CHAPTER 6

INTERMEDIATE APPROXIMATIONS

6.01 Introductory

The equations of the first-order approximation take account of four types of motion of an infinite plate. In order of decreasing frequency they are: thickness-stretch, thickness-shear, face-extension and thickness-flexure. In an isotropic plate, thickness-shear is coupled with thickness-flexure and thickness-stretch is coupled with face-extension. In a crystal plate, all four motions are coupled. There are many applications in which it is not necessary to take the full coupling into account. For example, if the terms accommodating the thickness-stretch mode are dropped, the remaining equations are useful up to frequencies somewhat above that of the thickness-shear mode. Further simplification, by neglect of the rotatory inertia, leads to equations which accommodate face-extension and moderately high-frequency thickness-flexure. Finally, suppression of the thickness-shear deformation leaves equations suitable for low-frequency thickness-flexure. At any stage of the process of reduction, the coupling between extension and flexure, in a crystal, may be eliminated with resulting simplification of the equations and with, nevertheless, some remaining useful ranges of application. Also, at any stage, another expansion may be made in a series of powers of a face-coordinate. This restricts the useful range to modes having a limited number of phase reversals along that coordinate in addition to the limitation with respect to the thickness-coordinate.

6.02 Thickness-Shear, Thickness-Flexure and Face-Extension

The thickness-stretch vibrations may be eliminated from the first-order approximation by setting

$$\ddot{u}_2^{(1)} = F_2^{(1)} = T_6^{(1)} = T_4^{(1)} = T_2^{(0)} = 0 \tag{6.021}$$

These conditions, rather than the condition $u_2^{(1)} = 0$, are adopted in order to permit free development of the strains illustrated in Fig. 4.011. As a result of (6.021) the fifth of the six equations of motion (5.0117) disappears. The extensional motions that are left are those of the zero-order approximation while the thickness-shear and flexure remain at the first-order approximation.

With $T_2^{(0)} = 0$, the zero-order stress-strain relations are obtained in the usual manner by using the second of (5.041) to eliminate the strain $S_2^{(0)}$. In the case of the first-order stress-strain relations, three components of stress ($T_2^{(1)}, T_4^{(1)}, T_6^{(1)}$) are zero. When so many components of stress are absent, simpler expressions for stress in terms of strain are obtained by employing elastic compliances s_{pq} instead of stiffnesses c_{pq}. In the present case we have

$$\begin{aligned}
\tfrac{2}{3}b^3 S_1^{(1)} &= s_{11}T_1^{(1)} + s_{13}T_3^{(1)} + s_{15}T_5^{(1)} \\
\tfrac{2}{3}b^3 S_3^{(1)} &= s_{31}T_1^{(1)} + s_{33}T_3^{(1)} + s_{35}T_5^{(1)} \\
\tfrac{2}{3}b\, S_5^{(1)} &= s_{51}T_1^{(1)} + s_{53}T_3^{(1)} + s_{55}T_5^{(1)}
\end{aligned} \tag{6.022}$$

When these are solved for the components of stress, there result

$$\begin{aligned}
T_1^{(1)} &= \tfrac{2}{3}b^3(\gamma_{11}S_1^{(1)} + \gamma_{13}S_3^{(1)} + \gamma_{15}S_5^{(1)}) \\
T_3^{(1)} &= \tfrac{2}{3}b^3(\gamma_{31}S_1^{(1)} + \gamma_{33}S_3^{(1)} + \gamma_{35}S_5^{(1)}) \\
T_5^{(1)} &= \tfrac{2}{3}b^3(\gamma_{51}S_1^{(1)} + \gamma_{53}S_3^{(1)} + \gamma_{55}S_5^{(1)})
\end{aligned} \tag{6.023}$$

where

$$
\begin{aligned}
\gamma_{11} &= (s_{33}s_{55} - s_{35}^2)/\Delta & \gamma_{35} &= \gamma_{53} = (s_{13}s_{15} - s_{11}s_{35})/\Delta \\
\gamma_{33} &= (s_{55}s_{11} - s_{51}^2)/\Delta & \gamma_{51} &= \gamma_{15} = (s_{35}s_{31} - s_{33}s_{51})/\Delta \qquad (6.024) \\
\gamma_{55} &= (s_{11}s_{33} - s_{13}^2)/\Delta & \gamma_{13} &= \gamma_{31} = (s_{51}s_{53} - s_{55}s_{13})/\Delta
\end{aligned}
$$

$$\Delta = s_{11}s_{33}s_{55} + 2s_{35}s_{51}s_{13} - s_{11}s_{35}^2 - s_{33}s_{51}^2 - s_{55}s_{13}^2$$

Hence, for the triclinic crystal, the stress-displacement relations (5.051) reduce to

$$
\begin{aligned}
T_1^{(0)} &= 2b\left[\bar{c}_{11}\frac{\partial u_1^{(0)}}{\partial x_1} + \bar{c}_{13}\frac{\partial u_3^{(0)}}{\partial x_3} + \bar{c}_{14}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + \bar{c}_{15}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + \bar{c}_{16}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right] \\
T_3^{(0)} &= 2b\left[\bar{c}_{31}\frac{\partial u_1^{(0)}}{\partial x_1} + \bar{c}_{33}\frac{\partial u_3^{(0)}}{\partial x_3} + \bar{c}_{34}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + \bar{c}_{35}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + \bar{c}_{36}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right] \\
T_4^{(0)} &= 2b\left[\bar{c}_{41}\frac{\partial u_1^{(0)}}{\partial x_1} + \bar{c}_{43}\frac{\partial u_3^{(0)}}{\partial x_3} + \bar{c}_{44}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + \bar{c}_{45}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + \bar{c}_{46}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right] \\
T_5^{(0)} &= 2b\left[\bar{c}_{51}\frac{\partial u_1^{(0)}}{\partial x_1} + \bar{c}_{53}\frac{\partial u_3^{(0)}}{\partial x_3} + \bar{c}_{54}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + \bar{c}_{55}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + \bar{c}_{56}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right] \\
T_6^{(0)} &= 2b\left[\bar{c}_{61}\frac{\partial u_1^{(0)}}{\partial x_1} + \bar{c}_{63}\frac{\partial u_3^{(0)}}{\partial x_3} + \bar{c}_{64}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + \bar{c}_{65}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + \bar{c}_{66}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right]
\end{aligned}
\qquad (6.025)
$$

$$
\begin{aligned}
T_1^{(1)} &= \frac{2b^3}{3}\left[\gamma_{11}\frac{\partial u_1^{(1)}}{\partial x_1} + \gamma_{13}\frac{\partial u_3^{(1)}}{\partial x_3} + \gamma_{15}\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right)\right] \\
T_3^{(1)} &= \frac{2b^3}{3}\left[\gamma_{31}\frac{\partial u_1^{(1)}}{\partial x_1} + \gamma_{33}\frac{\partial u_3^{(1)}}{\partial x_3} + \gamma_{35}\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right)\right] \\
T_5^{(1)} &= \frac{2b^3}{3}\left[\gamma_{51}\frac{\partial u_1^{(1)}}{\partial x_1} + \gamma_{53}\frac{\partial u_3^{(1)}}{\partial x_3} + \gamma_{55}\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right)\right]
\end{aligned}
$$

The displacement-equations of motion (5.061) reduce to

$$
\begin{aligned}
&\frac{\partial}{\partial x_1}\left[\bar{c}_{11}\frac{\partial u_1^{(0)}}{\partial x_1}+\bar{c}_{13}\frac{\partial u_3^{(0)}}{\partial x_3}+\bar{c}_{14}\left(\frac{\partial u_2^{(0)}}{\partial x_3}+u_3^{(1)}\right)+\bar{c}_{15}\left(\frac{\partial u_3^{(0)}}{\partial x_1}+\frac{\partial u_1^{(0)}}{\partial x_3}\right)+\bar{c}_{16}\left(\frac{\partial u_2^{(0)}}{\partial x_1}+u_1^{(1)}\right)\right]\\
&\quad+\frac{\partial}{\partial x_3}\left[\bar{c}_{51}\frac{\partial u_1^{(0)}}{\partial x_1}+\bar{c}_{53}\frac{\partial u_3^{(0)}}{\partial x_3}+\bar{c}_{54}\left(\frac{\partial u_2^{(0)}}{\partial x_3}+u_3^{(1)}\right)+\bar{c}_{55}\left(\frac{\partial u_3^{(0)}}{\partial x_1}+\frac{\partial u_1^{(0)}}{\partial x_3}\right)+\bar{c}_{56}\left(\frac{\partial u_2^{(0)}}{\partial x_1}+u_1^{(1)}\right)\right]+\frac{F_1^{(0)}}{2b}=\rho\ddot{u}_1^{(0)}\\[2ex]
&\frac{\partial}{\partial x_1}\left[\bar{c}_{61}\frac{\partial u_1^{(0)}}{\partial x_1}+\bar{c}_{63}\frac{\partial u_3^{(0)}}{\partial x_3}+\bar{c}_{64}\left(\frac{\partial u_2^{(0)}}{\partial x_3}+u_3^{(1)}\right)+\bar{c}_{65}\left(\frac{\partial u_3^{(0)}}{\partial x_1}+\frac{\partial u_1^{(0)}}{\partial x_3}\right)+\bar{c}_{66}\left(\frac{\partial u_2^{(0)}}{\partial x_1}+u_1^{(1)}\right)\right]\\
&\quad+\frac{\partial}{\partial x_3}\left[\bar{c}_{41}\frac{\partial u_1^{(0)}}{\partial x_1}+\bar{c}_{43}\frac{\partial u_3^{(0)}}{\partial x_3}+\bar{c}_{44}\left(\frac{\partial u_2^{(0)}}{\partial x_3}+u_3^{(1)}\right)+\bar{c}_{45}\left(\frac{\partial u_3^{(0)}}{\partial x_1}+\frac{\partial u_1^{(0)}}{\partial x_3}\right)+\bar{c}_{46}\left(\frac{\partial u_2^{(0)}}{\partial x_1}+u_1^{(1)}\right)\right]+\frac{F_2^{(0)}}{2b}=\rho\ddot{u}_2^{(0)}\\[2ex]
&\frac{\partial}{\partial x_1}\left[\bar{c}_{51}\frac{\partial u_1^{(0)}}{\partial x_1}+\bar{c}_{53}\frac{\partial u_3^{(0)}}{\partial x_3}+\bar{c}_{54}\left(\frac{\partial u_2^{(0)}}{\partial x_3}+u_3^{(1)}\right)+\bar{c}_{55}\left(\frac{\partial u_3^{(0)}}{\partial x_1}+\frac{\partial u_1^{(0)}}{\partial x_3}\right)+\bar{c}_{56}\left(\frac{\partial u_2^{(0)}}{\partial x_1}+u_1^{(1)}\right)\right]\\
&\quad+\frac{\partial}{\partial x_3}\left[\bar{c}_{31}\frac{\partial u_1^{(0)}}{\partial x_1}+\bar{c}_{33}\frac{\partial u_3^{(0)}}{\partial x_3}+\bar{c}_{34}\left(\frac{\partial u_2^{(0)}}{\partial x_3}+u_3^{(1)}\right)+\bar{c}_{35}\left(\frac{\partial u_3^{(0)}}{\partial x_1}+\frac{\partial u_1^{(0)}}{\partial x_3}\right)+\bar{c}_{36}\left(\frac{\partial u_2^{(0)}}{\partial x_1}+u_1^{(1)}\right)\right]+\frac{F_3^{(0)}}{2b}=\rho\ddot{u}_3^{(0)}\\[2ex]
&\frac{\partial}{\partial x_1}\left[\gamma_{11}\frac{\partial u_1^{(1)}}{\partial x_1}+\gamma_{13}\frac{\partial u_3^{(1)}}{\partial x_3}+\gamma_{15}\left(\frac{\partial u_3^{(1)}}{\partial x_1}+\frac{\partial u_1^{(1)}}{\partial x_3}\right)\right]+\frac{\partial}{\partial x_3}\left[\gamma_{51}\frac{\partial u_1^{(1)}}{\partial x_1}+\gamma_{53}\frac{\partial u_3^{(1)}}{\partial x_3}+\gamma_{55}\left(\frac{\partial u_3^{(1)}}{\partial x_1}+\frac{\partial u_1^{(1)}}{\partial x_3}\right)\right]\\
&\quad-\frac{3}{b^2}\left[\bar{c}_{61}\frac{\partial u_1^{(0)}}{\partial x_1}+\bar{c}_{63}\frac{\partial u_3^{(0)}}{\partial x_3}+\bar{c}_{64}\left(\frac{\partial u_2^{(0)}}{\partial x_3}+u_3^{(1)}\right)+\bar{c}_{65}\left(\frac{\partial u_3^{(0)}}{\partial x_1}+\frac{\partial u_1^{(0)}}{\partial x_3}\right)+\bar{c}_{66}\left(\frac{\partial u_2^{(0)}}{\partial x_1}+u_1^{(1)}\right)\right]+\frac{3F_1^{(1)}}{2b^3}=\rho_1\ddot{u}_1^{(1)}\\[2ex]
&\frac{\partial}{\partial x_1}\left[\gamma_{51}\frac{\partial u_1^{(1)}}{\partial x_1}+\gamma_{53}\frac{\partial u_3^{(1)}}{\partial x_3}+\gamma_{55}\left(\frac{\partial u_3^{(1)}}{\partial x_1}+\frac{\partial u_1^{(1)}}{\partial x_3}\right)\right]+\frac{\partial}{\partial x_3}\left[\gamma_{31}\frac{\partial u_1^{(1)}}{\partial x_1}+\gamma_{33}\frac{\partial u_3^{(1)}}{\partial x_3}+\gamma_{35}\left(\frac{\partial u_3^{(1)}}{\partial x_1}+\frac{\partial u_1^{(1)}}{\partial x_3}\right)\right]\\
&\quad-\frac{3}{b^2}\left[\bar{c}_{41}\frac{\partial u_1^{(0)}}{\partial x_1}+\bar{c}_{43}\frac{\partial u_3^{(0)}}{\partial x_3}+\bar{c}_{44}\left(\frac{\partial u_2^{(0)}}{\partial x_3}+u_3^{(1)}\right)+\bar{c}_{45}\left(\frac{\partial u_3^{(0)}}{\partial x_1}+\frac{\partial u_1^{(0)}}{\partial x_3}\right)+\bar{c}_{46}\left(\frac{\partial u_2^{(0)}}{\partial x_1}+u_1^{(1)}\right)\right]+\frac{3F_3^{(1)}}{2b^3}=\rho_1\ddot{u}_3^{(1)}
\end{aligned}
\qquad (6.026)
$$

The conditions for unique solutions remain the same as those in Section 5.03, except that the following are omitted:

a. Initial conditions on $u_2^{(1)}$ and $\dot{u}_2^{(1)}$.

b. Conditions on $F_2^{(1)}$ or $u_2^{(1)}$ throughout the plate.

c. Conditions on $T_{n2}^{(1)}$ or $u_2^{(1)}$ on the edge.

The stress-displacement relations for the monoclinic plate reduce to

$$T_1^{(0)} = 2b\left[\bar{c}_{11}\frac{\partial u_1^{(0)}}{\partial x_1} + \bar{c}_{13}\frac{\partial u_3^{(0)}}{\partial x_3} + \bar{c}_{14}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right)\right]$$

$$T_3^{(0)} = 2b\left[\bar{c}_{13}\frac{\partial u_1^{(0)}}{\partial x_1} + \bar{c}_{33}\frac{\partial u_3^{(0)}}{\partial x_3} + \bar{c}_{34}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right)\right]$$

$$T_4^{(0)} = 2b\left[\bar{c}_{14}\frac{\partial u_1^{(0)}}{\partial x_1} + \bar{c}_{34}\frac{\partial u_3^{(0)}}{\partial x_3} + \bar{c}_{44}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right)\right]$$

$$T_5^{(0)} = 2b\left[c_{55}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + c_{56}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right]$$

$$T_6^{(0)} = 2b\left[c_{56}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + c_{66}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right]$$

$$T_1^{(1)} = \frac{2b^3}{3}\left[\gamma_{11}\frac{\partial u_1^{(1)}}{\partial x_1} + \gamma_{13}\frac{\partial u_3^{(1)}}{\partial x_3}\right] \qquad (6.027)$$

$$T_3^{(1)} = \frac{2b^3}{3}\left[\gamma_{13}\frac{\partial u_1^{(1)}}{\partial x_1} + \gamma_{33}\frac{\partial u_3^{(1)}}{\partial x_3}\right]$$

$$T_5^{(1)} = \frac{2b^3}{3}\gamma_{55}\left[\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right]$$

where

$$\gamma_{11} = \frac{s_{33}}{s_{11}s_{33} - s_{13}^2} = \bar{c}_{11} - \frac{\bar{c}_{14}^2}{\bar{c}_{44}} \qquad \gamma_{33} = \frac{s_{11}}{s_{11}s_{33} - s_{13}^2} = \bar{c}_{33} - \frac{\bar{c}_{34}^2}{\bar{c}_{44}}$$

$$\gamma_{13} = -\frac{s_{13}}{s_{11}s_{33} - s_{13}^2} = \bar{c}_{13} - \frac{\bar{c}_{14}\bar{c}_{43}}{\bar{c}_{44}} \qquad \gamma_{55} = \frac{1}{s_{55}} = c_{55} - \frac{c_{56}^2}{c_{66}} \qquad (6.028)$$

and the equations of motion are

$$
\begin{aligned}
&\frac{\partial}{\partial x_1}\left[\bar{c}_{11}\frac{\partial u_1^{(0)}}{\partial x_1} + \bar{c}_{13}\frac{\partial u_3^{(0)}}{\partial x_3} + \bar{c}_{14}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right)\right] \\
&\qquad + \frac{\partial}{\partial x_3}\left[c_{55}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + c_{56}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right] + \frac{F_1^{(0)}}{2b} = \rho \ddot{u}_1^{(0)} \\[1em]
&\frac{\partial}{\partial x_1}\left[c_{56}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + c_{66}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right] \\
&\qquad + \frac{\partial}{\partial x_3}\left[\bar{c}_{14}\frac{\partial u_1^{(0)}}{\partial x_1} + \bar{c}_{34}\frac{\partial u_3^{(0)}}{\partial x_3} + \bar{c}_{44}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right)\right] + \frac{F_2^{(0)}}{2b} = \rho \ddot{u}_2^{(0)} \\[1em]
&\frac{\partial}{\partial x_1}\left[c_{55}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + c_{56}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right] \\
&\qquad + \frac{\partial}{\partial x_3}\left[\bar{c}_{13}\frac{\partial u_1^{(0)}}{\partial x_1} + \bar{c}_{33}\frac{\partial u_3^{(0)}}{\partial x_3} + \bar{c}_{34}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right)\right] + \frac{F_3^{(0)}}{2b} = \rho \ddot{u}_3^{(0)} \\[1em]
&\frac{\partial}{\partial x_1}\left(\gamma_{11}\frac{\partial u_1^{(1)}}{\partial x_1} + \gamma_{13}\frac{\partial u_3^{(1)}}{\partial x_3}\right) + \gamma_{55}\frac{\partial}{\partial x_3}\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right) \\
&\qquad - \frac{3}{b^2}\left[c_{56}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + c_{66}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right] + \frac{3F_1^{(1)}}{2b^3} = \rho_1 \ddot{u}_1^{(1)} \\[1em]
&\gamma_{55}\frac{\partial}{\partial x_1}\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right) + \frac{\partial}{\partial x_3}\left(\gamma_{13}\frac{\partial u_1^{(1)}}{\partial x_1} + \gamma_{33}\frac{\partial u_3^{(1)}}{\partial x_3}\right) \\
&\qquad - \frac{3}{b^2}\left[\bar{c}_{14}\frac{\partial u_1^{(0)}}{\partial x_1} + \bar{c}_{34}\frac{\partial u_3^{(0)}}{\partial x_3} + \bar{c}_{44}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right)\right] + \frac{3F_3^{(1)}}{2b^3} = \rho_1 \ddot{u}_3^{(1)}
\end{aligned}
\qquad (6.029)
$$

The conditions for unique solutions are the same as those following (6.026).

In an isotropic plate, extension and flexure are not coupled in the differential equations. The extensional equations reduce to the zero-order approximation (see Chapter 4) and the flexural equations are those of Chapter 5.

6.03 Thickness-Shear and Thickness-Flexure

Equations governing thickness-shear and flexural vibrations alone may be obtained by setting

$$u_1^{(0)} = u_3^{(0)} = u_2^{(1)} = 0 \qquad (6.031)$$

in the equations of the first-order approximation. Identical equations are obtained by setting

$$c_{14} = c_{24} = c_{34} = c_{54} = c_{16} = c_{26} = c_{36} = c_{56} = 0 \qquad (6.032)$$

in either the equations of the first-order approximation or the equations of Section 6.02.

In the case of the triclinic plate we obtain the stress-displacement relations

$$\begin{aligned}
T_4^{(0)} &= 2b\left[c_{44}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + c_{46}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right] \\
T_6^{(0)} &= 2b\left[c_{64}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + c_{66}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right] \\
T_1^{(1)} &= \frac{2b^3}{3}\left[\bar{c}_{11}\frac{\partial u_1^{(1)}}{\partial x_1} + \bar{c}_{13}\frac{\partial u_3^{(1)}}{\partial x_3} + \bar{c}_{15}\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right)\right] \\
T_3^{(1)} &= \frac{2b^3}{3}\left[\bar{c}_{31}\frac{\partial u_1^{(1)}}{\partial x_1} + \bar{c}_{33}\frac{\partial u_3^{(1)}}{\partial x_3} + \bar{c}_{35}\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right)\right] \\
T_5^{(1)} &= \frac{2b^3}{3}\left[\bar{c}_{51}\frac{\partial u_1^{(1)}}{\partial x_1} + \bar{c}_{53}\frac{\partial u_3^{(1)}}{\partial x_3} + \bar{c}_{55}\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right)\right]
\end{aligned} \qquad (6.033)$$

and the displacement-equations of motion

$$\frac{\partial}{\partial x_1}\left[c_{64}\left(\frac{\partial u_2^{(0)}}{\partial x_3}+u_3^{(1)}\right)+c_{66}\left(\frac{\partial u_2^{(0)}}{\partial x_1}+u_1^{(1)}\right)\right]$$
$$+\frac{\partial}{\partial x_3}\left[c_{44}\left(\frac{\partial u_2^{(0)}}{\partial x_3}+u_3^{(1)}\right)+c_{46}\left(\frac{\partial u_2^{(0)}}{\partial x_1}+u_1^{(1)}\right)\right]+\frac{F_2^{(0)}}{2b}=\rho\ddot{u}_2^{(0)}$$

$$\frac{\partial}{\partial x_1}\left[\bar{c}_{11}\frac{\partial u_1^{(1)}}{\partial x_1}+\bar{c}_{13}\frac{\partial u_3^{(1)}}{\partial x_3}+\bar{c}_{15}\left(\frac{\partial u_3^{(1)}}{\partial x_1}+\frac{\partial u_1^{(1)}}{\partial x_3}\right)\right]+\frac{\partial}{\partial x_3}\left[\bar{c}_{51}\frac{\partial u_1^{(1)}}{\partial x_1}+\bar{c}_{53}\frac{\partial u_3^{(1)}}{\partial x_3}+\bar{c}_{55}\left(\frac{\partial u_3^{(1)}}{\partial x_1}+\frac{\partial u_1^{(1)}}{\partial x_3}\right)\right]$$
$$-\frac{3}{b^2}\left[c_{64}\left(\frac{\partial u_2^{(0)}}{\partial x_3}+u_3^{(1)}\right)+c_{66}\left(\frac{\partial u_2^{(0)}}{\partial x_1}+u_1^{(1)}\right)\right]+\frac{3F_1^{(1)}}{2b^3}=\rho_1\ddot{u}_1^{(1)}$$

(6.034)

$$\frac{\partial}{\partial x_1}\left[\bar{c}_{51}\frac{\partial u_1^{(1)}}{\partial x_1}+\bar{c}_{53}\frac{\partial u_3^{(1)}}{\partial x_3}+\bar{c}_{55}\left(\frac{\partial u_3^{(1)}}{\partial x_1}+\frac{\partial u_1^{(1)}}{\partial x_3}\right)\right]+\frac{\partial}{\partial x_3}\left[\bar{c}_{31}\frac{\partial u_1^{(1)}}{\partial x_1}+\bar{c}_{33}\frac{\partial u_3^{(1)}}{\partial x_3}+\bar{c}_{35}\left(\frac{\partial u_3^{(1)}}{\partial x_1}+\frac{\partial u_1^{(1)}}{\partial x_3}\right)\right]$$
$$-\frac{3}{b^2}\left[c_{44}\left(\frac{\partial u_2^{(0)}}{\partial x_3}+u_3^{(1)}\right)+c_{46}\left(\frac{\partial u_2^{(0)}}{\partial x_1}+u_1^{(1)}\right)\right]+\frac{3F_3^{(1)}}{2b^3}=\rho_1\ddot{u}_3^{(1)}$$

For the monoclinic plate

$$c_{46}=c_{15}=c_{35}=0$$

and (6.033) and (6.034) reduce to (Mindlin, 1951 B)

$$T_4^{(0)}=2bc_{44}\left(\frac{\partial u_2^{(0)}}{\partial x_3}+u_3^{(1)}\right)$$

$$T_6^{(0)}=2bc_{66}\left(\frac{\partial u_2^{(0)}}{\partial x_1}+u_1^{(1)}\right)$$

$$T_1^{(1)}=\frac{2b^3}{3}\left(\bar{c}_{11}\frac{\partial u_1^{(1)}}{\partial x_1}+\bar{c}_{13}\frac{\partial u_3^{(1)}}{\partial x_3}\right) \qquad (6.035)$$

$$T_3^{(1)}=\frac{2b^3}{3}\left(\bar{c}_{31}\frac{\partial u_1^{(1)}}{\partial x_1}+\bar{c}_{33}\frac{\partial u_3^{(1)}}{\partial x_3}\right)$$

$$T_5^{(1)}=\frac{2b^3}{3}\bar{c}_{55}\left(\frac{\partial u_3^{(1)}}{\partial x_1}+\frac{\partial u_1^{(1)}}{\partial x_3}\right)$$

$$
\begin{aligned}
&c_{66}\frac{\partial}{\partial x_1}\left(\frac{\partial u_2^{(0)}}{\partial x_1}+u_1^{(1)}\right)+c_{44}\frac{\partial}{\partial x_3}\left(\frac{\partial u_2^{(0)}}{\partial x_3}+u_3^{(1)}\right)+\frac{F_2^{(0)}}{2b}=\rho\ddot{u}_2^{(0)}\\
&\bar{c}_{11}\frac{\partial^2 u_1^{(1)}}{\partial x_1^2}+\bar{c}_{55}\frac{\partial^2 u_1^{(1)}}{\partial x_3^2}+(\bar{c}_{13}+\bar{c}_{55})\frac{\partial^2 u_3^{(1)}}{\partial x_1\partial x_3}-\frac{3}{b^2}c_{66}\left(\frac{\partial u_2^{(0)}}{\partial x_1}+u_1^{(1)}\right)+\frac{3F_1^{(1)}}{2b^3}=\rho_1\ddot{u}_1^{(1)}\\
&\bar{c}_{33}\frac{\partial^2 u_3^{(1)}}{\partial x_3^2}+\bar{c}_{55}\frac{\partial^2 u_3^{(1)}}{\partial x_1^2}+(\bar{c}_{13}+\bar{c}_{55})\frac{\partial^2 u_1^{(1)}}{\partial x_1\partial x_3}-\frac{3}{b^2}c_{44}\left(\frac{\partial u_2^{(0)}}{\partial x_3}+u_3^{(1)}\right)+\frac{3F_3^{(1)}}{2b^3}=\rho_1\ddot{u}_3^{(1)}
\end{aligned}
\qquad (6.036)
$$

In the case of the isotropic plate the equations are the same as the flexural equations (5.065) of the first-order approximation.

An alternative procedure, which leads to equations of the same form, but with slightly different constants, is to set

$$T_1^{(0)} = T_3^{(0)} = T_5^{(0)} = 0 \qquad (6.037)$$

in (6.025) and eliminate the strains $S_1^{(0)}, S_3^{(0)}, S_5^{(0)}$. The only changes in (6.034) are the replacement of the $\bar{c}_{pq}$ by γ_{pq} and the replacement of c_{44}, c_{66} and c_{46} by

$$\frac{s_{66}}{s_{44}s_{66}-s_{46}^2}, \quad \frac{s_{44}}{s_{44}s_{66}-s_{46}^2}, \quad -\frac{s_{46}}{s_{44}s_{66}-s_{46}^2} \qquad (6.038)$$

respectively.

6.04 Classical Theory of Low-Frequency Vibrations of Thin Plates

The equations of the classical theory of thin plates are obtained by suppressing the thickness-shear strains in the equations of Section 6.02. The result is a set of coupled flexural and extensional equations of motion (Cauchy, 1829, p. 9). When the coupling coefficients are dropped, the equations separate into a single flexural equation and a pair of extensional equations (Cauchy, 1829, p. 13). In the isotropic case, the extensional equations are those of Poisson (1829, p. 499) and the flexural equation is that of Germain (1821) except for the term accounting for rotatory inertia, which was given later (and then dropped) by Poisson (1829, p. 533).

To effect the reduction of the equations of the preceding section to those of the classical theory, we begin by setting

$$S_4^{(0)} = 0\,, \quad S_6^{(0)} = 0 \tag{6.041}$$

so that

$$u_1^{(1)} = -\frac{\partial u_2^{(0)}}{\partial x_1}\,, \quad u_3^{(1)} = -\frac{\partial u_2^{(0)}}{\partial x_3} \tag{6.042}$$

It is to be noted that the components of rotation

$$\frac{1}{2}\left(\frac{\partial u_2^{(0)}}{\partial x_1} - u_1^{(1)}\right)\,, \quad \frac{1}{2}\left(\frac{\partial u_2^{(0)}}{\partial x_3} - u_3^{(1)}\right) \tag{6.043}$$

do not vanish as a result of (6.042).

In view of (6.042), the first-order stress-displacement relations (6.025) become

$$\begin{aligned}
T_1^{(1)} &= -\frac{2b^3}{3}\left(\gamma_{11}\frac{\partial^2 u_2^{(0)}}{\partial x_1^2} + \gamma_{13}\frac{\partial^2 u_2^{(0)}}{\partial x_3^2} + 2\gamma_{15}\frac{\partial^2 u_2^{(0)}}{\partial x_1 \partial x_3}\right)\\
T_3^{(1)} &= -\frac{2b^3}{3}\left(\gamma_{31}\frac{\partial^2 u_2^{(0)}}{\partial x_1^2} + \gamma_{33}\frac{\partial^2 u_2^{(0)}}{\partial x_3^2} + 2\gamma_{35}\frac{\partial^2 u_2^{(0)}}{\partial x_1 \partial x_3}\right)\\
T_5^{(1)} &= -\frac{2b^3}{3}\left(\gamma_{51}\frac{\partial^2 u_2^{(0)}}{\partial x_1^2} + \gamma_{53}\frac{\partial^2 u_2^{(0)}}{\partial x_3^2} + 2\gamma_{55}\frac{\partial^2 u_2^{(0)}}{\partial x_1 \partial x_3}\right)
\end{aligned} \tag{6.044}$$

The zero-order components of stress $T_2^{(0)}$ and $T_4^{(0)}$ may be expressed in terms of the displacement $u_2^{(0)}$ by means of (6.034) and the fourth and sixth of the stress-equations of motion. Thus, from (5.0117)

$$\begin{aligned}
T_6^{(0)} &= F_1^{(1)} + \frac{\partial T_1^{(1)}}{\partial x_1} + \frac{\partial T_5^{(1)}}{\partial x_3} - \frac{2b^3}{3}\rho_1 \ddot{u}_1^{(1)}\\
T_4^{(0)} &= F_3^{(1)} + \frac{\partial T_5^{(1)}}{\partial x_1} + \frac{\partial T_3^{(1)}}{\partial x_3} - \frac{2b^3}{3}\rho_1 \ddot{u}_3^{(1)}
\end{aligned} \tag{6.045}$$

The last terms in (6.045) account for rotatory inertia. They will be omitted (following Poisson) inasmuch as they provide a negligible contribution at the low frequencies to which the ensuing equations are restricted. Then, inserting (6.044) in (6.045), we have

$$
\begin{aligned}
T_6^{(0)} &= F_1^{(1)} - \frac{2b^3}{3}\left[\gamma_{11}\frac{\partial^3 u_2^{(0)}}{\partial x_1^3} + 3\gamma_{15}\frac{\partial^3 u_2^{(0)}}{\partial x_1^2\,\partial x_3} + (\gamma_{13}+2\gamma_{55})\frac{\partial^3 u_2^{(0)}}{\partial x_1\,\partial x_3^2} + \gamma_{35}\frac{\partial^3 u_2^{(0)}}{\partial x_3^3}\right] \\
T_4^{(0)} &= F_3^{(1)} - \frac{2b^3}{3}\left[\gamma_{15}\frac{\partial^3 u_2^{(0)}}{\partial x_1^3} + (\gamma_{13}+2\gamma_{55})\frac{\partial^3 u_2^{(0)}}{\partial x_1^2\,\partial x_3} + 3\gamma_{15}\frac{\partial^3 u_2^{(0)}}{\partial x_1\,\partial x_3^2} + \gamma_{33}\frac{\partial^3 u_2^{(0)}}{\partial x_3^3}\right]
\end{aligned}
\qquad (6.046)
$$

As for the remaining three components of zero-order stress, we first rewrite (6.025) in the form

$$
\begin{aligned}
2bS_1^{(0)} &= s_{11}T_1^{(0)} + s_{13}T_3^{(0)} + s_{14}T_4^{(0)} + s_{15}T_5^{(0)} + s_{16}T_6^{(0)} \\
2bS_3^{(0)} &= s_{31}T_1^{(0)} + s_{33}T_3^{(0)} + s_{34}T_4^{(0)} + s_{35}T_5^{(0)} + s_{36}T_6^{(0)} \\
2bS_5^{(0)} &= s_{51}T_1^{(0)} + s_{53}T_3^{(0)} + s_{54}T_4^{(0)} + s_{55}T_5^{(0)} + s_{56}T_6^{(0)}
\end{aligned}
\qquad (6.047)
$$

where

$$
S_1^{(0)} = \frac{\partial u_1^{(0)}}{\partial x_1}, \quad S_3^{(0)} = \frac{\partial u_3^{(0)}}{\partial x_3}, \quad S_5^{(0)} = \frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3} \qquad (6.048)
$$

Then, solving (6.047) for $T_1^{(0)}$, $T_3^{(0)}$ and $T_5^{(0)}$ in terms of the remaining quantities, we find

$$
\begin{aligned}
T_1^{(0)} &= 2b\left[\gamma_{11}\frac{\partial u_1^{(0)}}{\partial x_1} + \gamma_{13}\frac{\partial u_3^{(0)}}{\partial x_3} + \gamma_{15}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right)\right] - \alpha_{14}T_4^{(0)} - \alpha_{16}T_6^{(0)} \\
T_3^{(0)} &= 2b\left[\gamma_{31}\frac{\partial u_1^{(0)}}{\partial x_1} + \gamma_{33}\frac{\partial u_3^{(0)}}{\partial x_3} + \gamma_{35}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right)\right] - \alpha_{34}T_4^{(0)} - \alpha_{36}T_6^{(0)} \\
T_5^{(0)} &= 2b\left[\gamma_{51}\frac{\partial u_1^{(0)}}{\partial x_1} + \gamma_{53}\frac{\partial u_3^{(0)}}{\partial x_3} + \gamma_{55}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right)\right] - \alpha_{54}T_4^{(0)} - \alpha_{56}T_6^{(0)}
\end{aligned}
\qquad (6.049)
$$

where $T_4^{(0)}$ and $T_6^{(0)}$ are given by (6.036) and

$$
\begin{aligned}
\alpha_{14} &= \gamma_{11}s_{14} + \gamma_{13}s_{34} + \gamma_{15}s_{54} & \alpha_{16} &= \gamma_{11}s_{16} + \gamma_{13}s_{36} + \gamma_{15}s_{56} \\
\alpha_{34} &= \gamma_{31}s_{14} + \gamma_{33}s_{34} + \gamma_{35}s_{54} & \alpha_{36} &= \gamma_{31}s_{16} + \gamma_{33}s_{36} + \gamma_{35}s_{56} \\
\alpha_{54} &= \gamma_{51}s_{14} + \gamma_{53}s_{34} + \gamma_{55}s_{54} & \alpha_{56} &= \gamma_{51}s_{16} + \gamma_{53}s_{36} + \gamma_{55}s_{56}
\end{aligned}
\qquad (6.0410)
$$

It will be observed that the α_{pq} are constants which couple flexure with extension.

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To obtain the displacement-equations of motion we insert the stress-displacement relations (6.044), (6.046) and (6.049) in the remaining stress-equations of motion

$$\frac{\partial T_1^{(0)}}{\partial x_1} + \frac{\partial T_5^{(0)}}{\partial x_3} + F_1^{(0)} = 2b\rho\ddot{u}_1^{(0)}$$

$$\frac{\partial T_6^{(0)}}{\partial x_1} + \frac{\partial T_4^{(0)}}{\partial x_3} + F_2^{(0)} = 2b\rho\ddot{u}_2^{(0)} \qquad (6.0411)$$

$$\frac{\partial T_5^{(0)}}{\partial x_1} + \frac{\partial T_3^{(0)}}{\partial x_3} + F_3^{(0)} = 2b\rho\ddot{u}_3^{(0)}$$

and obtain

$$\begin{aligned}
&\frac{\partial^2}{\partial x_1^2}\left(\gamma_{11}u_1^{(0)} + \gamma_{15}u_3^{(0)}\right) + \frac{\partial^2}{\partial x_3^2}\left(\gamma_{55}u_1^{(0)} + \gamma_{35}u_3^{(0)}\right) + \frac{\partial^2}{\partial x_1\partial x_3}\left[2\gamma_{15}u_1^{(0)} + (\gamma_{13}+\gamma_{55})u_3^{(0)}\right] + \frac{F_1^{(0)}}{2b} \\
&+ \frac{b^2}{3}\left(\alpha_{14}\frac{\partial}{\partial x_1} + \alpha_{54}\frac{\partial}{\partial x_3}\right)\left[\gamma_{51}\frac{\partial^3 u_2^{(0)}}{\partial x_1^3} + (\gamma_{31}+2\gamma_{55})\frac{\partial^3 u_2^{(0)}}{\partial x_1^2\partial x_3} + 3\gamma_{35}\frac{\partial^3 u_2^{(0)}}{\partial x_1\partial x_3^2} + \gamma_{33}\frac{\partial^3 u_2^{(0)}}{\partial x_3^3} - \frac{3F_3^{(1)}}{2b^3}\right] \\
&+ \frac{b^2}{3}\left(\alpha_{16}\frac{\partial}{\partial x_1} + \alpha_{56}\frac{\partial}{\partial x_3}\right)\left[\gamma_{53}\frac{\partial^3 u_2^{(0)}}{\partial x_3^2} + (\gamma_{31}+2\gamma_{55})\frac{\partial^3 u_2^{(0)}}{\partial x_1\partial x_3^2} + 3\gamma_{51}\frac{\partial^3 u_2^{(0)}}{\partial x_1^2\partial x_3} + \gamma_{11}\frac{\partial^3 u_2^{(0)}}{\partial x_1^3} - \frac{3F_1^{(1)}}{2b^3}\right] = \rho\ddot{u}_1^{(0)}
\end{aligned}$$

$$\begin{aligned}
&\gamma_{11}\frac{\partial^4 u_2^{(0)}}{\partial x_1^4} + 4\gamma_{15}\frac{\partial^4 u_2^{(0)}}{\partial x_1^3\partial x_3} + 2(\gamma_{13}+2\gamma_{55})\frac{\partial^4 u_2^{(0)}}{\partial x_1^2\partial x_3^2} + 4\gamma_{35}\frac{\partial^4 u_2^{(0)}}{\partial x_1\partial x_3^3} + \gamma_{33}\frac{\partial^4 u_2^{(0)}}{\partial x_3^4} \\
&\qquad - \frac{3}{2b^3}\left(F_2^{(0)} + \frac{\partial F_1^{(1)}}{\partial x_1} + \frac{\partial F_3^{(1)}}{\partial x_3}\right) = \frac{3\rho}{b^2}\ddot{u}_2^{(0)}
\end{aligned} \qquad (6.0412)$$

$$\begin{aligned}
&\frac{\partial^2}{\partial x_1^2}\left(\gamma_{15}u_1^{(0)} + \gamma_{55}u_3^{(0)}\right) + \frac{\partial^2}{\partial x_3^2}\left(\gamma_{35}u_1^{(0)} + \gamma_{33}u_3^{(0)}\right) + \frac{\partial^2}{\partial x_1\partial x_3}\left[(\gamma_{13}+\gamma_{55})u_1^{(0)} + 2\gamma_{35}u_3^{(0)}\right] + \frac{F_3^{(0)}}{2b} \\
&+ \frac{b^2}{3}\left(\alpha_{54}\frac{\partial}{\partial x_1} + \alpha_{34}\frac{\partial}{\partial x_3}\right)\left[\gamma_{51}\frac{\partial^3 u_2^{(0)}}{\partial x_1^3} + (\gamma_{31}+2\gamma_{55})\frac{\partial^3 u_2^{(0)}}{\partial x_1^2\partial x_3} + 3\gamma_{35}\frac{\partial^3 u_2^{(0)}}{\partial x_1\partial x_3^2} + \gamma_{33}\frac{\partial^3 u_2^{(0)}}{\partial x_3^3} - \frac{3F_3^{(1)}}{2b^3}\right] \\
&+ \frac{b^2}{3}\left(\alpha_{56}\frac{\partial}{\partial x_1} + \alpha_{36}\frac{\partial}{\partial x_3}\right)\left[\gamma_{53}\frac{\partial^3 u_2^{(0)}}{\partial x_3^2} + (\gamma_{31}+2\gamma_{55})\frac{\partial^3 u_2^{(0)}}{\partial x_1\partial x_3^2} + 3\gamma_{51}\frac{\partial^3 u_2^{(0)}}{\partial x_1^2\partial x_3} + \gamma_{11}\frac{\partial^3 u_2^{(0)}}{\partial x_1^3} - \frac{3F_1^{(1)}}{2b^3}\right] = \rho\ddot{u}_3^{(0)}
\end{aligned}$$

Equations equivalent to (6.0412) were obtained by Cauchy (1829, p. 9).

Suppression of the thickness-shear deformation results in a reduction of the number of edge-conditions sufficient for a unique solution of the equations. Upon expansion of the terms in the integrand of the line integral in (5.035) we get

$$\oint \left(T_{nn}^{(0)} \dot{u}_n^{(0)} + T_{ns}^{(0)} \dot{u}_s^{(0)} + T_{n2}^{(0)} \dot{u}_2^{(0)} + T_{nn}^{(1)} \dot{u}_n^{(1)} + T_{ns}^{(1)} u_s^{(1)} \right) ds \tag{6.0413}$$

The first two products relate to extension and the last three to flexure. The term $T_{n2}^{(1)} \dot{u}_2^{(1)}$ is not present because we have set $T_4^{(1)}$ and $T_6^{(1)}$ equal to zero (see Section 6.02). Now, by virtue of (6.042), we have

$$\dot{u}_s^{(1)} = -\frac{\partial \dot{u}_2^{(0)}}{\partial s} \tag{6.0414}$$

Hence

$$\oint T_{ns}^{(1)} \dot{u}_s^{(1)} ds = -\oint T_{ns}^{(1)} \frac{\partial \dot{u}_2^{(0)}}{\partial s} ds = -\left[T_{ns}^{(1)} \dot{u}_2^{(0)}\right]_0^{\circ} + \oint \dot{u}_2^{(0)} \frac{\partial T_{ns}^{(1)}}{\partial s} ds \tag{6.0415}$$

The term outside the integral vanishes if the traction and displacement are continuous. Then (6.0413) reduces to

$$\oint \left[T_{nn}^{(0)} \dot{u}_n^{(0)} + T_{ns}^{(0)} \dot{u}_s^{(0)} + \left(T_{n2}^{(0)} + \frac{\partial T_{ns}^{(1)}}{\partial s}\right) \dot{u}_2^{(0)} + T_{nn}^{(1)} \dot{u}_n^{(1)} \right] ds \tag{6.0416}$$

that is, the three edge-conditions relating to flexure reduce to two (Kirchhoff, 1850).

If the coupling coefficients (α_{pq}) are omitted, (6.0412) separate into a pair of extensional equations and a single flexural equation (Cauchy, 1829, p. 13; Voigt, 1928, pp. 675-698).

The extensional equations are

$$\begin{aligned} &\frac{\partial^2}{\partial x_1^2}\left(\gamma_{11} u_1^{(0)} + \gamma_{15} u_3^{(0)}\right) + \frac{\partial^2}{\partial x_3^2}\left(\gamma_{55} u_1^{(0)} + \gamma_{35} u_3^{(0)}\right) + \frac{\partial^2}{\partial x_1 \partial x_3}\left[2\gamma_{15} u_1^{(0)} + (\gamma_{13}+\gamma_{55}) u_3^{(0)}\right] + \frac{F_1^{(0)}}{2b} = \rho \ddot{u}_1^{(0)} \\ &\frac{\partial^2}{\partial x_1^2}\left(\gamma_{15} u_1^{(0)} + \gamma_{55} u_3^{(0)}\right) + \frac{\partial^2}{\partial x_3^2}\left(\gamma_{35} u_1^{(0)} + \gamma_{33} u_3^{(0)}\right) + \frac{\partial^2}{\partial x_1 \partial x_3}\left[(\gamma_{13}+\gamma_{55}) u_1^{(0)} + 2\gamma_{35} u_3^{(0)}\right] + \frac{F_3^{(0)}}{2b} = \rho \ddot{u}_3^{(0)} \end{aligned} \tag{6.0417}$$

and the associated stress-displacement relations are given by (6.049) with $\alpha_{pq} = 0$. In the isotropic case, (6.0417) reduce to the zero-order extensional equations, i.e., the first and third of (4.043). The relation between

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frequency and wave-length is given by the curve marked "Zero-Order (Classical)" in Fig. 5.072. It may be seen that the approximation is very good for $\omega/\omega_s < 1$.

The flexural equation is the same as the second of (6.0412). In the isotropic case this reduces to (Germain, 1821)

$$D\nabla_1^4 u_2^{(0)} + 2\rho b \ddot{u}_2^{(0)} = F_2^{(0)} + \frac{\partial F_1^{(1)}}{\partial x_1} + \frac{\partial F_3^{(1)}}{\partial x_3} \qquad (6.0418)$$

If we set the face-tractions equal to zero and

$$u_2^{(0)} = A \sin \xi x_1 e^{i\omega t} \qquad (6.0419)$$

we obtain

$$\omega = \xi^2 b \sqrt{\frac{E}{3\rho(1-\nu^2)}} \qquad (6.0420)$$

This is the frequency for long wave-lengths of the lowest mode in the three-dimensional theory. Equation (6.0420) is plotted in Fig. 5.071 as the curve marked "Classical". It may be seen that the approximation is very good for $\omega/\omega_s < 0.1$.

6.05 Moderately-High-Frequency Vibrations of Thin Plates

In the reduction of the equations of Section 6.02 to those of Section 6.04, the range of the flexural equations is reduced to $\omega/\omega_s < 0.1$ while the extensional equations are usable up to $\omega/\omega_s = 1$. It is worthwhile to form equations in which the flexural part has a wider range. These have applications to flexure of isotropic plates, and they are especially useful in crystal plates where extension and flexure are coupled.

The equations are obtained from those of Section 6.02 simply by dropping the remaining terms of the first-order kinetic energy. That is, we

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drop the rotatory inertia terms but retain the thickness-shear strains. In so doing, we lose the correction factor κ, just as we did in the case of reduction to the classical theory of thin plates (Section 6.04). In the present case somewhat better results are obtained, especially for the flexural part, if the correction factor is transferred to the strain-energy-density, as described in Section 5.02. Then the stress-displacement relations (6.025) become, for the triclinic plate,

$$T_1^{(0)} = 2b\left[\bar{c}_{11}\frac{\partial u_1^{(0)}}{\partial x_1} + \bar{c}_{13}\frac{\partial u_3^{(0)}}{\partial x_3} + \kappa\bar{c}_{14}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + \bar{c}_{15}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + \kappa\bar{c}_{16}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right]$$

$$T_3^{(0)} = 2b\left[\bar{c}_{31}\frac{\partial u_1^{(0)}}{\partial x_1} + \bar{c}_{33}\frac{\partial u_3^{(0)}}{\partial x_3} + \kappa\bar{c}_{34}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + \bar{c}_{35}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + \kappa\bar{c}_{36}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right]$$

$$T_4^{(0)} = 2b\kappa\left[\bar{c}_{41}\frac{\partial u_1^{(0)}}{\partial x_1} + \bar{c}_{43}\frac{\partial u_3^{(0)}}{\partial x_3} + \kappa\bar{c}_{44}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + \bar{c}_{45}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + \kappa\bar{c}_{46}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right]$$

$$T_5^{(0)} = 2b\left[\bar{c}_{51}\frac{\partial u_1^{(0)}}{\partial x_1} + \bar{c}_{53}\frac{\partial u_3^{(0)}}{\partial x_3} + \kappa\bar{c}_{54}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + \bar{c}_{55}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + \kappa\bar{c}_{56}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right]$$

$$T_6^{(0)} = 2b\kappa\left[\bar{c}_{61}\frac{\partial u_1^{(0)}}{\partial x_1} + \bar{c}_{63}\frac{\partial u_3^{(0)}}{\partial x_3} + \kappa\bar{c}_{64}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + \bar{c}_{65}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + \kappa\bar{c}_{66}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right]$$

(6.051)

$$T_1^{(1)} = \frac{2b^3}{3}\left[\gamma_{11}\frac{\partial u_1^{(1)}}{\partial x_1} + \gamma_{13}\frac{\partial u_3^{(1)}}{\partial x_3} + \gamma_{15}\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right)\right]$$

$$T_3^{(1)} = \frac{2b^3}{3}\left[\gamma_{31}\frac{\partial u_1^{(1)}}{\partial x_1} + \gamma_{33}\frac{\partial u_3^{(1)}}{\partial x_3} + \gamma_{35}\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right)\right]$$

$$T_5^{(1)} = \frac{2b^3}{3}\left[\gamma_{51}\frac{\partial u_1^{(1)}}{\partial x_1} + \gamma_{53}\frac{\partial u_3^{(1)}}{\partial x_3} + \gamma_{55}\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right)\right]$$

It will be observed that these are the same as (6.025) except that the constants $\bar{c}_{pq}$ are multiplied by κ if one subscript is even and by κ^2 if both subscripts are even. Except for this alteration of constants and the omission of the accelerations $\ddot{u}_1^{(1)}$ and $\ddot{u}_3^{(1)}$, the equations of motion (6.026) remain unchanged.

In the isotropic case, the extensional part reduces to the zero-order approximation (Chapter 4) and the flexural part becomes (see (5.053) and (5.065))

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$$
\begin{aligned}
T_4^{(0)} &= 2bk^2\mu\left(\frac{\partial u_1^{(0)}}{\partial x_3} + u_3^{(1)}\right)\\
T_6^{(0)} &= 2bk^2\mu\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\\
T_1^{(1)} &= D\left(\frac{\partial u_1^{(1)}}{\partial x_1} + \nu\frac{\partial u_3^{(1)}}{\partial x_3}\right)\\
T_3^{(1)} &= D\left(\frac{\partial u_3^{(1)}}{\partial x_3} + \nu\frac{\partial u_1^{(1)}}{\partial x_1}\right)\\
T_5^{(1)} &= D\frac{(1-\nu)}{2}\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right)
\end{aligned}
\qquad (6.052)
$$

$$
\begin{aligned}
&2bk^2\mu\left(\nabla_1^2 u_2^{(0)} + e^{(1)}\right) + F_2^{(0)} = 2b\rho\ddot{u}_2^{(0)}\\
&\frac{D}{2}\left[(1-\nu)\nabla_1^2 u_1^{(1)} + (1+\nu)\frac{\partial e^{(1)}}{\partial x_1}\right] - 2bk^2\mu\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right) + F_1^{(1)} = 0\\
&\frac{D}{2}\left[(1-\nu)\nabla_1^2 u_3^{(1)} + (1+\nu)\frac{\partial e^{(1)}}{\partial x_3}\right] - 2bk^2\mu\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + F_3^{(1)} = 0
\end{aligned}
\qquad (6.053)
$$

Upon eliminating $u_1^{(1)}$ and $u_3^{(1)}$ from (6.053) we obtain, for comparison with (6.0418),

$$D\nabla_1^4 u_2^{(0)} + 2\rho b\ddot{u}_2^{(0)} - \frac{\rho D}{k^2\mu}\nabla_1^2\ddot{u}_2^{(0)} = F_2^{(0)} + \frac{\partial F_1^{(1)}}{\partial x_1} + \frac{\partial F_3^{(1)}}{\partial x_3} - \frac{D}{2bk^2\mu}\nabla_1^2 F_2^{(0)} \qquad (6.054)$$

To compare the frequency-wave-length relations given by (6.054) and (6.0418) we set the face-tractions equal to zero and

$$u_2^{(0)} = A\sin\xi x_1\, e^{i\omega t}$$

We obtain, from (6.054),

$$\omega = \xi^2 b\sqrt{\frac{E}{3\rho(1-\nu^2)}}\left[1 + \frac{2\xi^2 b^2}{3k^2(1-\nu)}\right]^{-1/2} \qquad (6.055)$$

When this relation is plotted on Fig. 5.071, the curve is indistinguishable from the corresponding curve for the exact solution.