Chapter 1

The Acoustic Equation of Motion

1.1 INTRODUCTION

In this chapter, we lay the groundwork for the study of crystal acoustics by developing fundamental mechanical equations in one dimension. We show that the presence of an unbalanced system of time-varying stresses results in the propagation of an acoustic wave with its propagation velocity dependent on the material properties of the body. Attenuation, material quality factor Q, and the energy relations are more easily handled in one dimension and can be extended to three dimensions if necessary. The strain and stress matrices are formulated in three dimensions, and rotational transformations, which will be increasingly important later, are developed. The three-dimensional equations of motion are developed by using the stress and strain matrices. The coupling between these matrices is not developed beyond the point of stating that they are linearly related. Thus, the dynamical three-dimensional equations of motion as developed in this chapter do not include the generalization of Hooke's law.

1.2 STRESS AND STRAIN IN ONE DIMENSION

In Newtonian mechanics, a *force* on a *rigid* body results in an *acceleration* of the body. Because the body is assumed to be rigid, the external force is instantaneously transmitted to all of the body's internal parts. No consideration is given to the *internal* structure of the body, nor to the bonding forces that hold the body together. These issues are dealt with in the science of *strength of materials* or *mechanics of deformable bodies*, which examine the relation between external forces, sometimes called *body forces*, and the resulting internal effects. The effect of the body forces is the creation of internal forces, called *stresses*, and deformations, called *strains*, in the atomic structure of the body.

In Newtonian mechanics, there is a causal relation between body forces and acceleration. If a body is accelerating, it must be acted on by a net force, but a body in equilibrium may be acted on by many forces while at rest. In this sense, we may think of force as the independent variable and acceleration as the dependent variable.

Stress does not cause strain (nor does strain cause stress), but the two are coupled to each other. Internal deformations for example, can be "caused" by thermal gradients, dislocations, and defects in the crystal lattice or by the presence of dopant atoms that are significantly larger or smaller than the host atoms and thus deform the lattice structure. In such cases, internal forces are established, and it would be proper to refer to these stresses as being the result of the strains. Nonetheless, it is usually more convenient (as well as precise) to refer to the coupling of stress and strain; the presence of either necessarily implies that the other is also present.

Because stress is intimately related to deformation or distortion in the internal structure of a body, the magnitude of stress is related to internal forces divided by the area over which the forces act. The nature of the deformation depends on the orientation of the area (recall that area is a vector with direction as defined by the surface normal) with respect to the stress. A compressive stress tends to push the internal particles together, a tensile stress tends to pull them apart, and a shear stress tends to cut. Compressive and tensile stresses form the class of *longitudinal stresses*. This is illustrated in Figure 1.1. Note that the orientation of the area (as defined by its normal vector) determines whether the stress is shear or longitudinal.

A further distinction between stress and force comes from the fact that stresses always occur in opposite (but not always equal) pairs. These stress components are individually referred to as traction forces, and, like stress, they are denoted by the letter T. A positive traction force points to the right, and a negative traction force points to the left, in agreement with conventional notation. The units of traction forces as well as stress are N/m^2. Both compressive and tensile stresses are clearly composed of two traction forces, one positive and one negative. We define a compressive stress as negative and a tensile stress as positive. This definition is quite logical because in a compressive stress the traction forces are both in a direction opposite to the area (defined as the outward normal). In the static case, the stresses are equal because there is no net motion of any internal volumes. In the dynamic case (e.g., the propagation of an acoustic wave), the opposite stresses are not generally equal.

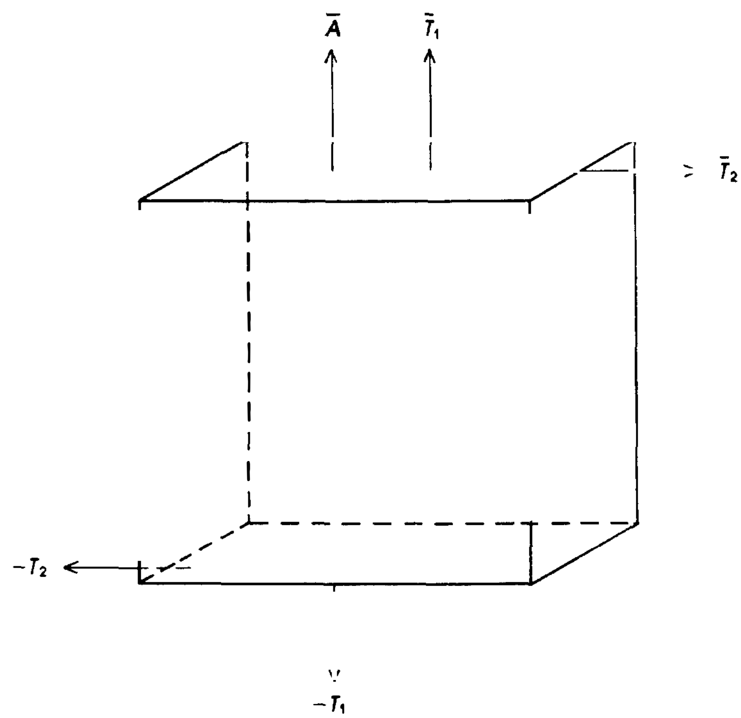

Figure 1.1 Orientation of traction forces relative to the area of an internal volume of an isotropic medium.

Consider Figure 1.2. There are two regions, labeled 1 and 2. Each region (which in general contains many particles) consists of a mass element connected by springs to two nearest neighbors. The equilibrium distance between them is denoted as ΔL, which is small enough so that the masses may be approximated by a continuum and $\Delta L \sim \mathrm{d}z$ (Figure 1.2(a)). If a z-directed external force, which may be either positive (directed toward the right) or negative (directed toward the left), is applied, internal forces will be established, moving the particles from their equilibrium positions. This situation is shown in Figure 1.2(b). The new distance between the masses is Δl, and the internal forces are described by stress components T_1 and T_2 (which are not necessarily equal) in Figure 1.2(b). The individual forces are given by

$$\mathrm{d}F_1 = \mathrm{d}A\ T_1$$
$$\mathrm{d}F_2 = \mathrm{d}A\ T_2$$

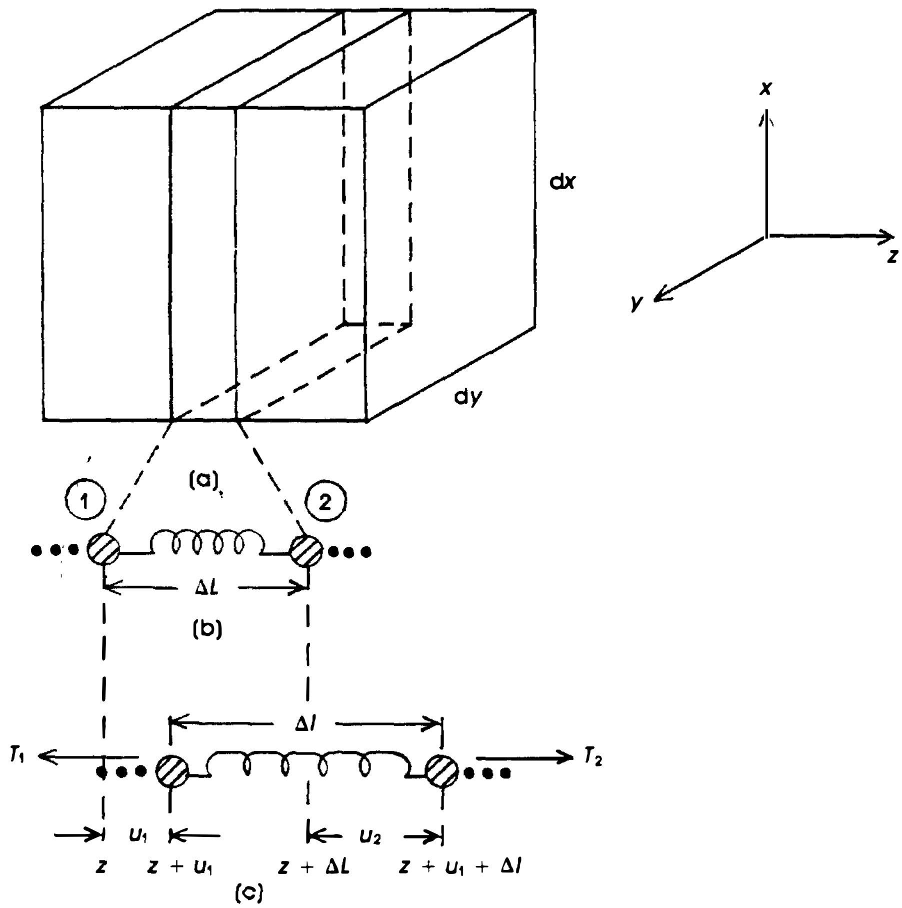

Figure 1.2 Section of internal volume element of isotropic medium: undistorted volume element; (a) representation of internal coupling as springs; (b) particle distortions due to traction forces.

where dA is the cross-sectional area with dimensions dx and dy. If the new positions are such that $u_1 = u_2$, then $\Delta L = \Delta l$, and there is no relative movement of the masses and thus no *distortion* of region 1 relative to region 2. This situation results from a translation of the body and is not of practical interest. If, however, $u_1 \neq u_2$, and $\Delta u \neq 0$, then we can define the distortion $\mathfrak{D}$ as

$$\mathcal{D} = (\Delta l)^2 - (\Delta L)^2$$
$$= (\Delta L + \Delta u)^2 - (\Delta L)^2$$
$$= (\Delta L + \frac{\partial u}{\partial z} \Delta L)^2 - (\Delta L)^2$$
$$= (\Delta L)^2 \left(\frac{\partial u}{\partial z}\right)^2 + 2 \frac{\partial u}{\partial z} (\Delta L)^2$$

$$\mathcal{D} = \frac{\partial u}{\partial z} \left(\frac{\partial u}{\partial z} + 2\right) (\Delta L)^2 \tag{1.1}$$

The strain $\mathbf{S}$ is defined as

$$\mathcal{D} = 2(\Delta L)^2 \mathbf{S} \tag{1.2}$$

If we assume that (u does not change rapidly with position) $\partial u/\partial z \ll 1$, then,

$$\mathcal{D} = 2 \frac{\partial u}{\partial z} (\Delta L)^2 = 2 \mathbf{S} (\Delta L)^2 \tag{1.3}$$

The assumption that the particle displacement u changes gradually with position is an example of linearization and is valid only if the strains are small, which may not be realistic in practical cases. We define the strain (from (1.3)) as

$$S = \frac{\partial u}{\partial z} \tag{1.4}$$

The presence of a strain implies that the particle displacement from equilibrium changes with position (i.e., there is a distortion in the body). From (1.3), we can also write the strain as

$$S = \frac{1}{2} \left(\frac{(\Delta l)^2 - (\Delta L)^2}{(\Delta L)^2}\right) \tag{1.5}$$

1.3 MECHANICAL EQUATIONS OF MOTION IN ONE DIMENSION

In this section we derive a self-consistent set of equations that describes the propagation of a mechanical strain in a one-dimensional solid. The mechanical variables corresponding to the electromagnetic variables E, D, H, and B are

stress $= \mathbf{T}$
strain $= \mathbf{S}$
particle displacement $= \mathbf{u}$
particle velocity $= \mathbf{v}$

Just as Maxwell's equations are a set of four relations between the four electromagnetic variables, we require four equations for completely characterizing the mechanical properties. They are as follows.

1. *Newton's law*: Consider the slab of Figure 1.2 of cross section $dA = dx\,dy$. If the stresses T_1 and T_2 are not equal, there is a net force on the slab given by

$$dF = \frac{\partial T}{\partial z}\,dz\,dA = |T_2 - T_1|\,dA$$

Newton's law is written as

$$\overbrace{\frac{\partial T}{\partial z}\,dzA}^{dF \uparrow} = \overbrace{\rho A\;\Delta z}^{m \uparrow}\;\overbrace{\frac{\partial^2 u}{\partial t^2}}^{a \uparrow} \qquad (F = AT)$$

or

$$\frac{\partial T}{\partial z} = \rho\,\frac{\partial^2 u}{\partial t^2} \tag{1.6}$$

where ρ is the density in kg/m^3.

2. *Particle velocity* is the time derivative of particle displacement:

$$v = \frac{\partial u}{\partial t} \tag{1.7}$$

3. Equation (1.4), which defines S as the *gradient* (spatial rate of change) of particle displacement with respect to position:

$$S = \frac{\partial u}{\partial z}$$

4. *Relation between the stress T and the strain S*: We assume there is a linear relation between the internal stresses and the deformation, and we write

$$T = CS \tag{1.8}$$

where C is called the *stiffness constant* and has units of stress (since strain is dimensionless). Equation (1.8) defines the properties of the connecting springs. For given stress components, a stiff spring results in a relatively small strain, whereas a compliant spring results in a large strain.

Equations (1.4), (1.6), (1.7), and (1.8) allow us to solve for the four variables T, S, v, and u. As for Maxwell's equations, there are two *fundamental* physical laws, (1.4) and (1.6), and two *constitutive* equations, (1.7) and (1.8). In the static case, the gradient of the stress is zero, just as in electrostatics where $\nabla \times \overline{E} = 0$. From (1.6) and (1.7):

$$\frac{\partial T}{\partial z} = \frac{\partial v}{\partial t}$$

Differentiating $S = \partial u/\partial z$ with respect to t and using (1.8), we obtain

$$\begin{aligned} \frac{\partial S}{\partial t} &= \frac{\partial^2 u}{\partial t \partial z} \\ &= \frac{\partial v}{\partial z} = \frac{1}{C}\frac{\partial T}{\partial t} \end{aligned} \tag{1.9}$$

We can now solve this set of equations by forming the one-dimensional wave equations. We differentiate (1.9) with respect to t and (1.6) (Newton's law) with respect to z:

$$\frac{\partial^2 v}{\partial t\, \partial z} = \frac{1}{C}\frac{\partial^2 T}{\partial t^2}$$

and

$$\frac{1}{\rho}\frac{\partial^2 T}{\partial z^2} = \frac{\partial^2 v}{\partial z\,\partial t}$$

Because the terms with cross derivatives are equal, we have

$$\frac{1}{\rho}\frac{\partial^2 T}{\partial z^2} = \frac{1}{C}\frac{\partial^2 T}{\partial t^2} \tag{1.10}$$

Equation (1.10) is the one-dimensional wave equation, the solution of which is a propagating function with phase velocity

$$v_a = \sqrt{\frac{C}{\rho}} \tag{1.11}$$

We should be careful not to confuse v (the particle velocity) with v_a (the phase velocity of the acoustic wave). The one-dimensional acoustic equations along with the well-known Maxwell equations are summarized in Table 1.1.

Table 1.1

One-Dimensional Acoustic Equations	*Maxwell Equations*
FUNDAMENTAL PHYSICAL LAWS	
$\frac{\partial T}{\partial z} = \rho \frac{\partial v}{\partial t}$ Newton's law	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ Faraday's law
$S = \frac{\partial u}{\partial z}$	$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$ Ampere's law
CONSTITUTIVE EQUATIONS	
$T = CS$ Hooke's law	$\mathbf{D} = \epsilon\, \mathbf{E}$
$v = \frac{\partial u}{\partial t}$	$\mathbf{B} = \mu\, \mathbf{H}$

Both the acoustic and electromagnetic systems are composed of two "fundamental" physical laws and two "constitutive" equations. In the electromagnetic system, the solutions of Maxwell's equations in Cartesian coordinates are plane waves with either one (for an optically isotropic medium) or two (for an optically anisotropic medium) polarizations. In the one-dimensional acoustic system, we have seen that the solution is also a plane wave with *acoustic* polarization (defined as the direction of either particle displacement or velocity) in the direction of wave propagation (a longitudinal wave). In the three-dimensional system, we will see that there are, in general, three possible acoustic polarizations.

Because the phase velocity of an electromagnetic wave is from four to six orders of magnitude greater than that of an acoustic wave, the presence of boundaries plays a much more significant role. On the other hand, acoustic propagation is complicated not only by the presence of three acoustic modes, but also because the phase velocities are complex functions of the propagation direction. This directional dependence is due, in part, to the relatively complex nature of Hooke's law (1.8) in three dimensions as compared to the corresponding constitutive relations for electromagnetics.

1.3.1 Phase Relations

For an electromagnetic wave propagating in an isotropic, lossless medium, the displacement **D**, the electric field **E**, the magnetic field **H**, and the induction **B** are all in phase. In the acoustic system, the particle displacement is not in phase with the particle velocity (1.7). If we let

$$u = u_0 \, e^{j(\omega t - \beta z)}$$

other phase relations are

$$\frac{\partial u}{\partial z} = -j\beta u_0 \, e^{j(\omega t - \beta z)}$$

$$S = \frac{\partial u}{\partial z} = -j\beta u$$

To find the phase relation between S and v, we use (1.9):

$$\frac{\partial S}{\partial t} = \frac{\partial^2 u}{\partial t \partial z} = \frac{\partial v}{\partial z} = -j\beta \frac{\partial u}{\partial t}$$

or

$$S = -\frac{\beta}{\omega}\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{\beta}{\omega} v$$

Because T and S are related by a constant (C), they are in phase. In summary, T and S are in phase, and all are 90° out of phase with u and 180° out of phase with v. The ratio of T and v is called the acoustic impedance of the medium (Z) and is

$$Z = -\frac{\mathbf{T}}{\mathbf{v}} = \frac{-C\mathbf{S}}{\mathbf{v}} = \frac{(-C)(-j\beta\mathbf{u})}{\mathbf{v}}$$

$$= \frac{jC\omega\mathbf{u}}{\mathbf{v}v_a}$$

(the minus sign is included so that the impedance will be positive, because T and v are 180° out of phrase), but $j\omega u = v$, so

$$Z = \frac{C\mathbf{v}}{\mathbf{v}v_a} = \frac{C}{v_a}$$

From (1.11), $v_a = \sqrt{C/\rho}$, so

$$Z = -\frac{\mathbf{T}}{\mathbf{v}} = C\sqrt{\rho/C} = \rho v_a \tag{1.12}$$

The units of impedance are kg/s m^2. Like the phase velocity, the acoustic impedance is a property of the medium. Corresponding to (1.11) and (1.12), the electromagnetic relations for phase velocity and impedance are given by the well-known formulas:

$$v_p = \sqrt{\frac{1}{\mu\epsilon}} \quad \text{phase velocity}$$

$$Z_e = \sqrt{\frac{\mu}{\epsilon}} \quad \text{impedance}$$

1.4 ABSORPTION OF AN ACOUSTIC WAVE

If a solid medium obeyed Hooke's law ($T = CS$) precisely, there would be no acoustic absorption. In a real medium, there are viscous damping forces and nonlinearities, which cause energy to be extracted from the wave in the form of heat. We have already encountered a nonlinearity in the definition of strain. We can include these forces in the wave equation by modifying Hooke's law:

$$T = CS + \eta \frac{dS}{dt} \tag{1.13}$$

where η is called the viscosity, and has the units $[\eta] = \mathrm{N\ s/m^2}$. The time derivative term in (1.13) implies that the strain tends to *relax* with time toward its equilibrium state. This is illustrated in Figure 1.3. The acoustic wave creates positions of high compression, which are physically located next to positions of high tension; the relaxation term tends to equalize these opposite stresses and thus reduces the energy in the wave. For a constant phase velocity, the peaks and valleys of the wave are closer together at higher acoustic frequencies, making it easier to extract energy from the wave; for constant frequency, a lower velocity implies a smaller wavelength, which also pushes the peaks and valleys closer to one another. We thus expect that the attenuation will be proportional to frequency and inversely proportional to velocity. We can formally demonstrate this by recalling the one-dimensional equations of motion (1.6):

$$\frac{\partial T}{\partial z} = \rho \frac{\partial v}{\partial t}$$

and (1.9):

$$\frac{\partial S}{\partial t} = \frac{\partial v}{\partial z}$$

Substituting (1.13) into (1.6), we obtain

$$\frac{\partial}{\partial z}(CS + \eta \dot{S}) = \rho \frac{\partial v}{\partial t} \tag{1.14}$$

Differentiating (1.14) with respect to t and (1.9) with respect to z gives

$$\frac{\partial^2}{\partial t \partial z}(CS + \eta \dot{S}) = \rho \frac{\partial^2 v}{\partial t^2}$$

and

$$\frac{\partial^2 S}{\partial z \partial t} = \frac{\partial^2 v}{\partial z^2}$$

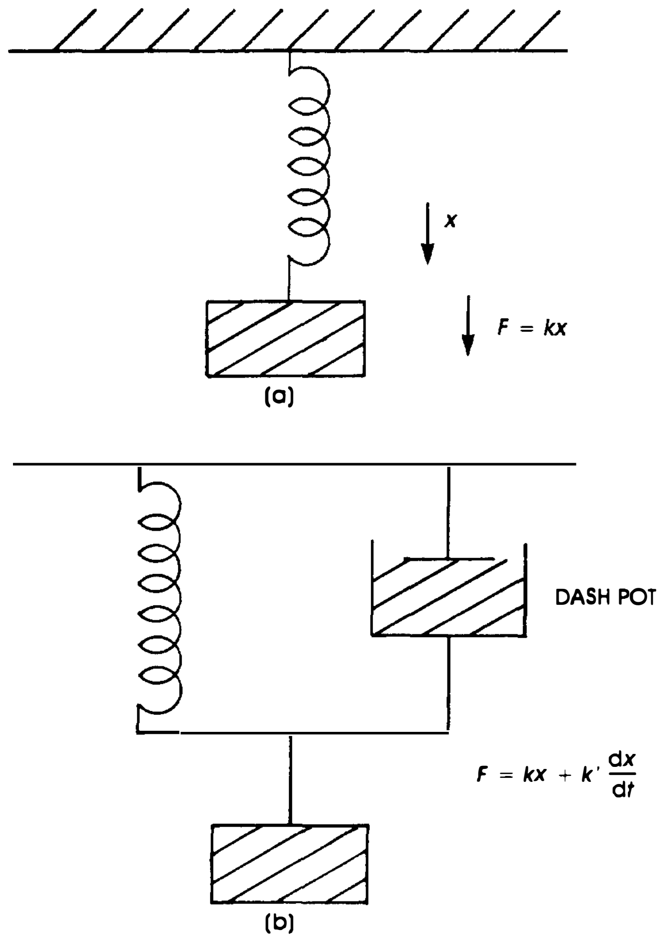

Figure 1.3 Model of acoustic wave: (a) with no absorption; (b) with absorption.

The dot in (1.14) signifies the time derivative, where $\dot{S} = \partial S/\partial t$. Combining these last equations and using (1.9), we get

$$\rho \frac{\partial^2 v}{\partial t^2} = C \frac{\partial^2 v}{\partial z^2} + \eta \frac{\partial^3 v}{\partial t \partial z^2} \tag{1.15}$$

Notice that if $\eta \to 0$, then (1.15) reduces to the ordinary wave equation (1.10). We assume a plane wave solution, but we allow for absorption by requiring a complex propagation constant:

$$v = A\, e^{j(\omega t - \hat{k}z)} \tag{1.16}$$

where $\hat{k} = \beta + j\alpha$. Thus,

$$\frac{\partial^2 v}{\partial t^2} \rightarrow -\omega^2 v \quad \text{and} \quad \frac{\partial^2 v}{\partial z^2} = -\hat{k}^2 v$$

The wave equation becomes

$$(-\omega^2)\rho v = C(-\hat{k}^2)v + j\eta\hat{k}^2\omega v \tag{1.17}$$

Because $\hat{k} = \beta + j\alpha$:

$$\hat{k}^2 = (\beta + j\alpha)^2 = \beta^2 - \alpha^2 + j2\beta\alpha \tag{1.18}$$

Substituting (1.18) into (1.17), we obtain

$$-\omega^2\rho = -C(\beta^2 - \alpha^2 + j2\beta\alpha) + j\eta(\beta^2 - \alpha^2 + j2\beta\alpha)\omega \tag{1.19}$$

Equation (1.19) is a complex equation with real part:

$$-\omega^2\rho = -C\beta^2 + C\alpha^2 - 2\eta\beta\alpha\omega \tag{1.20}$$

For all practical cases, η and thus α are very small compared with β and ω, so the last two terms can be neglected. Equation (1.20) reduces to

$$\frac{\omega}{\beta} = \sqrt{\frac{C}{\rho}} = v_a \quad \text{(the phase velocity)}$$

Thus, for very small absorption the phase velocity is not a function of frequency. The medium is said to be dispersionless.

Now consider the imaginary part of (1.19):

$$0 = -2C\beta\alpha + \eta\beta^2\omega - \eta\alpha^2\omega \tag{1.21}$$

As before, the last term is negligible ($\alpha \ll \beta$). Hence, solving for α, we have

$$\begin{aligned} \alpha &= \frac{\eta\omega\beta}{2C} = \frac{\eta\omega^2}{2v_a^3\rho} \\ &= \left(\frac{2\pi}{\lambda}\right)^2 \frac{\eta}{2\sqrt{C\rho}} \end{aligned} \tag{1.22}$$

From (1.22) the units of α are

$$[\alpha] = \frac{\text{N s/m}^2\ \text{s}^{-2}}{(\text{m/s})^3\ \text{kg/m}^3} = 1/\text{m}$$

The absorption is usually expressed in tables in either dB/cm or dB/μs (normalized to velocity). In calculations, it should be converted to Np/m (1 Np = 8.7 dB). From (1.22) it is clear that, as the wavelength decreases (high frequency or low velocity), the peaks and valleys of the wave get closer together, facilitating the interaction between them and thus increasing the absorption.

Equation (1.22) can be written in the form:

$$\begin{aligned}\alpha &= \frac{\omega\eta}{v_a^2\rho}\left(\frac{\omega}{2v_a}\right)\\ &= \frac{\omega}{2Qv_a}\end{aligned} \tag{1.23}$$

where we define the *quality factor* Q as

$$\begin{aligned}Q &\equiv \frac{v_a^2\rho}{\omega\eta}\\ &= \frac{\omega}{2\alpha v_a}\end{aligned} \tag{1.24}$$

Because α is proportional to ω^2 (1.22), Q is inversely proportional to frequency. When specifying the material Q, we must also specify the frequency. Quality factors of some important acoustic materials are given in Table 1.2.

The absorption values (and Q-factors) in Table 1.2 are approximations, because they vary with the quality of material as well as with the particular mode (longitudinal or shear) and propagating direction. For most materials, however, the Q does not vary with orientation by more than 5 to 10%. An important exception is paratellurite (TeO_2), for which the variation in Q with crystal orientation with acoustic mode is more than an order of magnitude. We will study the properties of this important crystal in detail later. In general, materials with high velocities tend to have low absorptions and high Q-factors.

Table 1.2
Absorption and Q at 1 GHz

Material	α(dB/cm)	Q
Sapphire (Al_2O_3	.2	$2 \cdot 10^5$
$LiNbO_3$	.3–.5	$1 \cdot 10^5$
$LiTaO_3$	.2	$2 \cdot 10^5$
TiO_2 (rutile)	.4	$1 \cdot 10^5$
SiO_2 (crystal quartz)	2	$2 \cdot 10^4$
SiO_2 (fused quartz)	14	2700
GaAs	30	1250
Aluminum	18	$2 \cdot 10^3$
Gold	80	450

The degree of compliance of acoustic attenuation with frequency is an excellent measure of material quality. A poorly grown crystal, e.g., will generally be highly stressed and will contain a high density of grain boundaries, air pockets, and impurities. These defects will usually have dimensions comparable with an acoustic wavelength and will scatter the wave, thus reducing the frequency variation to a linear dependence. In practice, then, the frequency dependence is ω^n, where $1.2 < n < 1.4$ for a poor-quality material and $n > 1.8$ for a high quality material. If $n < 1.7$, it is usually safe to assume that improved crystal growth conditions will result in a significant reduction in absorption.

Absorption is determined experimentally by performing a pulse echo measurement in which a crystal sample is excited by an impulse and the reduction in the resulting pulse train is observed. In some situations in which low absorption is critical, the sample is cooled to a very low temperature, which dramatically reduces α. For longitudinal modes at room temperature, the dependence, as derived by Woodruff and Ehrenreich, is proportional to $\gamma^2\omega^2 kT$, where γ is called the Gruneisen constant and is a more fundamental physical property of the crystal than α is, and k is the thermal conductivity (which is inversely proportional to T so there is no net temperature dependence) [5]. The dependence on k follows from the fact that high conductivity facilitates the transfer of energy from regions of compression to regions of extension. This mechanism is not operative for shear modes (because there is no compression or extension). Hence, it may be inferred that the absorption of shear modes is less than that of longitudinal modes; the lower acoustic velocities of shear modes usually more than compensate for the lower absorptions.

The dependence of α on thermal conductivity has led to a search for techniques to reduce k by, for example, doping the crystal; the results have been only partially successful. At very low temperatures, the absorption is extremely low and its dependence is of the form ωT^4. Between the high temperature and low temperature regions, the absorption follows a dependence characteristic of *relaxation* behavior as shown in Figure 1.4. Unfortunately, for most materials T_1 falls between 30 and 100 K.

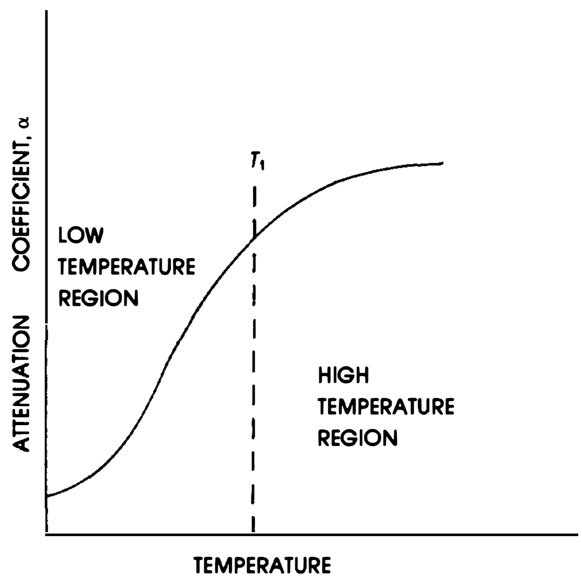

Figure 1.4 Variation of absorption with temperature for typical crystal medium.

1.5 POWER RELATIONS

In electromagnetic theory, the divergence of the Poynting vector $\mathbf{P} = (\mathbf{E} \times \mathbf{H})$ determines the power in the electromagnetic wave:

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\frac{\partial}{\partial t}\left(\underbrace{\tfrac{1}{2}\epsilon E^2}_{\frac{\text{energy in the electric field}}{\text{m}^3}} + \underbrace{\tfrac{1}{2}\mu H^2}_{\frac{\text{energy in the magnetic field}}{\text{m}^3}}\right) \tag{1.25}$$

Note that $\nabla \cdot \mathbf{P} = \nabla \cdot (\mathbf{E} \times \mathbf{H})$ is a scalar, but $\mathbf{E} \times \mathbf{H}$ is a vector with units W/m^2 and

$$\oint_{\substack{\text{closed}\\ \text{area}}} (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{ds}$$

is the power in watts crossing the closed surface. For complex fields in an isotropic medium:

$$\mathbf{E} = E_0\, e^{j\omega t}\hat{\mathbf{a}}_E$$

$$\mathbf{H} = H_0\, e^{j\omega t}\hat{\mathbf{a}}_H$$

where $\hat{\mathbf{a}}_E$ and $\hat{\mathbf{a}}_H$ are unit vectors in the directions of $\mathbf{E}$ and $\mathbf{H}$, respectively, and

$$\hat{\mathbf{a}}_E \cdot \hat{\mathbf{a}}_H = 0$$

and the Poynting vector is (because $\mathbf{E}$ and $\mathbf{H}$ are orthogonal)

$$P = \frac{1}{2} E_0 H_0 \tag{1.26}$$

Furthermore the $\mathbf{E}$ and $\mathbf{H}$ fields are in phase and related by the impedance Z_e:

$$Z_e = \frac{E}{H} = \sqrt{\frac{\mu}{\epsilon}}$$

From (1.25) and the definition of impedance, it is easy to see that the electric and magnetic energy densities are equal:

$$\frac{\epsilon E^2}{\mu H^2} = \frac{\epsilon}{\mu}\left(\frac{\mu}{\epsilon}\right) = 1$$

We seek an analogous acoustic relation. Recall (1.6) and (1.9):

$$\frac{\partial T}{\partial z} = \rho \frac{\partial v}{\partial t}$$

$$\frac{\partial S}{\partial t} = \frac{\partial v}{\partial z}$$

Multiplying (1.6) by v and (1.9) by T and adding gives

$$v\frac{\partial T}{\partial z} + T\frac{\partial v}{\partial z} = \rho v\frac{\partial v}{\partial t} + T\frac{\partial S}{\partial t} \tag{1.27}$$

Using (1.8) (Hooke's law) and assuming that ρ and C are time invariant, we can write (1.27) as

$$-\frac{\partial}{\partial z}(Tv) = -\frac{\partial}{\partial t}\left(\underset{\substack{\uparrow \\ \text{particle} \\ \text{kinetic energy} \\ \text{density}}}{\frac{1}{2}\rho v^2} + \underset{\substack{\uparrow \\ \text{strain energy} \\ \text{density}}}{\frac{1}{2}CS^2}\right) \tag{1.28}$$

Equation (1.28) is the acoustic analogue of (1.25). Comparing (1.28) with (1.25), we define the *acoustic Poynting vector* as

$$\mathbf{P}_{\text{ac}} = -Tv \tag{1.29}$$

In a lossy material $\eta \neq 0$, and (1.13) must be used in place of Hooke's law in (1.27). Equation (1.28) is then modified to

$$-\frac{\partial}{\partial z}(Tv) = -\frac{\partial}{\partial t}\underset{\substack{\uparrow \\ \text{stored} \\ \text{energy} \\ \text{density}}}{\left(\frac{1}{2}\rho v^2 + \frac{1}{2}CS^2\right)} - \underset{\substack{\uparrow \\ \text{dissipated} \\ \text{energy} \\ \text{density}}}{\frac{1}{2}\frac{\partial}{\partial t}\omega\eta S^2} \tag{1.30}$$

For a lossless medium using (1.28) and (1.29):

$$\frac{\rho v^2}{CS^2} = \frac{\rho v^2}{T^2/C} = \rho C\left(\frac{v}{T}\right)^2 = 1$$

As in the electromagnetic case the energy is equally divided between strain (potential) and particle velocity (kinetic) energy densities. In the acoustic case, however, the strain and velocity are 180° out of phase. Thus, the energy is continually transferred between potential and kinetic much the same way as in a mass-spring system. A common definition of Q is

$$Q \equiv \frac{\omega(\text{stored energy})}{\text{dissipated energy per cycle}}$$

Because the maximum strain and kinetic energies are equal, substituting from (1.30) yields

$$Q = \frac{CS^2}{\omega^2 \eta S^2} = \frac{C}{\omega\eta} = \frac{v_a^2 \rho}{\omega\eta} \tag{1.31}$$

which agrees with (1.23). As in the electromagnetic case, we can write the acoustic fields in complex notation:

$$T = T_0 \, \mathrm{e}^{j\omega t}$$

$$v = v_0 \, \mathrm{e}^{j\omega t}$$

The acoustic Poynting vector becomes

$$\mathbf{P}_{\mathrm{ac}} = -\frac{T_0 v_0}{2} \ \mathrm{W/m^2} \tag{1.32}$$

Now recall the definition of wave impedance (1.12):

$$Z = -\frac{T}{v} = \rho v_a$$

Substituting (1.12) into (1.32) gives

$$\mathbf{P}_{\mathrm{ac}} = \frac{Z v_0^2}{2} \tag{1.33}$$

where v_0 is the amplitude of the particle velocity. The particle displacement is given by

$$u = u_0 \, \mathrm{e}^{j\omega t}$$

where u_0 is the particle displacement amplitude (maximum displacement). Because the particle velocity is the derivative of particle displacement, we write

$$v = \frac{\mathrm{d}u}{\mathrm{d}t} = \omega u \rightarrow v_0 = \omega u_0 \tag{1.34}$$

and (1.33) can be written as

$$\mathbf{P}_{ac} = \frac{Z\omega^2 u_0^2}{2} \tag{1.35}$$

Example 1.1. Consider a longitudinal acoustic wave with a power of 20 dBm (0.1 W) propagating in ⟨x⟩ lithium niobate ($v_a = 6.6 \cdot 10^3$ m/s and $\rho = 4.6 \cdot 10^3$ kg/m^3). Find the particle displacement at 1 GHz, 100 MHz, and 10 MHz.

Because (1.32) and (1.35) require the acoustic intensity, we must make reasonable assumptions about the cross-sectional area of the acoustic waves. At 1 GHz the area will be between $1 \cdot 10^{-8}$ and $2 \cdot 10^{-8}$ m^2 (5 mils × 5 mils). The acoustic impedance is

$$Z = (6.6 \cdot 10^3)(4.6 \cdot 10^3) = 3 \cdot 10^7$$

From (1.35), the particle displacement is (if we assume an area of $1.5 \cdot 10^{-8}$ m^2)

$$u_0^2 = \frac{2\mathbf{P}_{ac}}{Z\omega^2} = \frac{2(.1/1.5 \cdot 10^{-8})}{(3 \cdot 10^7)(2\pi \cdot 10^9)^2} = 1.1 \cdot 10^{-20} \text{ m}^2$$

and

$$u_0 = 1 \cdot 10^{-10} \text{ m or } 1 \text{ Å}$$

At 100 MHz the acoustic wave cross-sectional area would typically be about 1 mm × 1 mm, yielding a displacement of 1.3 Å. Even at 10 MHz (with an area of perhaps 2 mm × 2 mm), the displacement is less than 10 Å! For less acoustic power or larger radiating area, the displacements are even smaller. The particle velocity is given by (1.34) and typically ranges from 10 to 100 cm/s; because the displacements are so small, the particle accelerations are on the order of 10^7 m/s^2!

1.6 STRAIN IN THREE DIMENSIONS

In the previous section we dealt with strains of the form $\partial u/\partial z$. Depending on the coordinates of the one-dimensional mass-spring system, there are three such terms: $\partial u_x/\partial x$, $\partial u_y/\partial x$, and $\partial u_z/\partial z$. Now consider Figure 1.5. The definitions are identical to the one-dimensional case. We allow the displacement ($\mathbf{u}$) to be a function of y and z as well as x. This is illustrated in Figure 1.6. We write

$$\begin{aligned} \mathcal{D} &= (\Delta l)^2 - (\Delta L)^2 \\ &= (\mathrm{d}l)^2 - (\mathrm{d}L)^2 \end{aligned} \tag{1.36}$$

(in the limit of infinitesimal displacements). From Figure 1.5, we have immediately,

$$\mathrm{d}l_x = \mathrm{d}L_x + \mathrm{d}u_x \tag{1.37}$$

We allow a deformation ($\mathrm{d}l_x$) in x if there is a displacement in y or z:

$$\mathrm{d}l_x = \mathrm{d}L_x + \frac{\partial u_x}{\partial L_x}\,\mathrm{d}L_x + \frac{\partial u_x}{\partial L_y}\,\mathrm{d}L_y + \frac{\partial u_x}{\partial L_z}\,\mathrm{d}L_z \tag{1.38}$$

and

$$\mathrm{d}l_y = \mathrm{d}L_y + \frac{\partial u_y}{\partial L_x}\,\mathrm{d}L_x + \frac{\partial u_y}{\partial L_y}\,\mathrm{d}L_y + \frac{\partial u_y}{\partial L_z}\,\mathrm{d}L_z \tag{1.39}$$

From (1.36), the deformation $\mathfrak{D}$ is (we consider only two dimensions):

$$\mathfrak{D} = (\mathrm{d}l_x)^2 + (\mathrm{d}l_y)^2 - (\mathrm{d}L_x)^2 - (\mathrm{d}L_y)^2 \tag{1.40}$$

Performing the operations in (1.40), we have, using (1.38):

$$\mathrm{d}l_x^2 = \mathrm{d}L_x^2 + \left(\frac{\partial u_x}{\partial L_x}\,\mathrm{d}L_x + \frac{\partial u_x}{\partial L_y}\,\mathrm{d}L_y\right)^2 + 2\mathrm{d}L_x\left(\frac{\partial u_x}{\partial L_x}\,\mathrm{d}L_x + \frac{\partial u_x}{\partial L_y}\,\mathrm{d}L_y\right) \tag{1.41}$$

and

$$\mathrm{d}l_y^2 = \mathrm{d}L_y^2 + \left(\frac{\partial u_y}{\partial L_x}\,\mathrm{d}L_x + \frac{\partial u_y}{\partial L_y}\,\mathrm{d}L_y\right)^2 + 2\mathrm{d}L_y\left(\frac{\partial u_y}{\partial L_y}\,\mathrm{d}L_y + \frac{\partial u_y}{\partial L_x}\,\mathrm{d}L_x\right) \tag{1.42}$$

From (1.40) and the definition of the deformation, we have

$$\begin{aligned}\mathfrak{D} = {} & \left(\frac{\partial u_x}{\partial L_x}\right)^2 \mathrm{d}L_x^2 + \left(\frac{\partial u_x}{\partial L_x}\right)^2 \mathrm{d}L_y^2 + 2\,\frac{\partial u_x}{\partial L_x}\,\frac{\partial u_x}{\partial L_y}\,\mathrm{d}L_x\mathrm{d}L_y + 2\,\frac{\partial u_x}{\partial L_x}\,\mathrm{d}L_x^2 \\ & + 2\,\frac{\partial u_x}{\partial L_y}\,\mathrm{d}L_x\mathrm{d}L_y + \left(\frac{\partial u_y}{\partial L_x}\right)^2 \mathrm{d}L_x^2 + \left(\frac{\partial u_y}{\partial L_y}\right)^2 \mathrm{d}L_y^2 \\ & + 2\,\frac{\partial u_y}{\partial L_x}\,\frac{\partial u_y}{\partial L_y}\,\mathrm{d}L_x\mathrm{d}L_y + 2\,\frac{\partial u_y}{\partial L_y}\,\mathrm{d}L_y^2 + 2\,\frac{\partial u_y}{\partial L_x}\,\mathrm{d}L_x\mathrm{d}L_y\end{aligned} \tag{1.43}$$

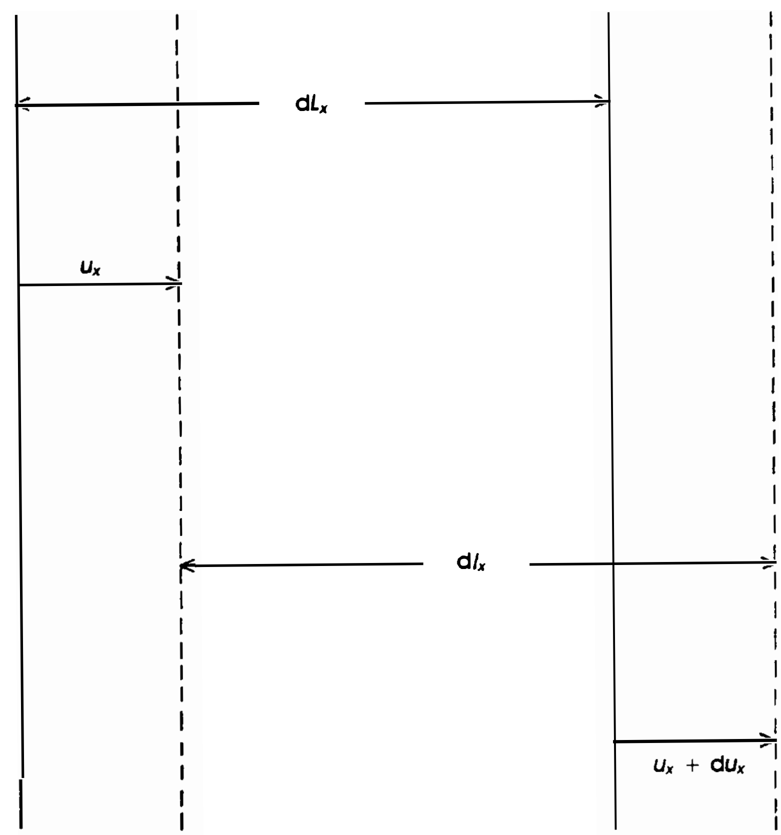

Figure 1.5 Definition of particle displacements. *Note*: In three dimensions, we generalize the displacement u_x so that it depends on y and z coordinates.

We now group the terms in (1.43):

$$\begin{aligned}\mathfrak{D} = &\left[\left(\frac{\partial u_x}{\partial L_x}\right)^2 + 2\frac{\partial u_x}{\partial L_x} + \left(\frac{\partial u_y}{\partial L_x}\right)^2\right] dL_x^2 \quad \text{(term 1)}\\ &+\left[\left(\frac{\partial u_x}{\partial L_y}\right)^2 + 2\frac{\partial u_y}{\partial L_y} + \left(\frac{\partial u_y}{\partial L_y}\right)^2\right] dL_y^2 \quad \text{(term 2)}\\ &+\left(2\frac{\partial u_x}{\partial L_x}\frac{\partial u_x}{\partial L_y} + 2\frac{\partial u_y}{\partial L_x}\frac{\partial u_y}{\partial L_y} + \frac{\partial u_x}{\partial L_y} + \frac{\partial u_y}{\partial L_x}\right) dL_x dL_y \quad \text{(term 3)}\end{aligned} \tag{1.44}$$

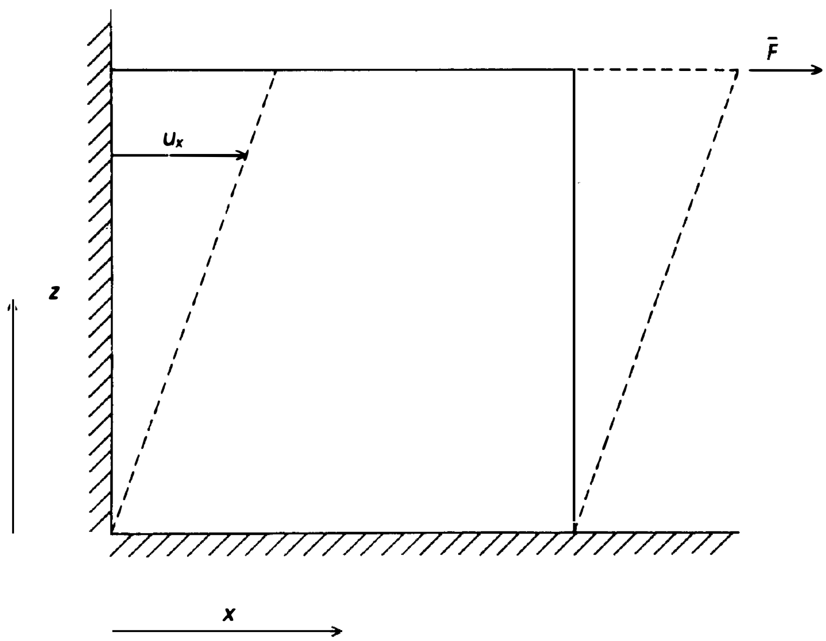

Figure 1.6 Relation of particle displacement u_x with y in simple shear.

Finally, this expression can be put into matrix form:

$$\mathfrak{D} = 2\,[\mathrm{d}L_x \quad \mathrm{d}L_y] \begin{bmatrix} S_{xx} & S_{xy} \\ S_{yx} & S_{yy} \end{bmatrix} \begin{bmatrix} \mathrm{d}L_x \\ \mathrm{d}L_y \end{bmatrix} \tag{1.45}$$

where

$$S_{xx} = \frac{\text{term 1,}}{\mathrm{d}L_x^2} \quad S_{yy} = \frac{\text{term 2,}}{\mathrm{d}L_y^2} \quad S_{xy} = S_{yx} = (\text{term 3})/2(\mathrm{d}L_x \mathrm{d}L_y)$$

We transformed the strain into a 2×2 *symmetric* matrix. The presence of off-diagonal terms is a consequence of (1.38); these terms are associated with shear, as shown in Figure 1.6. As in the one-dimensional case, we linearize the expressions for the strain components by assuming that all square terms are negligible:

$$\left(\frac{\partial u_i}{\partial L_j}\right)^2 \ll 1, \quad \frac{\partial u_i}{\partial L_j}\frac{\partial u_j}{\partial L_i} \ll 1$$

where i and j represent x, y, and z.

The components of the *strain matrix* become

$$S_{xx} = \frac{\partial u_x}{\partial L_x}, \quad S_{yy} = \frac{\partial u_y}{\partial L_y}$$
$$S_{xy} = S_{yx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial L_y} + \frac{\partial u_y}{\partial L_x}\right) \tag{1.46}$$

In a continuum, we let $L_x \to x$, $L_y \to y$, and write

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i}\right) \tag{1.47}$$

In the three-dimensional case, the strain can be written as a 3×3 *symmetric* matrix:

$$\mathbf{S} = \begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{bmatrix} \tag{1.48}$$

where

$$S_{yx} = S_{xy}, \quad S_{xz} = S_{zx}, \quad S_{yz} = S_{zy}$$

Thus, there are only six independent components of the strain matrix. For example,

$$S_{xy} = S_{yx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

If we know the displacement as a function of positions ($\mathbf{u} = f(\mathbf{r})$), we can form the terms $\partial\mathbf{u}/\partial\mathbf{r}$. We define a displacement gradient matrix:

$$\mathbf{E} = \begin{bmatrix} \dfrac{\partial u_x}{\partial x} & \dfrac{\partial u_x}{\partial y} & \dfrac{\partial u_x}{\partial z} \\ \dfrac{\partial u_y}{\partial x} & \dfrac{\partial u_y}{\partial y} & \dfrac{\partial u_y}{\partial z} \\ \dfrac{\partial u_z}{\partial x} & \dfrac{\partial u_z}{\partial y} & \dfrac{\partial u_z}{\partial z} \end{bmatrix} = \nabla\mathbf{u} \tag{1.49}$$

Unlike **S**, **E** is *not* symmetric. This fact severely limits its usefulness. We can, however, break up any matrix into symmetric and antisymmetric components. Formally, this is accomplished by

$$\mathbf{E} = \frac{1}{2}(\mathbf{E} + \tilde{\mathbf{E}}) + \frac{1}{2}(\mathbf{E} - \tilde{\mathbf{E}}) \tag{1.50}$$

where $\tilde{\mathbf{E}}$ is the transpose of $\mathbf{E}$. Using this formalism, we see immediately that $\mathbf{S}$ is the symmetric part of $\mathbf{E}$. The antisymmetric part will not concern us further, but it can be interpreted as a rigid rotation.

1.7 THE STRESS MATRIX

Although the strain is a dimensionless number in one dimension, it must be written as a symmetric matrix in three dimensions. Because Hook's law (1.8) relates stress to strain, it is not unreasonable to assume that stress also is a 3×3 matrix in three dimensions. To confirm this fact, consider Figure 1.7: The stress components T_{xx} and T_{yy} are the familiar *longitudinal* stresses that tend to distort the body by extending or compressing it. The stresses T_{xy} and T_{yx} tend to *rotate* and cut the body and are associated with shear. Because the body must be in both lateral and rotational equilibrium, we have

$$\Sigma M_A = 0$$

or

$$T_{xy} = T_{yx} \tag{1.51}$$

Equation (1.51) expresses the equilibrium condition that the sum of the moments about any axis is zero. This condition is true only for all instants of time in the static case. In the dynamic case, the body will, in general, compress, expand, and rotate, but it will do so in an oscillatory manner such that the *average* motion will be zero, and thus the preceding formulation is still valid. This two-dimensional case is easily extended to three dimensions with the conclusion that the stress can be expressed by a 3×3 matrix with six independent components: T_{xx}, T_{yy}, T_{zz}, T_{xy}, T_{xz}, T_{yz}, and

$$T_{xy} = T_{yx}, \; T_{yz} = T_{zy}, \; T_{xz} = T_{zx}$$

Example 1.2. Consider Figure 1.8. There is an external force on the rod, which is attached at one end. The displacement is measured relative to this point; thus, at $z = 0$, $\mathbf{u}(0) = 0$. At $z = L$, the displacement is

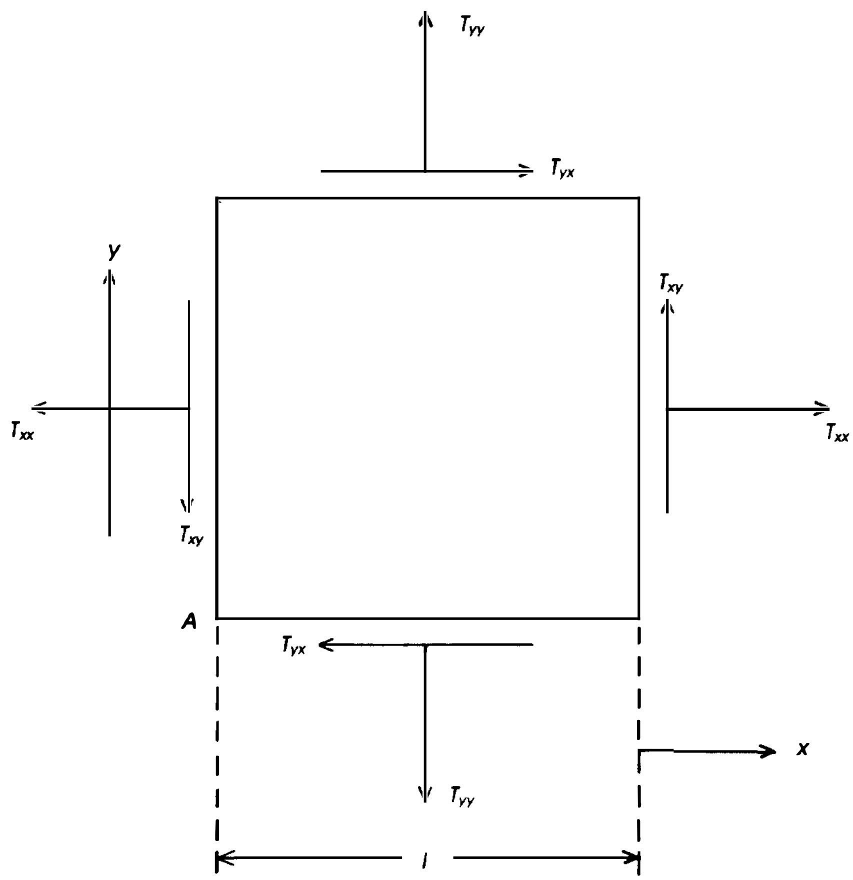

Figure 1.7 Definition of stress components in two dimensions. *Note*: In equilibrium, the stress matrix, like the strain matrix, is symmetric.

maximum, and at any intermediate point we may assume a linear relationship:

$$\mathbf{u}(L) = L' - L$$

and

$$\mathbf{u}(z) = \frac{(L' - L)z}{L} \tag{1.52}$$

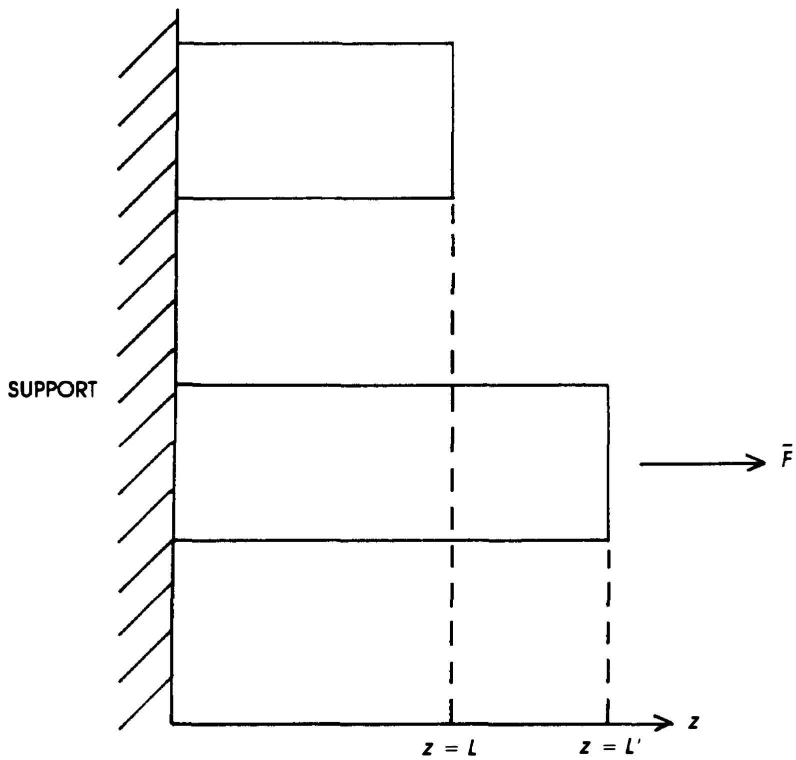

Figure 1.8 Displacement in one dimension.

Because there is tension only in the z direction:

$$\mathbf{u}(z) = \mathbf{u}_z(z)$$

and all other components are zero.

The displacement is linear in z, but the strain is constant:

$$\mathbf{S} = \frac{\mathrm{d}\mathbf{u}_z(z)}{\mathrm{d}z} = \left(\frac{L' - L}{L}\right) \tag{1.53}$$

Note that because the strain is constant, so is the stress (because the ratio of stress to strain is always constant). Thus, if the force is greater than the ability of the rod to hold itself together, there is no preferred point of failure. The strain matrix is

$$\mathbf{S} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (L' - L)/L \end{bmatrix} \tag{1.54}$$

Example 1.3. Simple shear strain (Figure 1.6): In this case, there is a displacement in the x direction ($\mathbf{u}_x$) that is a function of y:

$$\mathbf{u}_x(y) = 2Cy\hat{\mathbf{i}} \tag{1.55}$$

where C is a constant and $\mathbf{i}$ is a unit vector in the x direction. Note that this configuration is associated with a deformation that tends to cut or shear the body. There is no change in the volume of the body; only its shape changes. The diagonal components of the strain matrix (associated with longitudinal strain) are thus zero:

$$S_{xx} = \frac{\partial u_x}{\partial x} = 0 \quad (\text{since } u_x \neq f(x)) \tag{1.56}$$

Two off-diagonal terms are nonzero:

$$S_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = S_{yx} = C \tag{1.57}$$

The strain matrix is

$$\mathbf{S} = \begin{bmatrix} 0 & C & 0 \\ C & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.58}$$

Note that $\mathbf{S}$ is a symmetric matrix with only off-diagonal terms. The displacement gradient matrix (given in (1.49)) is:

$$\mathbf{E} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & 2C & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.59}$$

The transpose of **E** is

$$\tilde{\mathbf{E}} = \begin{bmatrix} 0 & 0 & 0 \\ 2C & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.60}$$

The symmetric part of **E** is the strain matrix (recall (1.50)):

$$\mathbf{S} = \frac{1}{2}(\mathbf{E} + \tilde{\mathbf{E}}) = \frac{1}{2}\left(\begin{bmatrix} 0 & 2C & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 2C & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0 & C & 0 \\ C & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.61}$$

Example 1.4. Pure shear: Consider Figure 1.9. In this case there are displacements $\mathbf{u}_x = f(y)$ and $u_y = f(x)$. The total displacement is

$$\mathbf{u}(x, y) = Cy\,\hat{\mathbf{i}} + Cx\,\hat{\mathbf{j}} \tag{1.62}$$

As in Example 1.2, the longitudinal stresses (diagonal terms of the matrix) are zero. There are two cross terms:

$$S_{xy} = S_{yx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = C \tag{1.63}$$

The stress matrix is identical to the case of pure shear and is equal to the gradient displacement matrix:

$$\mathbf{S} = \begin{bmatrix} 0 & C & 0 \\ C & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{E} = \tilde{\mathbf{E}} \tag{1.64}$$

Example 1.5. Dynamic case or stress wave: Consider Figure 1.10. In this case the external force is an impulse applied at $t = t_1$. The rod is broken up into slabs of thickness dz and cross-sectional area A. At $t = t_1$ the end slab (initially in equilibrium) accelerates under the influence of F_{ext} to the left. At this instant, there is no tension in the slab. As the slab moves to the left, a counterforce is set up by the left neighboring element, which creates a tension in the slab. By Newton's third law, an equal and opposite force must act on the left neighboring slab and start it moving to the left.

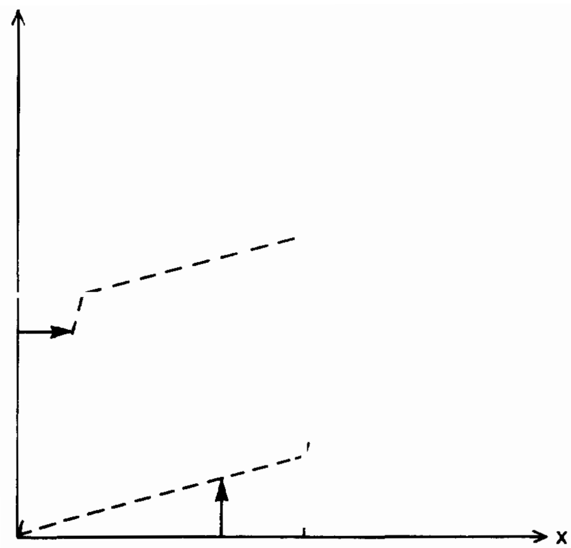

Figure 1.9 Pur(

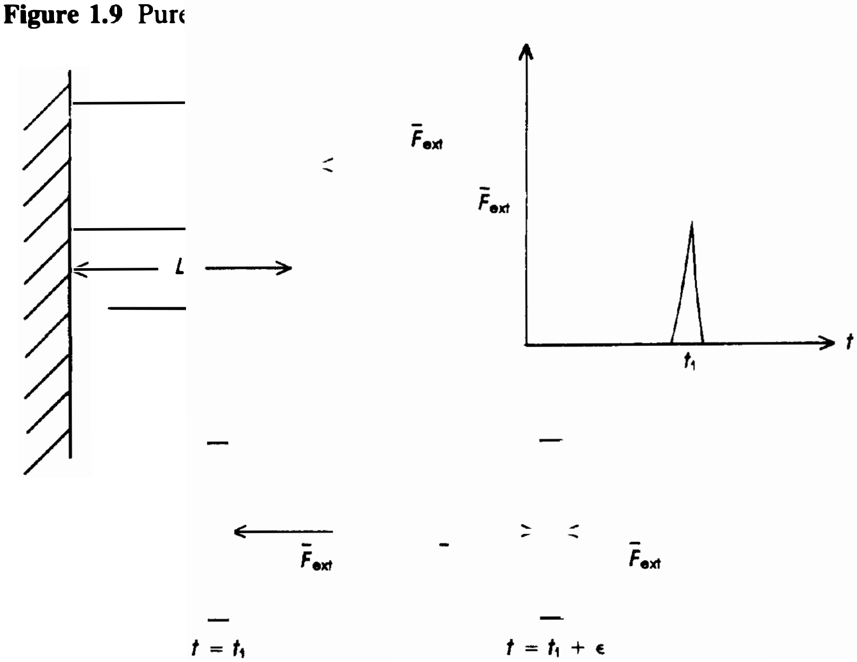

Figure 1.10 Simple model of the impulse response in one dimension. *Note*: At $t = t_1$ an external force in the form of an impulse is applied to the rod, exciting a mechanical wave.

In this way, the impulse is transmitted down the rod. The velocity of transmission is a function of the "stiffness" of the medium. A compliant medium such as rubber requires a longer time to set up internal counter-forces than does a stiff medium such as steel or glass, and thus the velocity of an acoustic wave in rubber is much lower than in glass. Note that the conditions for wave propagation are present in this example:

1. Each slab in the medium will eventually experience a net force (i.e., an imbalance of traction forces).
2. There is a tension in each slab caused by counteracting internal forces, which distort the neighboring slabs.

These two conditions lead to the wave equation (1.10). The phase velocity of the wave is given by (1.11):

$$v_a = \sqrt{\frac{C}{\rho}}$$

which is a function only of the medium.

The stiffness constant C depends not only on the properties of the medium but also on the form of the propagating wave. For example, if the external impulse was directed along the x-axis in Figure 1.10, internal shear forces would be set up in each internal slab and a shear wave would have propagated in the rod. The stress transmission for this configuration would probably be different than that for the longitudinal wave, resulting in a different phase velocity. Also, in an acoustically anisotropic medium (in which the stiffness depends on the directions), the phase velocity of both shear and longitudinal waves depends on the orientation of the medium with respect to the external stimulus.

1.8 TRANSFORMATIONS

We will often have to transform between coordinate systems. In general, there are three types of transformations: (1) linear displacement, (2) change of coordinates (e.g., from Cartesian to polar), and (3) rotation. Only rotational transformation will concern us. Consider Figure 1.11. The vector $\mathbf{v}$ has the components v_x and v_y in the unprimed coordinates and $v_{x'}$, and $v_{y'}$ in the primed coordinates. Our task is to express the primed coordinates of $\mathbf{v}$ in terms of the unprimed coordinates. Because $\mathbf{v}$ is a three-dimensional vector, we have

$$\mathbf{v}' = \begin{bmatrix} v_{x'} \\ v_{y'} \\ v_{z'} \end{bmatrix} = \begin{bmatrix} a_{xx} & a_{xy} & 0 \\ a_{yx} & a_{yy} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \tag{1.65}$$

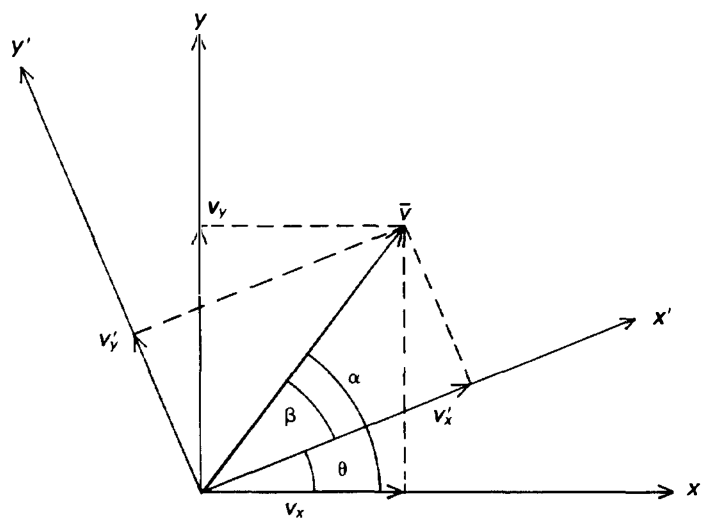

Figure 1.11 Definition of rotated coordinate system. *Note*: The x'-y' system is rotated counterclockwise about the z-axis.

or

$$\mathbf{v}' = \mathbf{A} : \mathbf{v} \tag{1.66}$$

Note that $v_{z'} = v_z$ because the rotation is around the z-axis. From Figure 1.11, we project v_x and v_y onto $v_{x'}$. The sum of the projections is

$$v_x \cos\theta + v_y \sin\theta$$

From Figure 1.12, we have

$$\begin{aligned} v_x \cos\theta + v_y \sin\theta &= \mathbf{v} \cos\alpha[\cos(\alpha - \beta)] + \mathbf{v} \sin\alpha[\sin(\alpha - \beta)] \\ &= \mathbf{v} \cos\alpha(\cos\alpha \cos\beta + \sin\alpha \sin\beta) \\ &\quad + \mathbf{v} \sin\alpha(\sin\alpha \cos\beta - \cos\alpha \sin\beta) \\ &= \mathbf{v} \cos\beta = v_{x'} \end{aligned} \tag{1.67}$$

Thus,

$$v_{x'} = v_x \cos\theta + v_y \sin\theta \tag{1.68}$$

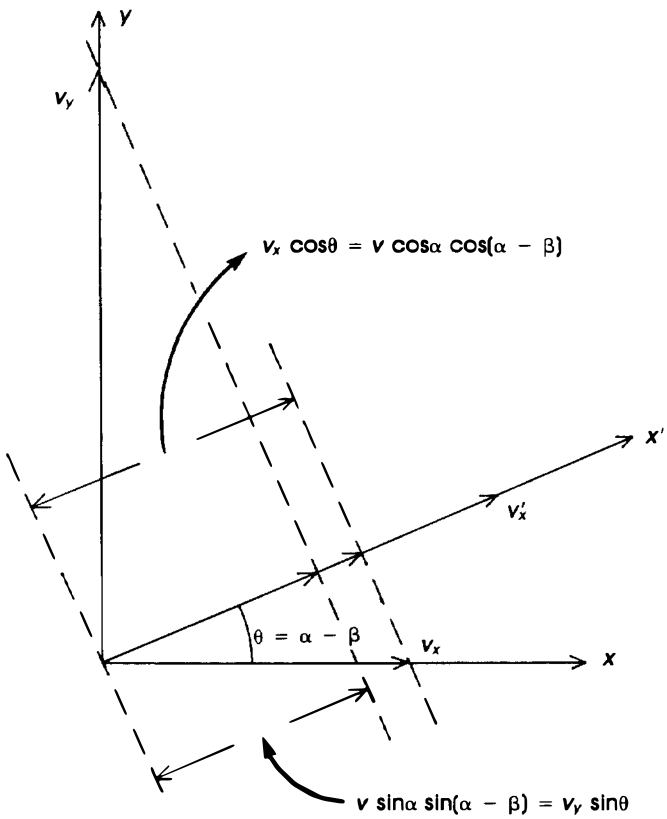

Figure 1.12 Projection of the vector components in the unprimed system onto the primed (rotated) system (in two dimensions).

Similarly, we can show that

$$v_{y'} = -v_y \sin\theta + v_x \cos\theta \tag{1.69}$$

The rotation matrix for a rotation about the z-axis is therefore

$$\mathbf{A} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.70}$$

It is easy to convince ourselves that a rotation about the x-axis would be given by the rotation matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{1.71}$$

and a rotation about the *y*-axis by the rotation matrix:

$$\mathbf{A} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{1.72}$$

Note that the matrices **A** are not symmetric; for all i, $j = x$, y, or z, we have

$$a_{ij} = -a_{ji}, \quad i \neq j \tag{1.73}$$

where a_{ij} is any element of the matrix **A**.

An obvious but essential requirement for any vector rotation is that the vector magnitude remain constant in the two rotated systems. Formally, we require that

$$\underset{\substack{\text{original}\\ \text{system}}}{\tilde{\mathbf{v}} \cdot \mathbf{v}} = [v_x \quad v_y \quad v_z] \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = v_x^2 + v_y^2 + v_z^2 = \underset{\substack{\text{rotated}\\ \text{system}}}{\tilde{\mathbf{v}}' \cdot \mathbf{v}'} \tag{1.74}$$

where $\tilde{\mathbf{v}}$ is the transpose $\mathbf{v}$. From (1.66), we have

$$\tilde{\mathbf{v}}' \cdot \mathbf{v}' = (\widetilde{\mathbf{A}:\mathbf{v}}):\mathbf{A}:\mathbf{v} \tag{1.75}$$

An important matrix identity states that

$$[\widetilde{\mathbf{A}\mathbf{v}}] \equiv \tilde{\mathbf{v}}\tilde{\mathbf{A}} \tag{1.76}$$

(note that the order of multiplication is reversed). Using (1.76) and (1.74), we have

$$\tilde{\mathbf{v}}' \cdot \mathbf{v}' = \tilde{\mathbf{v}}:\tilde{\mathbf{A}}:\mathbf{A}:\mathbf{v} = \tilde{\mathbf{v}} \cdot \mathbf{v} \tag{1.77}$$

where the last equality is true only if

$$\tilde{\mathbf{A}}:\mathbf{A} = \mathbf{I}$$

where **I** is the identity matrix. Finally,

$$\tilde{\mathbf{A}} = \mathbf{A}^{-1} \tag{1.78}$$

where $\mathbf{A}^{-1}$ is the *inverse* of $\mathbf{A}$.

If the transpose of a matrix is equal to its inverse, the matrix is said to be orthogonal. All matrices that we will deal with possess this property. It is easy to show that the rotation matrix is orthogonal. Consider a general rotation about the z-axis:

$$\mathbf{A} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{A}} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Forming the product $\tilde{\mathbf{A}}\mathbf{A}$, we have

$$\begin{aligned} \tilde{\mathbf{A}}{:}\mathbf{A} &= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta & 0 \\ \cos\theta\sin\theta - \sin\theta\cos\theta & \cos^2\theta + \sin^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I} \end{aligned}$$

Example 1.6. Rotation of $\pi/2$ about the z-axis:

$$\mathbf{A} = \begin{bmatrix} \cos\pi/2 & \sin\pi/2 & 0 \\ -\sin\pi/2 & \cos\pi/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can consider this rotation as a transformation (see Figure 1.13) in which $x \to y$, $y \to -x$, and $z \to z$. Another notation which will be quite useful later substitutes 1, 2, and 3 for x, y, and z. The previous transformation becomes $1 \to 2$, $2 \to -1$, and $3 \to 3$.

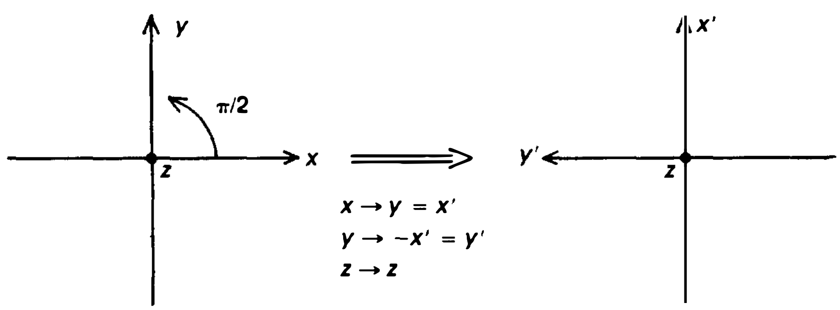

Figure 1.13 Coordinate axes transformation for $\pi/2$ rotation about the z-axis.

Example 1.7. Rotation of $\pi/2$ about the y-axis:

$$\begin{array}{rl} x \to z & (1 \to 3) \\ y \to y & (2 \to 2) \\ z \to -x & (3 \to -1) \end{array}$$

The rotation matrix is then

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Example 1.8. Rotation of $\pi/3$ around y:

$$\mathbf{A} = \begin{bmatrix} \cos\pi/3 & 0 & \sin\pi/3 \\ 0 & 1 & 0 \\ -\sin\pi/3 & 0 & \cos\pi/3 \end{bmatrix} = \begin{bmatrix} .5 & 0 & .866 \\ 0 & 1 & 0 \\ -.866 & 0 & .5 \end{bmatrix}$$

Example 1.9. Double rotation: Consider the following series of rotations:

1. $\pi/2$ about the z-axis ($x \to y$, $y \to -x$, $z \to z$) followed by
2. $\pi/2$ about the y-axis: ($y \to y$, $x \to z$, $z \to -x$); the combination of rotations can be expressed as $x \to y$, $y \to -z$, $z \to -x$. Thus, the rotation matrix is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

The final transformation does not conform to (1.73), indicating that it is not a pure rotation. The two rotation matrices are

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Multiplying the two matrices together yields

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \text{ which agrees with the transformation matrix.}$$

Note that the *order* of the matrices is extremely important:

$$\mathbf{A} = \mathbf{A}_1{:}\mathbf{A}_2$$

Reversing the order results in

$$\mathbf{A}' = \mathbf{A}_2{:}\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

Matrix $\mathbf{A}'$ represents the transformation ($x \rightarrow z$, $y \rightarrow -x$, $z \rightarrow -y$), or a $\pi/2$ rotation about y followed by a $\pi/2$ rotation about z.

The transformation rule for vectors is given by (1.66). Now we consider the transformation rule for 3×3 matrices. Examples of these matrices are the gradient displacement matrix $\mathbf{E}$ and the stress and strain matrices $\mathbf{T}$ and $\mathbf{S}$. Consider the gradient displacement matrix:

$$\mathbf{E} = \frac{d\mathbf{u}}{d\mathbf{r}}$$

where $\mathbf{u}$ and $\mathbf{r}$ are vectors. In a rotated coordinate system, we have

$$d\mathbf{u}' = \mathbf{A}\,d\mathbf{u} = \mathbf{A}\mathbf{E}\,d\mathbf{r} \tag{1.79}$$

but

$$d\mathbf{r}' = \mathbf{A}\,d\mathbf{r} \quad \text{or} \quad d\mathbf{r} = \mathbf{A}^{-1}\,d\mathbf{r}' \tag{1.80}$$

Substituting (1.80) into (1.79), we have

$$d\mathbf{u}' = \mathbf{A}\mathbf{E}\mathbf{A}^{-1}\,d\mathbf{r}'$$

or

$$\frac{d\mathbf{u}'}{d\mathbf{r}'} = \mathbf{E}' = \mathbf{A}{:}\mathbf{E}{:}\mathbf{A}^{-1} \tag{1.81}$$

Example 1.10. As an example of the transformation of a 3×3 matrix, consider the pure shear strain. Recall that

$$\mathbf{S} = \begin{bmatrix} 0 & C & 0 \\ C & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let us apply a rotation of $\pi/4$ to this matrix and use the transformation rule given by (1.81). The transformed strain is

$$\mathbf{S}' = \begin{array}{ccccc} \mathbf{A} & : & \mathbf{S} & : & \tilde{\mathbf{A}} \\ \downarrow & & \downarrow & & \downarrow \end{array}$$

$$\mathbf{S}' \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & C & 0 \\ C & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Carrying out the matrix multiplication, we arrive at

$$\mathbf{S}' = \begin{bmatrix} C & 0 & 0 \\ 0 & -C & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The resulting strain matrix thus possesses only diagonal (longitudinal) terms. This has a simple and elegant physical interpretation. If we pull (stretch) along the x'-axis and push (compress) along the y'-axis, we can produce the pure shear where the $x' - y'$ system is rotated at an angle of 45° from the x-y system.

1.9 CONTRACTED NOTATION AND THE DYNAMICAL EQUATIONS IN THREE DIMENSIONS

The strain **S** and stress **T** have the properties of matrices (they transform according to (1.81), but they possess only six independent elements. It is convenient to write them as 6×1 column vectors (but we must remember that they are not vectors!):

$$\mathbf{S} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} \quad \text{and} \quad \mathbf{T} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} \tag{1.82}$$

Comparing with our previous formulations for **S** and **T**, we require that

$$\begin{aligned} S_1 &= S_{xx} \qquad & S_4 &= 2S_{yz} = 2S_{zy} \\ S_2 &= S_{yy} \qquad & S_5 &= 2S_{xz} = 2S_{zx} \\ S_3 &= S_{zz} \qquad & S_6 &= 2S_{xy} = 2S_{yx} \end{aligned} \tag{1.83}$$

and

$$\begin{aligned} T_1 &= T_{xx} \qquad & T_4 &= T_{yz} = T_{zy} \\ T_2 &= T_{yy} \qquad & T_5 &= T_{xz} = T_{zx} \\ T_3 &= T_{zz} \qquad & T_6 &= T_{xy} = T_{yx} \end{aligned} \tag{1.84}$$

Notice that the first three equations establish conditions for the *longitudinal* components of the stress and strain, and the last three equations connect the *shear* components. Also notice the factor of 2 in the shear strain components, which has its origin in the definition of shear strain (1.47) and is therefore not present in the stress. Finally, it is important to realize that these conditions are a matter of definitions and thus cannot be derived from basic principles. They are the result of convention and convenience, but we will refer to them often, so they should be readily accessible.

Now recall that

$$S_1 = S_{xx} = \frac{\partial u_x}{\partial x}, \quad S_{yy} = \frac{\partial u_y}{\partial_y}, \quad S_5 = 2S_{xz} = \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right), \ldots$$

The six equations for the six independent strain components can be written as the product of a *differential operator* matrix on a vector:

$$\mathbf{S} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \tag{1.85}$$

This matrix equation contains the six strain equations. For example,

$$S_4 = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}, \quad S_6 = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \ldots$$

Alternatively, we say that the gradient operator "transforms the displacement *vector* **u** into the strain *matrix* S. Equation (1.85) may be written in a shorthand notation as

$$\mathbf{S} = \nabla_s \mathbf{u} \tag{1.86}$$

where

$$\nabla_s \rightarrow \nabla_{Ij} \quad I = 1 \text{ to } 6 \quad \text{(six rows)}$$
$$j = 1 \text{ to } 3 \quad \text{(three columns)}$$

Equation (1.86) is the three-dimensional analogue of (1.4).

Recall that in one dimension Newton's law was written as (1.6):

$$\frac{\partial \mathbf{T}}{\partial z} = \rho \frac{\partial \mathbf{v}}{\partial t}$$

In three dimensions, the right side is still a vector (the derivative of a vector with respect to a scalar is a vector), so the left side must also be a vector. Now we know that **T** is a matrix; thus we need a gradient operation (because of the spatial derivative in (1.6)) that transforms a matrix into a vector (compare to (1.86), where the operator transforms a vector into a matrix). The required operator is $\nabla \cdot \mathbf{T}$.

As in vector analysis where the divergence of a vector results in a scalar, here the "divergence" of a matrix produces a vector. First we recall the divergence of a vector:

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \quad \text{(a scalar)} \tag{1.87}$$

The matrix analogue of (1.87) is

$$\begin{aligned}\nabla \cdot \mathbf{T} = &\left(\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z}\right)\hat{\mathbf{i}} \\ &+ \left(\frac{\partial T_{yx}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{yz}}{\partial z}\right)\hat{\mathbf{j}} \\ &+ \left(\frac{\partial T_{zx}}{\partial x} + \frac{\partial T_{zy}}{\partial y} + \frac{\partial T_{zz}}{\partial z}\right)\hat{\mathbf{k}}\end{aligned} \tag{1.88}$$

The operation transformed **T** into a vector in which each component corresponds to a "divergence" on a different row. For example, the x component is the analogue of the divergence on the first row of **T**. This can be written in a more elegant form as

$$\nabla \cdot \mathbf{T} = \nabla \cdot \begin{array}{cl} \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix} & \begin{array}{l} \leftarrow i \text{ component} \\ \leftarrow j \text{ component} \\ \leftarrow k \text{ component} \end{array} \end{array} \tag{1.89}$$

In summary, just as a gradient transforms a scalar into a vector, and a divergence transforms a vector into a scalar, the gradient matrix operator transforms a vector into a matrix, and the divergence matrix operator transforms a matrix into a vector.

Now we substitute the definitions of the six independent stress components into (1.89):

$$\nabla \cdot \mathbf{T} = \begin{bmatrix} \frac{\partial}{\partial x} T_1 + \frac{\partial}{\partial y} T_6 + \frac{\partial}{\partial z} T_5 \\ \frac{\partial}{\partial x} T_6 + \frac{\partial}{\partial y} T_2 + \frac{\partial}{\partial z} T_4 \\ \frac{\partial}{\partial x} T_5 + \frac{\partial}{\partial y} T_4 + \frac{\partial}{\partial z} T_3 \end{bmatrix} \begin{array}{l} \leftarrow x \text{ component} \\ \leftarrow y \text{ component} \\ \leftarrow z \text{ component} \end{array} \tag{1.90}$$

Using (1.90), we can write (1.6) in matrix form:

$$\nabla \cdot \mathbf{T} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \rho \frac{\partial \mathbf{v}}{\partial t} \tag{1.91}$$

Equation (1.91) is a set of six equations corresponding to (1.6). For example, consider the y (or $\hat{\mathbf{j}}$) component:

$$\left(\frac{\partial T_2}{\partial y} + \frac{\partial T_4}{\partial z} + \frac{\partial T_6}{\partial x}\right)\hat{\mathbf{j}} = \Bigg(\underbrace{\frac{\partial T_{yy}}{\partial y}}_{\substack{\uparrow \\ \text{long.} \\ \text{force}}} + \underbrace{\frac{\partial T_{yz}}{\partial z} + \frac{\partial T_{xy}}{\partial x}}_{\substack{\uparrow \\ \text{shear force}}}\Bigg) = \rho \frac{\partial v_y}{\partial t} \tag{1.92}$$

The longitudinal term is associated with a longitudinal acoustic wave and the shear term is associated with a shear acoustic wave. The divergence matrix operator is 3×6 matrix:

$$\nabla \cdot \rightarrow \nabla_{iJ} \quad \text{where } i = 1 \text{ to } 3,\ J = 1 \text{ to } 6$$

Compare this to the gradient matrix operator (in (1.86)), which is a 6×3 matrix. The divergence operator matrix is the transpose of the gradient matrix operator matrix.

The dynamical equations of motion for the one- and three-dimensional cases are shown together with Maxwell's equations in Table 1.3.

Table 1.3

Acoustic Equations		*Electromagnetic Equations*
One-dimensional	*Three-Dimensional*	*Maxwell Equations*
	Newton's law	Faraday's law
$\frac{\partial T}{\partial z} = \rho \frac{\partial v}{\partial t}$	$\nabla \cdot \mathbf{T} = \rho \frac{\partial \mathbf{v}}{\partial t}$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
	Definition of stress	Ampere's law
$S = \frac{\partial u}{\partial z}$	$\mathbf{S} = \nabla_s \mathbf{u}$	$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$
	CONSTITUTIVE RELATIONS	
Hooke's law		
$T = CS$	$\mathbf{T} = \mathbf{c}:\mathbf{S}$	$\mathbf{D} = \epsilon \mathbf{E}$
Definition of particle velocity		
$v = \frac{\partial u}{\partial t}$	$\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t}$	$\mathbf{B} = \mu \mathbf{H}$
Wave impedance		
	$Z = \sqrt{\rho C}$	$Z_e = \sqrt{\frac{\mu}{\epsilon}}$
Phase velocity		
	$v_a = \sqrt{\frac{C}{\rho}}$	$v_p = \sqrt{\frac{1}{\mu\epsilon}}$

In Chapter 2 we examine the innocent-looking constitutive relation $\mathbf{T} = \mathbf{c}:\mathbf{S}$ in more detail.

PROBLEMS

1.1 Show that the particle displacement and particle velocity satisfy the one-dimensional wave equation.

1.2 Show that an arbitrary 3×3 matrix can be written as the sum of symmetric and antisymmetric matrices.

1.3 Show that the combination of a $\pi/2$ rotation about the y-axis followed by a $\pi/2$ rotation about the z-axis can be represented by the matrix:

$$\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

1.4 Write the three components of Newton's law:

$$\nabla \cdot \mathbf{T} = \rho \frac{\partial \mathbf{v}}{\partial t}$$

Identify the stress components.

1.5 Complete the derivation of (1.30).

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