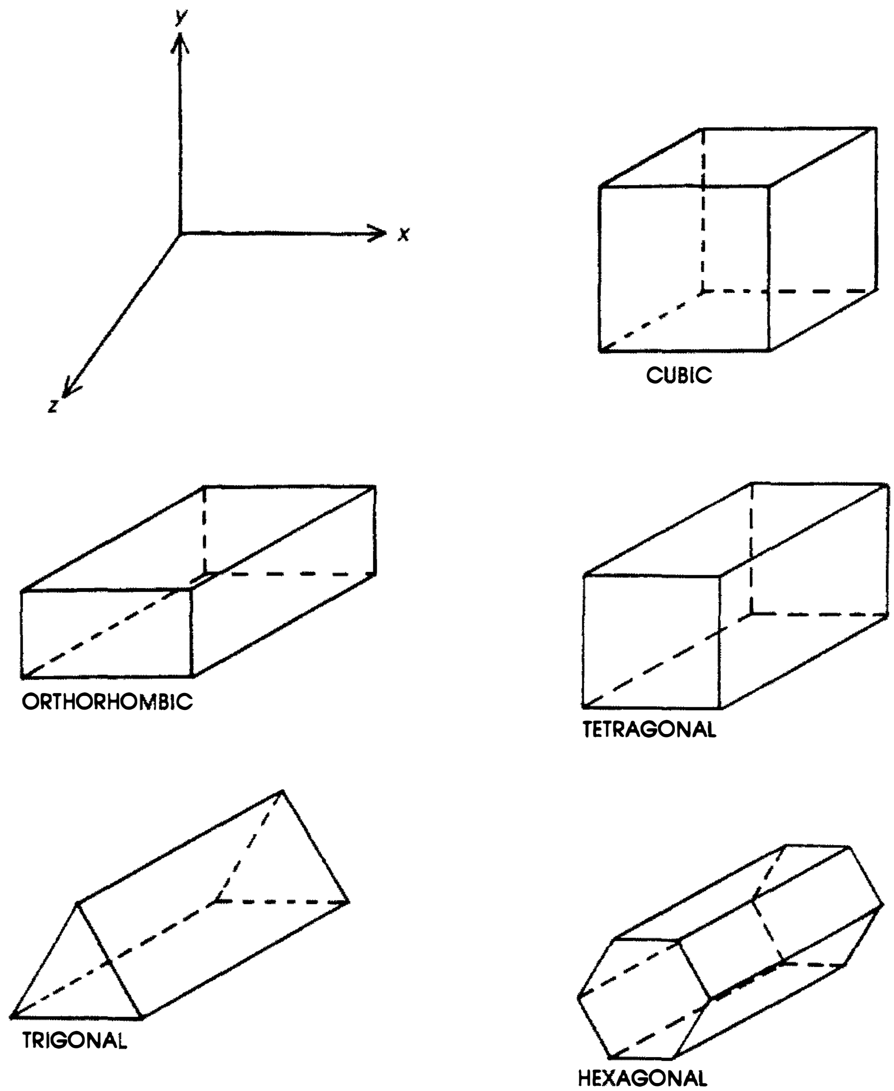


**Figure 2.2** The major symmetry systems. The lines connecting individual particles represent springs with stiffnesses proportional to their lengths. Thus, the cubic system has equal stiffness properties along all major axes, whereas in the tetragonal system only the stiffness properties $x$- and $y$-axes are equivalent.

2. *Cubic*: The $x$-, $y$-, and $z$-axes are equivalent, and each is a fourfold axis of symmetry. This means that a 90° rotation leaves the properties of the crystals unchanged. These crystals are optically isotropic but acoustically anisotropic. Acoustically important members include the III–V semiconductors (gallium arsenide, gallium phosphide, indium phosphide, and indium antimony) and silicon, germanium, and bismuth germanium oxide (BGO). The stiffness matrix contains three independent components. There are five classes in the cubic system.

3. *Hexagonal*: There is sixfold symmetry around the $z$- (optic) axis. These crystals are acoustically and optically anisotropic. Optically they are uniaxial. Important acoustic members include cadmium sulfide and zinc oxide. The stiffness matrix **c** contains five independent components and there are seven classes in the system.

4. *Trigonal*: There is threefold symmetry around the $z$- (optic) axis; i.e., a rotation of 120° about the $z$-axis leaves the mechanical properties unchanged. These crystals are acoustically and optically anisotropic (optically they are uniaxial). Important acoustic members include sapphire ($Al_2O_3$), lithium niobate ($LiNbO_3$), lithium tantalate ($LiTaO_3$), and *crystal* quartz ($SiO_2$). There five classes in the system, and the stiffness matrix contains either six or seven independent components, depending on the particular class.

5. *Tetragonal*: There are two equivalent axes (the $x$- and $y$-axes) separated by 90°. There is fourfold symmetry around the $z$-axis. The crystals are also acoustically and optically anistropic and optically uniaxial. Important members of this class include paratellurite ($TeO_2$), rutile ($TiO_2$), and lead molybdate ($PbMoO_4$). There are seven classes, and the stiffness matrix possesses six or seven independent components, depending on the particular symmetry class.

6. *Orthorhombic*: There are three major axes, none of which are equal; each axis possesses twofold symmetry. Important examples include barium sodium niobate ($Ba_2NaNb_5O_{15}$), lithium gallate ($LiGaO_3$), and the sulfosalt chalcogenides. Like the hexagonal, trigonal, and tetragonal, these crystals are both acoustically and optically anisotropic; optically they are biaxial, however, (they possess two optic axes). This system contains three classes, and the stiffness matrix has nine independent components for all classes.

### 2.5.1 Cubic Symmetry

We first consider the cubic symmetry; because it has three equivalent axes ($x$, $y$, and $z$), we can replace each in turn without affecting the material properties. For example, if $x \to y$, $y \to z$, and $z \to x$, then **c** is unchanged. That the form of stiffness matrix (a physical property of a particular crystal) depends on the symmetrical properties of the system to which the crystal belongs is a result of Neumann's principle [3]. A matrix that performs this operation is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x \to y & y \to z & z \to x \\ 1 \to 2 & 2 \to 3 & 3 \to 1 \end{matrix} \tag{2.73}$$

In the double subscript notation, we write

$$11 \to 22 \quad 22 \to 33 \quad 33 \to 11 \quad 12 \to 23 \quad 23 \to 31 \quad 31 \to 12$$

or

$$1 \to 2 \qquad 2 \to 3 \qquad 3 \to 1 \qquad 6 \to 4 \qquad 4 \to 5 \qquad 5 \to 6$$

(Recall from Chapter 1 that 12 $(xy)$ is equivalent to 6 in the single subscript notation.)

We next write the **c** matrix with all places filled (21 possible independent components) and apply the symmetry operation. Because **c** is a symmetric matrix, we need only deal with the top right-hand part. Instead of writing the $c$ values explicitly, we choose, for convenience, to label them by their numerical places:

$$\mathbf{c} = \begin{bmatrix} 11 & 12 & 13 & 14 & 15 & 16 \\ & 22 & 23 & 24 & 25 & 26 \\ & & 33 & 34 & 35 & 36 \\ & & & 44 & 45 & 46 \\ & & & & 55 & 56 \\ & & & & & 66 \end{bmatrix}$$

$$\xrightarrow{\text{apply symmetry operation}} \mathbf{c}' = \begin{bmatrix} 22 & 23 & 21 & 25 & 26 & 24 \\ & 33 & 31 & 35 & 36 & 34 \\ & & 11 & 15 & 16 & 14 \\ & & & 55 & 56 & 54 \\ & & & & 66 & 64 \\ & & & & & 44 \end{bmatrix} \tag{2.74}$$

11 refers to $c_{11}$, *et cetera*. Because the transformation leaves **c** unchanged, the components of **c** and **c**′ must be individually equal term by term.

first row: $c_{11} = c'_{22}$, $c_{12} = c'_{23}$, $c_{13} = c'_{21}$, $c_{14} = c'_{25}$, $c_{15} = c'_{26}$, $c_{16} = c'_{24}$
second row: $c_{22} = c'_{33}$, $c_{23} = c'_{31}$, $c_{24} = c'_{35}$, $c_{25} = c'_{36}$, $c_{26} = c'_{34}$
third row: $c_{33} = c'_{11}$, $c_{34} = c'_{15}$, $c_{35} = c'_{16}$, $c_{36} = c'_{14}$
fourth row: $c_{44} = c'_{55}$, $c_{45} = c'_{56}$, $c_{46} = c'_{54}$ (but $c_{54} = c_{45}$ because **c** is symmetric)
fifth row: $c_{55} = c'_{66}$, $c_{56} = c'_{64}$
sixth row: $c_{66} = c'_{44}$

We now substitute these results into **c**:

$$\mathbf{c} = \begin{bmatrix} c_{11} & c_{12} & c_{12} & c_{14} & c_{15} & c_{24} \\ & c_{11} & c_{12} & c_{24} & c_{14} & c_{15} \\ & & c_{11} & c_{15} & c_{24} & c_{14} \\ & & & c_{44} & c_{45} & c_{45} \\ & & & & c_{44} & c_{45} \\ & & & & & c_{44} \end{bmatrix} \tag{2.75}$$

We next apply the symmetry operation that rotates the unit cell 180° about the $z$-axis:

$$\begin{array}{lll} x \to -x & y \to -y & z \to z \\ 1 \to -1 & 2 \to -2 & 3 \to 3 \end{array}$$

The matrix operator is simply

$$\mathbf{A} = \begin{bmatrix} \cos\pi & \sin\pi & 0 \\ -\sin\pi & \cos\pi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.76}$$

Applying the same rules we used in the previous symmetry operation, we find that

$$\begin{aligned} 11 \to (-1)(-1) &= 11,\ 22 \to (-2)(-2) \\ &= 22,\ 33 \to 33\ 12 \to 12,\ 23 \to -23,\ 31 \to -31 \end{aligned}$$

or, in single subscript notation:

$$1 \to 1,\quad 2 \to 2,\quad 3 \to 3,\quad 6 \to 6,\quad 4 \to -4,\quad 5 \to -5$$

We can easily apply these changes to **c** directly and equate individual terms. We find that

$$c_{11} = c'_{11} \text{ and } c_{12} = c'_{12}$$

but

$$\begin{aligned}
c_{14} &= -c'_{14} \rightarrow c_{14} = 0\\
c_{15} &= -c'_{15} \rightarrow c_{15} = 0\\
c_{24} &= -c'_{24} \rightarrow c_{24} = 0\\
c_{46} &= -c'_{46} \rightarrow c_{46} = 0
\end{aligned}$$

Finally, from the first operation, we know that

$$c_{45} = c_{54} = c_{46} \rightarrow c_{45} = 0$$

Thus, the final form of the stiffness matrix is

$$\mathbf{c} = \begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0\\ & c_{11} & c_{12} & 0 & 0 & 0\\ & & c_{11} & 0 & 0 & 0\\ & & & c_{44} & 0 & 0\\ & & & & c_{44} & 0\\ & & & & & c_{44} \end{bmatrix} \tag{2.77}$$

There are three independent components; indeed, it is the presence of the third component (or, equivalently, the lack of the isotropy condition) that makes the cubic class crystals acoustically anisotropic. However, the form of the stiffness matrix is identical to that of an isotropic medium, therefore, it is clear that the form of the Christoffel matrix will also be identical to that of an isotropic medium. We repeat (2.51) for convenience:

$$\Gamma = \begin{bmatrix} c_{11}l_x^2 + c_{44}(l_y^2 + l_z^2) & (c_{12} + c_{44})l_xl_y & (c_{12} + c_{44})l_xl_z\\ (c_{12} + c_{44})l_yl_x & c_{11}l_y^2 + c_{44}(l_x^2 + l_z^2) & (c_{12} + c_{44})l_yl_z\\ (c_{12} + c_{44})l_zl_x & (c_{12} + c_{44})l_zl_y & c_{11}l_z^2 + c_{44}(l_x^2 + l_y^2) \end{bmatrix}$$

where now $c_{12} \neq c_{11} - 2c_{44}$.

We now consider propagation along a major axis, the $x$ axis:

$$\hat{\mathbf{l}} = \hat{\mathbf{i}} \quad (l_y = l_z = 0)$$

The Christoffel matrix immediately reduces to

$$\Gamma = \begin{bmatrix} c_{11} & 0 & 0\\ 0 & c_{44} & 0\\ 0 & 0 & c_{44} \end{bmatrix} \tag{2.78}$$

In accordance with (2.51), the Christoffel equation reduces to three uncoupled equations, yielding the eigenvalues:

$$\text{for } v_x\text{:} \quad c_{11} = \frac{\omega^2}{k^2}\rho \rightarrow v_a = \sqrt{\frac{c_{11}}{\rho}} \tag{2.79}$$

Equation (2.79) is obviously the longitudinal mode because the polarization is in the $x$ direction ($v_x$), as is the propagation direction.

$$\text{for } v_y\text{:} \quad c_{44} = \frac{\omega^2}{k^2}\rho \rightarrow v_a = \sqrt{\frac{c_{44}}{\rho}} \tag{2.80}$$

Equation (2.80) is a shear mode because the polarization ($v_y$) is normal to $\hat{\mathbf{i}}$.

$$\text{for } v_z\text{:} \quad v_a = \sqrt{\frac{c_{44}}{\rho}} \tag{2.81}$$

which is also a pure shear wave.

It is clear that similar results would be obtained for propagation along the $y$- or $z$-axis. In each case, there are three orthogonally polarized modes (one longitudinal and two shear) with a shear degeneracy (equal shear mode phase velocities).

Next we consider propagation in the $xy$ plane at an angle of 45° to the $x$-axis:

$$l_z = 0, \quad \hat{\mathbf{i}} = \frac{\hat{\mathbf{i}}}{\sqrt{2}} + \frac{\hat{\mathbf{j}}}{\sqrt{2}}$$

The Christoffel matrix is

$$\Gamma = \frac{1}{2}\begin{bmatrix} c_{11} + c_{44} & c_{12} + c_{44} & 0 \\ c_{12} + c_{44} & c_{11} + c_{44} & 0 \\ 0 & 0 & 2c_{44} \end{bmatrix} \tag{2.82}$$

For the $v_z$ polarized mode, we have

$$v_a = \sqrt{\frac{c_{44}}{\rho}} \tag{2.83}$$

which is a pure shear wave.

The other modes are coupled; $\Gamma$ takes the form:

$$\Gamma = \begin{bmatrix} A & B \\ B & A \end{bmatrix} \quad \begin{aligned} A &= c_{11} + c_{44} \\ B &= c_{12} + c_{44} \end{aligned} \tag{2.84}$$

We follow the procedure for the isotropic medium and diagonalize the characteristic equation:

$$\begin{vmatrix} A - \lambda & B \\ B & A - \lambda \end{vmatrix} = 0 \tag{2.85}$$

The eigenvalues are

$$\begin{aligned} \lambda_1 &= c_{11} + c_{12} + 2c_{44} \\ \lambda_2 &= c_{11} - c_{12} \end{aligned} \tag{2.86}$$

The phase velocities are

$$\text{for } \lambda_1\text{:} \quad v_a = \sqrt{\frac{c_{11} + c_{12} + 2c_{44}}{2\rho}} \tag{2.87}$$

$$\text{for } \lambda_2\text{:} \quad v_a = \sqrt{\frac{c_{11} - c_{12}}{2\rho}} \tag{2.88}$$

For an isotropic medium, we apply the isotropy condition (2.35); the phase velocities then reduce to

$$\lambda_1\text{:} \quad v_a \rightarrow \sqrt{\frac{c_{11} + (c_{11} - 2c_{44}) + 2c_{44}}{2\rho}} = \sqrt{\frac{c_{11}}{\rho}}$$

$$\lambda_2\text{:} \quad v_a \rightarrow \sqrt{\frac{c_{44}}{\rho}}$$

To find the eigenvectors (the polarizations), we substitute $\lambda_1$ and $\lambda_2$ in the characteristic equations. For $\lambda_1$:

$$\begin{bmatrix} A - \lambda_1 & B \\ B & A - \lambda_1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = 0 \tag{2.89}$$

As before, we obtain two linearly dependent equations upon substituting for $A$, $B$, and $\lambda_1$:

$$(c_{12} + c_{44})(-v_x + v_y) = 0 \rightarrow v_x = v_y \tag{2.90}$$

A suitable polarization is thus (including the zero $z$ component)

$$\mathbf{v}_1 \propto \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ or, normalized: } \hat{\mathbf{v}}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{2.91}$$

Because the normalized propagation vector $\hat{\mathbf{l}}$ is identical to $\mathbf{v}_1$, clearly this mode represents the longitudinal mode. Next we consider $\lambda_2$; carrying out the same procedure, we obtain

$$(v_x + v_y) = 0 \rightarrow \hat{\mathbf{v}}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \tag{2.92}$$

Because $\hat{\mathbf{l}} \cdot \hat{\mathbf{v}}_2 = 0$, this mode is a pure shear mode.

Finally, we consider propagation in the $xy$ plane in a direction neither along one of the principal axes nor at 45°. Because $l_z$ is zero, there is a pure shear mode polarized in the $z$ direction, as before. Equation (2.84) takes the form:

$$\Gamma = \begin{bmatrix} A & B \\ B & C \end{bmatrix} \tag{2.93}$$

With $A$, $B$, and $C$ given by (2.54). This form is precisely the same as in the isotropic case, but here the isotropy condition (2.35) does not apply. The eigenvalue solution for (2.93) in terms of the stiffness constants is quite complex. In addition, the mode corresponding to $\lambda_1$ is no longer precisely parallel to the propagation direction, and the shear mode (corresponding to $\lambda_2$) is no longer precisely perpendicular. Indeed, the deviation of these angles depends on the difference $A - C$. From (2.54):

$$A - C = (c_{11} - c_{44})(l_x^2 - l_y^2) \tag{2.94}$$

The deviation from pure mode direction depends on both the material properties (through $c_{11}$–$c_{44}$) and the direction of propagation. Because the angle of deviation from the pure mode direction vanishes for propagation along the $x$- or $y$-axes as well as at 45°, it is reasonable to assume that it is a maximum near 22.5°, and this is usually the case (the precise value of the angle depends on the stiffness constants). For typical values of stiffness constants, the maximum deviation (near 22.5°) varies from about 5° to 15°.

The three modes are, however, orthogonal; the orthogonality is a consequence of the symmetry of the Christoffel matrix. Thus the pure shear mode, in the $xy$ plane, is polarized along $z$; the quasilongitudinal and quasishear modes must be polarized *in* the $xy$ plane as shown in Figure 2.3. The deviation angle $\alpha$ is dependent on $\hat{\mathbf{l}}$ and the *material constants*. Propagation in the $xy$ plane in a cubic crystal is summarized in Table 2.3.

**Table 2.3**

| *Propagation Direction* | *Modes* | *Phase Velocities* | *Polarizations* |
|---|---|---|---|
| Principal axis | | | |
| $\langle 1, 0, 0\rangle$ | | | |
| all modes | Quasilong. | $\sqrt{c_{11}/\rho}$ | $\langle 1, 0, 0\rangle$ |
| pure | Quasishear | $\sqrt{c_{44}/\rho}$ | $\langle 0, 1, 0\rangle$ |
| shear wave | Pure shear | $\sqrt{c_{44}/\rho}$ | $\langle 0, 0, 1\rangle$ |
| degeneracy | | | |
| Face diagonal | Quasilong. | $\sqrt{(c_{11} + c_{12} + 2c_{44})/\rho}$ | $\langle 1, 1, 0\rangle$ |
| $\langle 1, 1, 0\rangle$ | | | |
| all modes | Quasishear | $\sqrt{(c_{11} - c_{12})/\rho}$ | $\langle -1, 1, 0\rangle$ |
| pure | Pure shear | $\sqrt{c_{44}/\rho}$ | $\langle 0, 0, 1\rangle$ |
| Arbitrary direction | Quasilong. | Between $v_a$ for $\langle 1, 0, 0\rangle$ and $\langle 1, 1, 0\rangle$ | Deviates from pure mode |
| | Quasishear | Between $v_a$ for $\langle 1, 0, 0\rangle$ and $\langle 1, 1, 0\rangle$ | Deviates from pure mode |
| | Pure shear | $\sqrt{c_{44}/\rho}$ | $\langle 0, 0, 1\rangle$ |

Identical results are obtained in the $xz$ and $yz$ planes. In the $xz$ plane, e.g., for propagation in the $\langle 1, 0, 1\rangle$ direction, the quasilongitudinal and quasishear modes are pure, with the polarization in the $\langle 1, 0, 1\rangle$ and $\langle -1, 0, 1\rangle$ directions, respectively, and the pure shear mode is polarized $\langle 0, 1, 0\rangle$ (normal to the $xz$ plane). Phase velocities are identical to those of the $\langle 1, 1, 0\rangle$ direction.

One other direction of interest is the $\langle 1, 1, 1\rangle$ or body diagonal. Direct substitution into the Christoffel equation shows that all modes are pure and that there is a shear degeneracy:

$$\text{longitudinal mode:} \quad v_a = \sqrt{\frac{c_{11} + 2c_{12} + 4c_{44}}{3\rho}} \tag{2.95}$$

$$\text{shear modes:} \quad v_a = \sqrt{\frac{c_{11} - c_{12} + c_{44}}{3\rho}} \tag{2.96}$$

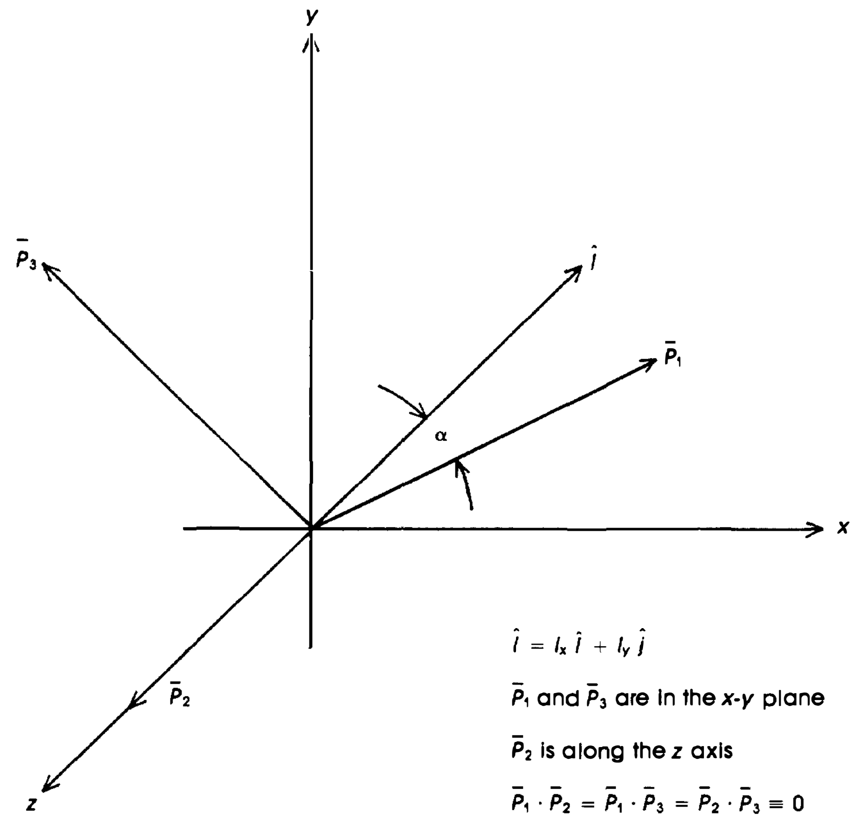


**Figure 2.3** Acoustic polarization vectors for propagation in the $xy$ plane of a cubic system. There is a pure mode polarized along $z$ and two quasimodes with polarizations that lie in the $xy$ plane.

### 2.5.2 Tetragonal Symmetry

We next consider the somewhat more complex symmetry of the tetragonal class of crystals. A number of important materials in this class are invariant under the symmetry operation called $\bar{4}$ around the $z$-axis; the operation consists of the 90° rotation around the $z$-axis and then an inversion through it. The operation is illustrated in Figure 2.4. From the figure, it is clear that

$$\begin{array}{lll} x \rightarrow y & y \rightarrow -x & z \rightarrow -z \\ 1 \rightarrow 2 & 2 \rightarrow -1 & 3 \rightarrow -3 \end{array}$$

The matrix operator is easily written as

$$\bar{4} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{2.97}$$

In the $IJ$ notation, we have

$$11 \rightarrow 22 \quad 22 \rightarrow 11 \quad 33 \rightarrow 33 \quad 23 \rightarrow 13 \quad 31 \rightarrow -32 \quad 12 \rightarrow -21$$

or

$$1 \rightarrow 2 \qquad 2 \rightarrow 1 \qquad 3 \rightarrow 3 \qquad 4 \rightarrow 5 \qquad 5 \rightarrow -4 \qquad 6 \rightarrow -6$$

the stiffness matrix (2.17) transforms under the operation $\bar{4}$ to

$$\mathbf{c}' = \begin{bmatrix} 11 & 21 & 23 & 25 & -24 & -26 \\ & 11 & 13 & 15 & -14 & -16 \\ & & 33 & 35 & -34 & -36 \\ & & & 55 & -54 & -56 \\ & & & & 44 & 46 \\ & & & & & 66 \end{bmatrix} \tag{2.98}$$

We can easily check the validity of (2.98) by using (2.97); for example:

$$\begin{array}{rl} c_{14} \rightarrow c'_{25} & \text{because } 1 \rightarrow 2 \text{ and } 4 \rightarrow 5 \\ c_{15} \rightarrow -c'_{24} & \text{because } 1 \rightarrow 2 \text{ and } 5 \rightarrow -4 \\ c_{46} \rightarrow -c'_{56} & \text{because } 4 \rightarrow 5 \text{ and } 6 \rightarrow -6, \ldots \end{array}$$

Equating (2.74) and (2.75), term by term, we find

first row: $c_{11} = c'_{11}$, $c_{12} = c'_{21}$, $c_{13} = c'_{23}$, $c_{14} = c'_{25}$, $c_{15} = c'_{24}$, $c_{16} = c'_{26}$
second row: $c_{22} = c'_{11}$, $c_{23} = c'_{13}$, $c_{24} = c'_{15}$, $c_{25} = c'_{14}$, $c_{26} = -c'_{16}$

third row: $c_{33} = c'_{33}$, $c_{34} = c'_{35}$, $c_{35} = -c'_{34}$, $c_{36} = -c'_{36}$
fourth row: $c_{44} = c'_{55}$, $c_{45} = -c'_{54}$, $c_{46} = c'_{56}$
fifth row: $c_{55} = c'_{44}$, $c_{56} = c'_{46}$
sixth row: $c_{66} = c'_{66}$

These conditions can be summarized as follows:

$$c_{11} = c'_{11},\ c_{66} = c'_{66},\ c_{33} = c'_{33} \quad \text{(no conditions)}$$

$$\left.\begin{matrix} c_{13} = c'_{23} \\ c_{23} = c'_{13} \end{matrix}\right] \rightarrow c_{13} = c_{23}, \quad \left.\begin{matrix} c_{14} = c'_{25} \\ c_{25} = c'_{14} \end{matrix}\right] \rightarrow c_{14} = c_{25} = 0$$

$$c_{15} = -c'_{24} = -c'_{15} = 0, \quad \left.\begin{matrix} c_{16} = -c'_{26} \\ c_{26} = c'_{16} \end{matrix}\right] \rightarrow c_{16} = -c_{26}$$

$$\left.\begin{matrix} c_{46} = -c'_{56} \\ c_{56} = c'_{46} \end{matrix}\right] \rightarrow c_{56} = c_{46} = 0, \quad \left.\begin{matrix} c_{34} = c'_{35} \\ c_{35} = c'_{34} \end{matrix}\right] \rightarrow c_{34} = c_{35} = 0$$

$$c_{36} = -c_{36} = 0, \quad c_{55} = c'_{44} \rightarrow c_{55} = c_{44}$$

Finally, we obtain the form of the stiffness matrix:

$$\mathbf{c} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{11} & c_{13} & 0 & 0 & -c_{16} \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ c_{16} & -c_{16} & 0 & 0 & 0 & c_{66} \end{bmatrix} \tag{2.99}$$

Comparing the form of $\mathbf{c}$ for the isotropic cubic with that for tetragonal symmetries, we notice that the upper left portion is filled; this is true for all symmetries. For the tetragonal symmetry, however, $c_{12} \neq c_{13}$ (there are two different Poisson ratios), $c_{33} \neq c_{11}$, and $c_{66} \neq c_{44}$; these changes reflect the uniqueness of the $z$-axis for this symmetry. The elements $c_{13}$ and $c_{23}$ are identical because the $x$- (1) and $y$- (2) axes are identical. The two new elements in the lower left and upper right corners represent coupling between shear stresses and longitudinal strains, and *vice versa*. These components are present in the classes $\bar{4}$, 4, and $4/m$, but not in classes 422 and $\bar{4}2m$, and thus are not essential for the symmetry. Classes 422 and $\bar{4}2m$ both possess the symmetry operation 2 (a 180° rotation about the $z$-axis). Referring to (2.76), we have already shown that this operation eliminates the stiffness components $c_{16}$ and $c_{25}$. In the following discussion,

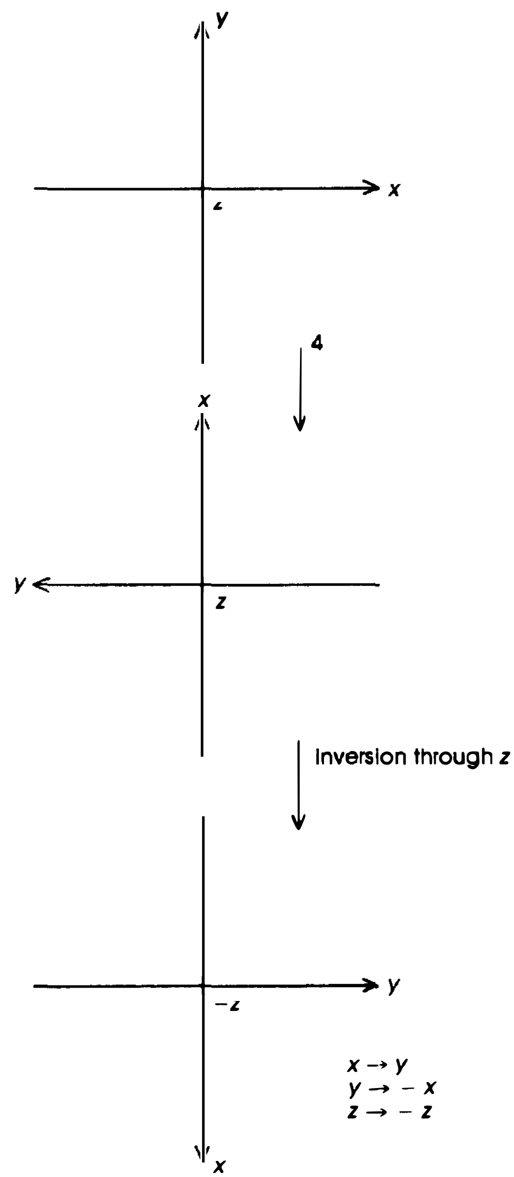


**Figure 2.4** The symmetry transformation $\bar{4}$ can be represented by two transformations: a $\pi/2$ rotation around the $z$-axis and an inversion through $z$.

these components will be omitted. Thus, the tetragonal stiffness matrix possesses either six or seven independent components.

Forming the Christoffel matrix for the tetragonal symmetry (2.99) and omitting the terms containing $c_{16}$, we get

$$\Gamma = \begin{bmatrix} c_{11}l_x^2 + c_{44}l_z^2 + c_{66}l_y^2 & c_{12}l_xl_y + c_{66}l_xl_y & c_{13}l_xl_z + c_{44}l_xl_z \\ c_{12}l_xl_y + c_{66}l_xl_y & c_{11}l_y^2 + c_{44}l_z^2 + c_{66}l_x^2 & c_{13}l_yl_z + c_{44}l_yl_z \\ c_{13}l_xl_z + c_{44}l_xl_z & c_{13}l_yl_z + c_{44}l_yl_z & c_{33}l_z^2 + c_{44}l_y^2 + c_{44}l_x^2 \end{bmatrix} \tag{2.100}$$

For propagation in the $x$ direction:

$$\mathbf{l} = \hat{\mathbf{i}}$$

$$\begin{bmatrix} c_{11} & 0 & 0 \\ 0 & c_{66} & 0 \\ 0 & 0 & c_{44} \end{bmatrix}$$

$$\begin{aligned} v_x : v_a &= \sqrt{\frac{c_{11}}{\rho}} \quad \text{(longitudinal)} \\ v_y : v_a &= \sqrt{\frac{c_{66}}{\rho}} \quad \text{(shear)} \\ v_z : v_a &= \sqrt{\frac{c_{44}}{\rho}} \quad \text{(shear)} \end{aligned} \tag{2.101}$$

For propagation in the $y$ direction:

$$\hat{\mathbf{l}} = \hat{\mathbf{j}}$$

$$\Gamma = \begin{bmatrix} c_{66} & 0 & 0 \\ 0 & c_{11} & 0 \\ 0 & 0 & c_{44} \end{bmatrix}$$

$$\begin{aligned} v_y : v_a &= \sqrt{\frac{c_{11}}{\rho}} \quad \text{(longitudinal)} \\ v_x : v_a &= \sqrt{\frac{c_{66}}{\rho}} \quad \text{(shear)} \\ v_z : v_a &= \sqrt{\frac{c_{44}}{\rho}} \quad \text{(shear)} \end{aligned} \tag{2.102}$$

For propagation in the $z$ direction:

$$\hat{\mathbf{l}} = \hat{\mathbf{k}}$$

$$\begin{bmatrix} c_{44} & 0 & 0 \\ 0 & c_{44} & 0 \\ 0 & 0 & c_{33} \end{bmatrix}$$

$$\begin{aligned} v_z &: v_a = \sqrt{\frac{c_{33}}{\rho}} \quad \text{(longitudinal)} \\ v_x &: v_a = \sqrt{\frac{c_{44}}{\rho}} \quad \text{(shear)} \\ v_y &: v_a = \sqrt{\frac{c_{44}}{\rho}} \quad \text{(shear)} \end{aligned} \tag{2.103}$$

We note that the longitudinal waves have identical velocities in the $x$- and $y$-axes, but a different velocity in the $z$-axis, which is consistent with the fact that the "spring" constant or stiffness is different in the $z$ direction. The $z$ stiffness $c_{33}$ may be greater or less than $c_{11}$, depending on the particular crystal. In the $xy$ plane, the Christoffel matrix is

$$\Gamma = \begin{bmatrix} c_{11}l_x^2 + c_{66}l_y^2 & c_{12}l_xl_y + c_{66}l_xl_y & 0 \\ c_{12}l_xl_y + c_{66}l_xl_y & c_{11}l_y^2 + c_{66}l_x^2 & 0 \\ 0 & 0 & c_{44}(l_x^2 + l_y^2) \end{bmatrix} \tag{2.104}$$

Comparing (2.104) with (2.53), we note that the tetragonal symmetry looks "cubic" in the $xy$ plane, with some important differences. Like the cubic symmetry, there is a pure shear mode that is $z$-polarized with velocity independent of direction. In the cubic symmetry, however, there is a shear degeneracy for the $x$- and $y$-axes because both involve particle motions that "see" identical stiffnesses (along the $y$- and $z$-axes, respectively). In the tetragonal case, the $x$ shear wave involves the $y$-directed stiffness, and the $y$ shear wave involves the $z$-directed stiffness; because they are unequal, a shear wave degeneracy does not exist. However, for a $z$-propagating shear wave, we see, by the same reasoning, that a shear degeneracy does exist because both waves have equal stiffness constants.

Let us now solve for the velocities in the direction 45° from the $x$- to the $y$-axis in the $xy$ plane:

$$\hat{\mathbf{l}} = \frac{\hat{\mathbf{i}}}{\sqrt{2}} + \frac{\hat{\mathbf{j}}}{\sqrt{2}}$$

The Christoffel matrix is

$$\Gamma = \begin{bmatrix} c_{11} + c_{66} & c_{12} + c_{66} \\ c_{12} + c_{66} & c_{11} + c_{66} \end{bmatrix} \tag{2.105}$$

which is in the form of

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

We have previously solved the eigenvalue problem for a matrix of this form (2.84). The solutions are

$$v_1 = \sqrt{\frac{c_{11} + c_{12} + 2c_{66}}{2\rho}} \quad \text{(longitudinal mode)} \tag{2.106}$$

$$v_2 = \sqrt{\frac{c_{11} - c_{12}}{2\rho}} \quad \text{(shear mode)} \tag{2.107}$$

Compared with (2.87) (for cubic symmetry), $c_{44}$ has been replaced by $c_{66}$, but the shear velocities are identical.

Now consider the $xz$ plane ($l_y = 0$). The Christoffel matrix is

$$\Gamma = \begin{bmatrix} c_{11}l_x^2 + c_{44}l_z^2 & 0 & (c_{13} + c_{44})l_xl_z \\ 0 & c_{66}l_x^2 + c_{44}l_z^2 & 0 \\ (c_{13} + c_{44})l_xl_z & 0 & c_{44}l_x^2 + c_{33}l_z^2 \end{bmatrix} \tag{2.108}$$

The middle term represents the $y$-polarized (and therefore pure shear) mode. Unlike the pure shear mode in isotropic and cubic sysems, its phase velocity varies from

$$v_a(x) = \sqrt{\frac{c_{66}}{\rho}} \tag{2.109}$$

for an $x$-directed wave to

$$v_a(z) = \sqrt{\frac{c_{44}}{\rho}} \tag{2.110}$$

for a $z$-propagating wave. Furthermore, as we already noted, there is shear degeneracy along the $z$-axis because the second shear mode ($x$-polarized) also has velocity $v_a(z)$. For the $\langle 1, 0, 1\rangle$ direction (45° from the $x$- to the $z$-axis for which $|l_y| = |l_z|$), the $y$-polarized (pure) shear mode has velocity:

$$v_a = \sqrt{\frac{c_{66} + c_{44}}{2\rho}} \tag{2.111}$$

Equation (2.111) reduces to (2.83) for cubic symmetry (because $c_{66} \rightarrow c_{44}$). Equation (2.111) also provides the velocity of the (pure $x$-polarized) shear mode in the $\langle 0, 1, 1\rangle$ direction because of the equality between $x$ and $y$ in the tetragonal class; it does not provide the velocity for the ($z$-directed) shear in the $\langle 1, 1, 0\rangle$ direction.

Returning to the $xz$ plane, the Christoffel matrix for quasimodes:

$$\Gamma = \begin{bmatrix} c_{11}l_x^2 + c_{44}l_z^2 & (c_{13} + c_{44})l_xl_z \\ (c_{13} + c_{44})l_xl_z & c_{44}l_x^2 + c_{33}l_z^2 \end{bmatrix} \tag{2.112}$$

This matrix has the form:

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

which is the same as in the $yz$ plane with the substitution $l_y \rightarrow l_z$. Compared with the cubic or tetragonal symmetry in the $xy$ plane, this matrix has diagonal components that are not equal; this fact significantly complicates the eigenvalue calculation even for the relatively simple case of a $\langle 1\ 0\ 1\rangle$

directed wave. The closed-form expressions for the velocities are quite involved.

In the $yz$ plane, the form of the Christoffel matrix is identical to (2.108), due to the equality of the $x$- and $y$-axes. Thus, there is an $x$-polarized (pure) shear mode with a velocity that varies from

$$v_a(y) = \sqrt{\frac{c_{66}}{\rho}}$$

to

$$v_a(z) = \sqrt{\frac{c_{44}}{\rho}}$$

### 2.5.3 Orthorhombic Symmetry

In the orthorhombic class, the form of the stiffness matrix can also be deduced from Neumann's principle by using the symmetry operator 2:

$$2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.113}$$

This matrix represents a rotation of 180° about the $z$-axis. A matrix representing a rotation about the $x$- or $y$-axis would yield the same result for the stiffness matrix, which is

$$\mathbf{c} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \tag{2.114}$$

Compared with the cubic and tetragonal classes, all three major axes are unique in the orthorhombic symmetry. Equation (2.114) shows this by the presence of three separate longitudinal diagonal terms and three separate shear diagonal terms. There are also three different Poisson ratio terms, $c_{12}$, $c_{13}$, and $c_{23}$. In all, there are nine independent components to the stiffness matrix; the solutions of the characteristic equation are thus quite complex except for the simplest cases. Based on our experience with tetragonal symmetry, we surmise that there are no shear degeneracies along the major axes.

The general form of the Christoffel matrix is easily derived from (2.49):

$$\Gamma = \begin{bmatrix} c_{11}l_x^2 + c_{66}l_y^2 + c_{55}l_z^2 & (c_{12} + c_{66})l_xl_y & (c_{13} + c_{55})l_xl_z \\ (c_{12} + c_{66})l_xl_y & c_{66}l_x^2 + c_{22}l_y^2 + c_{44}l_z^2 & (c_{23} + c_{44})l_yl_z \\ (c_{13} + c_{55})l_xl_z & (c_{23} + c_{44})l_yl_z & c_{55}l_x^2 + c_{44}l_y^2 + c_{33}l_z^2 \end{bmatrix} \tag{2.115}$$

For the principal axes, the velocities are:

$x$-axis:

longitudinal mode: $v_a = \sqrt{\dfrac{c_{11}}{\rho}}$ $\langle 1, 0, 0\rangle$

shear mode: $v_a = \sqrt{\dfrac{c_{66}}{\rho}}$ $\langle 0, 1, 0\rangle$

shear mode: $v_a = \sqrt{\dfrac{c_{55}}{\rho}}$ $\langle 0, 0, 1\rangle$

$y$-axis:

longitudinal mode: $v_a = \sqrt{\dfrac{c_{22}}{\rho}}$ $\langle 0, 1, 0\rangle$

shear mode: $v_a = \sqrt{\dfrac{c_{66}}{\rho}}$ $\langle 1, 0, 0\rangle$

shear mode: $v_a = \sqrt{\dfrac{c_{44}}{\rho}}$ $\langle 0, 0, 1\rangle$

$z$-axis:

longitudinal mode: $v_a = \sqrt{\dfrac{c_{33}}{\rho}}$ $\langle 0, 0, 1\rangle$

shear mode: $v_a = \sqrt{\dfrac{c_{55}}{\rho}}$ $\langle 1, 0, 0\rangle$

shear mode: $v_a = \sqrt{\dfrac{c_{44}}{\rho}}$ $\langle 0, 1, 0\rangle$

In the orthorhombic symmetry, the form the Christoffel matrix in all three principle planes resembles that of (2.108). In each plane, there is a pure shear mode with a velocity that varies with direction. Although closed-form solutions exist for arbitrary directions in the principal planes, they are quite cumbersome. In Chapter 3, we investigate computer-aided solutions.

## PROBLEMS

**2.1** Show that the phase velocities for propagation along the $\langle 1, 1, 1\rangle$ direction in an isotropic medium are given by

$$v_a = \sqrt{\frac{c_{11}}{\rho}} \qquad \text{longitudinal mode}$$

$$v_a = \sqrt{\frac{c_{44}}{\rho}} \qquad \text{shear mode}$$

Find the polarizations and verify that the modes are pure.

**2.2** Derive (2.115) by following the procedure of (2.49) and (2.51).

**2.3** Write the Christoffel matrix for the orthorhombic symmetry in the *xy* plane and determine the stiffness component of the pure shear mode for an arbitrary direction in the plane.

**2.4** Find the inverse of the compliance matrix **s** for an isotropic medium and verify (2.32) and (2.33).

**2.5** Verify that the Christoffel equation for the particle velocity has the same form as (2.40).

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