CHAPTER 5

FIRST-ORDER APPROXIMATION

5.01 Separation of Zero- and First-Order Terms from Series

The establishment of a set of approximate equations, in which only the zero-order and first-order displacements appear, follows a course similar to that employed in obtaining the zero-order approximation. Thus, corresponding to (4.011) we begin by setting (see Fig. 3.011)

$$u_1^{(n)} = 0, \quad u_3^{(n)} = 0, \quad n > 1$$
$$u_2^{(n)} = 0, \quad n > 2 \qquad (5.011)$$

The retention of $u_2^{(2)}$ is necessary in order to accommodate the thickness-strains which accompany low-frequency flexure. Consider, for example, an isotropic plate, in a horizontal position, subjected to simple, static bending such that the upper face of the plate is concave. Then the upper half of the plate is in compression (in its plane) and the lower half is in tension. Due to the effect of Poisson's ratio, the upper half must expand in thickness and the lower half contract. This is a strain of the nature of $S_2^{(1)}$ ($=2u_2^{(2)}$, see Figs. 3.032). In addition, if we consider variation of bending with x_1 and x_3, we see that we must permit the contributions $\partial u_2^{(2)}/\partial x_1$ and $\partial u_2^{(2)}/\partial x_3$ to the strains $S_4^{(0)}$ and $S_6^{(0)}$ (see Fig. 3.032).

The assumptions (5.011) reduce the kinetic energy-density (3.0514) to

$$\bar{K} = \rho b \dot{u}_j^{(0)} \dot{u}_j^{(0)} + \tfrac{1}{3}\rho b^3 \dot{u}_j^{(1)} \dot{u}_j^{(1)} + \tfrac{2}{3}\rho b^3 \dot{u}_2^{(2)} \dot{u}_2^{(0)} + \tfrac{1}{5}\rho b^5 \dot{u}_2^{(2)} \dot{u}_2^{(2)} \qquad (5.012)$$

and the strain-energy-density (3.0511) to

$$\bar{U} = \frac{1}{2}\Big[T_p^{(0)} S_p^{(0)} + T_1^{(1)} \frac{\partial u_1^{(1)}}{\partial x_1} + T_3^{(1)} \frac{\partial u_3^{(1)}}{\partial x_3} + T_4^{(1)} \frac{\partial u_2^{(1)}}{\partial x_3}$$
$$+ T_5^{(1)} \left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3} \right) + T_6^{(1)} \frac{\partial u_2^{(1)}}{\partial x_1} \tag{5.013}$$
$$+ 2 T_2^{(1)} u_2^{(2)} + T_4^{(2)} \frac{\partial u_2^{(2)}}{\partial x_3} + T_6^{(2)} \frac{\partial u_2^{(2)}}{\partial x_1} \Big]$$

The presence of $u_j^{(1)}$, $T_2^{(1)}$, $T_4^{(2)}$ and $T_6^{(2)}$ in the energies accommodates motions which represent the three fundamental thickness-modes, i.e., the lowest x_1 and x_3 simple thickness-shear modes and the lowest simple thickness-stretch mode (see Figs. 2.031, 2.032 and 3.011). The lowest, symmetric, simple thickness-shear mode (Fig. 2.032, $q=2$) is not included; that is, its frequency is assumed to be higher than that of the lowest symmetric, simple thickness-stretch mode (Fig. 2.031, $p=1$). In an isotropic plate, for example, this limits the applicability of the equations to $\nu < 1/3$. The displacement $u_2^{(2)}$, which appears in both energies, is associated with the second thickness-stretch mode (see Fig. 2.031, $p=2$, and Fig. 3.011). We choose to eliminate this mode from the first-order approximation, just as we have already eliminated the third and higher thickness-stretch modes and the second and higher thickness-shear modes by the assumptions (5.011). To accomplish the elimination without suppressing $u_2^{(2)}$ completely, we set

$$\dot{u}_2^{(2)} = 0 \tag{5.014}$$

in (5.012) and

$$T_2^{(1)} = T_4^{(2)} = T_6^{(2)} = 0 \tag{5.015}$$

in (5.013). In this way we permit the free development of the first-order thickness-stretch ($S_2^{(1)} = 2u_2^{(2)}$) and the portions $\partial u_2^{(2)}/\partial x_1$ and $\partial u_2^{(2)}/\partial x_3$ of the second-order shears $S_4^{(2)}$ and $S_6^{(2)}$, as shown in Fig. 5.011.

As a result of (5.014) the kinetic energy-density (5.012) becomes

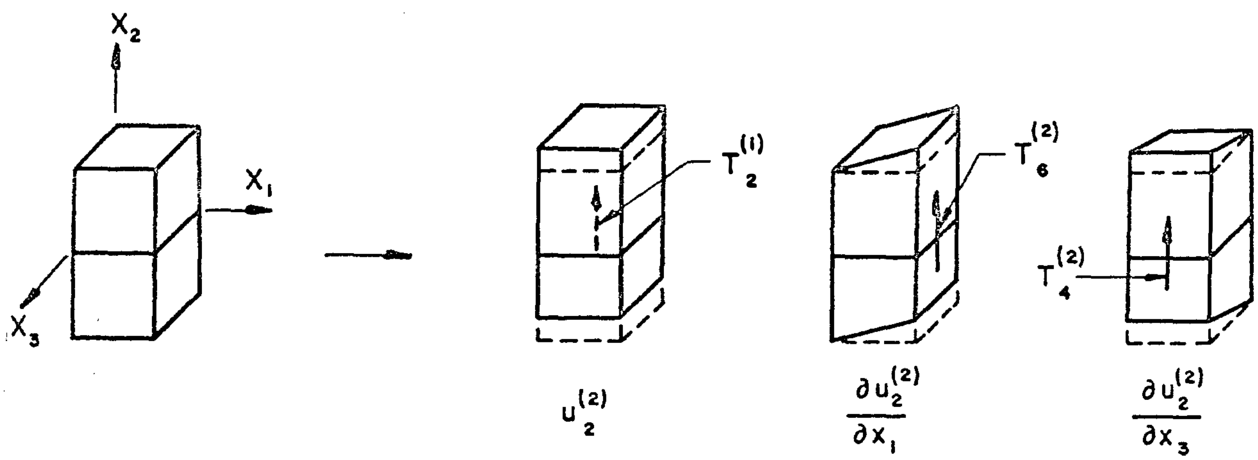


Fig. 5.011

Strains which are permitted to develop freely in the first-order approximation.

$$\bar{K} = \rho b \dot{u}_j^{(0)} \dot{u}_j^{(0)} + \tfrac{1}{3} \rho_1 b^3 \dot{u}_j^{(1)} \dot{u}_j^{(1)} \tag{5.016}$$

In anticipation of an alteration, to be made later, we have identified the density, in the first-order kinetic energy, by writing it as ρ_1.

As a result of (5.015) the strain-energy-density (5.013) becomes

$$\bar{U} = \frac{1}{2}\left[T_p^{(0)} S_p^{(0)} + T_1^{(1)} \frac{\partial u_1^{(1)}}{\partial x_1} + T_3^{(1)} \frac{\partial u_3^{(1)}}{\partial x_3} + T_4^{(1)} \frac{\partial u_2^{(1)}}{\partial x_3} + T_5^{(1)} \left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3} \right) + T_6^{(1)} \frac{\partial u_2^{(1)}}{\partial x_1} \right] \tag{5.017}$$

while the zero-order and first-order stress-strain relations are

$$\begin{aligned} &T_p^{(0)} = 2b\, c_{pq} S_q^{(0)}, \quad p, q = 1,2,3,4,5,6 \\ &\left.\begin{aligned} T_p^{(1)} &= \tfrac{2}{3} b^3 c_{pq} S_q^{(1)} \\ T_2^{(1)} &= \tfrac{2}{3} b^3 c_{2q} S_q^{(1)} \end{aligned}\right\} \left\{ \begin{aligned} p &= 1,3,4,5,6 \\ q &= 1,2,3,4,5,6 \end{aligned}\right. \end{aligned} \tag{5.018}$$

where the $S_q^{(0)}$ are given by their full expressions in terms of the displacements (see Fig. 3.031) but the first-order strains are now

$$\begin{aligned} S_1^{(1)} &= \frac{\partial u_1^{(1)}}{\partial x_1} & S_4^{(1)} &= \frac{\partial u_2^{(1)}}{\partial x_3} \\ S_2^{(1)} &= 2u_2^{(2)} & S_5^{(1)} &= \frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3} \\ S_3^{(1)} &= \frac{\partial u_3^{(1)}}{\partial x_3} & S_6^{(1)} &= \frac{\partial u_2^{(1)}}{\partial x_1} \end{aligned} \tag{5.019}$$

In the first of (5.018) the terms

$$\frac{2b^3}{3}\left(c_{p4} \frac{\partial u_2^{(2)}}{\partial x_3} + c_{p6} \frac{\partial u_2^{(2)}}{\partial x_1} \right)$$

are omitted, as it is desired to permit the free development of these strains, without contribution to the strain-energy (see Fig. 5.011).

The second and third of (5.018) may be reduced to the single expression

$$T_p^{(1)} = \tfrac{2}{3} b^3 \bar{c}_{pq} S_q^{(1)} \tag{5.0110}$$

where

$$\bar{c}_{pq} = c_{pq} - \frac{c_{p2} c_{2q}}{c_{22}}$$

by the same procedure as that followed in the reduction of (4.016) to (4.019).

In unabbreviated notation, (5.018) become

$$\begin{aligned} T_{ij}^{(0)} &= 2b c_{ijkl} S_{kl}^{(0)} \\ T_{ij}^{(1)} &= \tfrac{2}{3} b^3 \bar{c}_{ijkl} S_{kl}^{(1)} \end{aligned} \tag{5.0111}$$

where

$$\bar{c}_{ijkl} = c_{ijkl} - \frac{c_{ij22} c_{22kl}}{c_{2222}}$$

The stress-displacement relations are, from (5.0111), (3.037) and (5.019),

$$\begin{aligned} T_{ij}^{(0)} &= b c_{ijkl} \left(u_{k,l}^{(0)} + u_{l,k}^{(0)} + \delta_{2l} u_k^{(1)} + \delta_{2k} u_l^{(1)} \right) \\ T_{ij}^{(1)} &= \tfrac{1}{3} b^3 \bar{c}_{ijkl} \left(u_{k,l}^{(1)} + u_{l,k}^{(1)} \right) \end{aligned} \tag{5.0112}$$

At this stage, the strain-energy-density in terms of the strains is

$$\begin{aligned} \bar{U} &= b c_{pq} S_p^{(0)} S_q^{(0)} + \tfrac{1}{3} b^3 \bar{c}_{pq} S_p^{(1)} S_q^{(1)} \\ &= b c_{ijkl} S_{ij}^{(0)} S_{kl}^{(0)} + \tfrac{1}{3} b^3 \bar{c}_{ijkl} S_{ij}^{(1)} S_{kl}^{(1)} \end{aligned} \tag{5.0113}$$

Then

$$\begin{aligned} T_p^{(0)} &= \frac{\partial \bar{U}}{\partial S_p^{(0)}} \\ T_p^{(1)} &= \frac{\partial \bar{U}}{\partial S_p^{(1)}} \end{aligned} \tag{5.0114}$$

or

$$\begin{aligned} T_{ij}^{(0)} &= \frac{\partial \bar{U}}{\partial S_{ij}^{(0)}} , \quad \left(\frac{\partial S_{ij}^{(0)}}{\partial S_{ji}^{(0)}} = 0 \, ; \; i \neq j \right) \\ T_{ij}^{(1)} &= \frac{\partial U}{\partial S_{ij}^{(1)}} , \quad \left(\frac{\partial S_{ij}^{(1)}}{\partial S_{ji}^{(1)}} = 0 \, ; \; i \neq j \right) \end{aligned} \tag{5.0115}$$

The zero-order and first-order stress equations of motion (3.0212) reduce to

$$T_{ij,i}^{(0)} + F_j^{(0)} = 2b\rho \ddot{u}_j^{(0)}$$
$$T_{ij,i}^{(1)} - T_{2j}^{(0)} + F_j^{(1)} = \tfrac{2}{3}b^3\rho_1 \ddot{u}_j^{(1)} \qquad (5.0116)$$

or

$$\frac{\partial T_1^{(0)}}{\partial x_1} + \frac{\partial T_5^{(0)}}{\partial x_3} + F_1^{(0)} = 2b\rho\ddot{u}_1^{(0)}$$
$$\frac{\partial T_6^{(0)}}{\partial x_1} + \frac{\partial T_4^{(0)}}{\partial x_3} + F_2^{(0)} = 2b\rho\ddot{u}_2^{(0)}$$
$$\frac{\partial T_5^{(0)}}{\partial x_1} + \frac{\partial T_3^{(0)}}{\partial x_3} + F_3^{(0)} = 2b\rho\ddot{u}_3^{(0)}$$
$$\frac{\partial T_1^{(1)}}{\partial x_1} + \frac{\partial T_5^{(1)}}{\partial x_3} - T_6^{(0)} + F_1^{(1)} = \tfrac{2}{3}b^3\rho_1\ddot{u}_1^{(1)} \qquad (5.0117)$$
$$\frac{\partial T_6^{(1)}}{\partial x_1} + \frac{\partial T_4^{(1)}}{\partial x_3} - T_2^{(0)} + F_2^{(1)} = \tfrac{2}{3}b^3\rho\ddot{u}_2^{(1)}$$
$$\frac{\partial T_5^{(1)}}{\partial x_1} + \frac{\partial T_3^{(1)}}{\partial x_3} - T_4^{(0)} + F_3^{(1)} = \tfrac{2}{3}b^3\rho\ddot{u}_3^{(1)}$$

Finally, from (5.0116) and (5.0112) the displacement equations of motion are

$$c_{ijk\ell}\left[b\left(u_{k,\ell}^{(0)} + u_{\ell,k}^{(0)} + \delta_{2\ell}u_k^{(1)} + \delta_{2k}u_\ell^{(1)}\right]_{,i} + F_j^{(0)} = 2b\rho\ddot{u}_j^{(0)}$$
$$\bar{c}_{ijk\ell}\left[b^3\left(u_{k,\ell}^{(1)} + u_{\ell,k}^{(1)}\right)\right]_{,i} - 3bc_{2jk\ell}\left(u_{k,\ell}^{(0)} + u_{\ell,k}^{(0)} + \delta_{2\ell}u_k^{(1)} + \delta_{2k}u_\ell^{(1)}\right] + \underset{3}{F_j^{(1)}} = 2b^3\rho\ddot{u}_j^{(1)} \qquad (5.0118)$$

5.02 Adjustment of Upper Modes

The six equations of motion (5.0118) on the six displacements $(u_j^{(0)}, u_j^{(1)})$ yield six modes of vibration of an infinite plate, each mode with its own relation between frequency and wave-length. As the wave-length approaches infinity, three of the frequencies approach zero and three do not. The frequencies for the limiting case $\xi = 0$ are readily found from (5.0118) by setting

$$F_j^{(0)} = F_j^{(1)} = 0$$
$$u_j^{(0)} = A_j^{(0)} e^{i\omega_r t}, \quad r = 1,2,3 \qquad (5.021)$$
$$u_j^{(1)} = A_j^{(1)} e^{i\omega_s t}, \quad s = 1,2,3$$

where $A_j^{(0)}$ and $A_j^{(1)}$ are constants. Then (5.0118) reduce to

$$A_j^{(0)} \omega_r^2 = 0$$
$$3 c_{2jk\ell}(\delta_{2\ell} A_k^{(1)} + \delta_{2k} A_\ell^{(1)}) = 2\rho_1 b^2 \omega^2 A_j^{(1)} \qquad (5.022)$$

Hence, when $A_j^{(0)} \neq 0$, $A_j^{(1)} \neq 0$,

$$\omega_r = 0, \quad r = 1,2,3$$
$$\omega_s = \frac{\sqrt{3}}{b}\sqrt{\frac{c_s}{\rho_1}}, \quad s = 1,2,3 \qquad (5.023)$$

where the c_s are the roots of

$$\begin{vmatrix} c_{66} - c_s & c_{26} & c_{46} \\ c_{26} & c_{22} - c_s & c_{24} \\ c_{46} & c_{24} & c_{44} - c_s \end{vmatrix} = 0 \qquad (5.024)$$

The three non-zero frequencies should be those of the three fundamental modes of simple thickness-vibration, that is, they should be given by (2.051) with $n = 1$. However, it may be seen that the frequencies are in error by a factor $\pi/\sqrt{12}$. The reasons for the discrepancy, and its remedy, can be found from a study of the limiting frequencies and mode-shapes obtained from the solution of the three-dimensional equations. It is sufficient to examine the case of isotropic, plane strain (Section 2.11).

In the case of flexural vibrations of an isotropic plate, in a state of plane strain, the relation between frequency and wave-length of the fundamental mode is illustrated, for example, by the lowest dashed curve in Fig. 2.113(a). As the wave-length increases, the frequency approaches zero. At the same time the displacement normal to the plate (u_2) approaches the form $u_2^{(0)} + x_2^2 u_2^{(2)}$ and the displacement parallel to the plate (u_1) approaches the form $x_2 u_1^{(1)}$. Similarly, for the fundamental extensional mode, as $\xi b \to 0$, $u_2 \to x_2 u_2^{(1)}$ and $u_1 \to u_1^{(0)}$. The relation between frequency and wave-length of the fundamental extensional mode is illustrated by the lowest full curve in Fig. 2.113(a). Thus, for both flexural and extensional fundamental modes, the exact solution, for long wave-lengths, is described by just those displacements that appear in the first-order approximation. Consequently, we may expect the first-order approximation to be very good for long waves of the lower modes (see Section 5.7).

Turning, now, to the second flexural and extensional modes, the relations between frequency and wave-length, in the exact solution, are given by the next higher dashed and full curves in Fig. 2.113(a). As $\xi b \to 0$ these modes approach the fundamental simple thickness-shear and thickness-stretch modes. In this case, as $\xi b \to 0$,

$$
\begin{aligned}
u_1 &\to A_1 \sin\frac{\pi x_2}{2b}, \qquad \omega \to \frac{\pi}{2b}\sqrt{\frac{\mu}{\rho}} \\
u_2 &\to A_2 \sin\frac{\pi x_2}{2b}, \qquad \omega \to \frac{\pi}{2b}\sqrt{\frac{\lambda+2\mu}{\rho}}
\end{aligned}
\qquad (5.025)
$$

The corresponding displacements in the first-order approximation are $u_1^{(1)}$ and $u_2^{(1)}$ which represent displacements linear in x_2 and lead to limiting thickness-frequencies

$$
\begin{aligned}
\omega &\to \frac{\sqrt{3}}{b}\sqrt{\frac{\mu}{\rho}} \\
\omega &\to \frac{\sqrt{3}}{b}\sqrt{\frac{\lambda+2\mu}{\rho}}
\end{aligned}
\qquad (5.026)
$$

The error, a factor $\pi/\sqrt{12}$, is seen to be due to the incorrect variation of displacement through the thickness.

If an expansion in a series of trigonometric functions, instead of powers, had been employed, we could have obtained correct frequencies for the simple thickness-modes but the frequencies of the fundamental flexural and extensional modes would have been in error. We can, however, make adjustments which will correct the limiting frequencies of the upper modes without affecting the fundamental flexural and extensional frequencies at long wave-lengths.

We proceed to trace the influence of the displacements on the simple thickness-frequencies. The latter are determined by the kinetic energy and strain-energy densities. In the kinetic energy only the first-order terms $\frac{1}{3} b^3 \rho_1 \dot{u}_j^{(1)} \dot{u}_j^{(1)}$ contribute to the simple thickness-modes and, in the strain-energy, only the zero-order strain-components $S_{2j}^{(0)}$ contribute. In the expression (5.023) for the simple thickness-frequencies, the kinetic energy terms are represented by the first-order density and the strain-energy terms are represented by the elastic constants c_s which, in turn, depend only on c_{pq}, $p,q = 2,4,6$ as shown in (5.024).

Thus, we can correct the limiting frequencies of the upper modes by replacing ρ_1, in the kinetic energy-density, by ρ/κ^2 where $\kappa = \pi/\sqrt{12}$, or by replacing the $S_{2j}^{(0)}$, in the strain-energy, by $\kappa\, S_{2j}^{(0)}$. Neither of these alterations affects the fundamental flexural and extensional modes at long wave-lengths inasmuch as the altered terms contribute negligible amounts to the total energy-densities of the lower modes when the wave-length is long. In this way, we may adjust the higher modes without appreciable influence on the lower modes in the range of wave-lengths to which the equations are restricted in any event.

To effect the adjustment of the strain-energy, we have to replace the strain-energy-density (5.0113) by

$$\overline{U} = b\,\kappa^{\alpha}\kappa^{\beta} c_{ijkl} S_{ij}^{(0)} S_{kl}^{(0)} + \frac{1}{3} b^3 \overline{c}_{ijkl} S_{ij}^{(1)} S_{kl}^{(1)} \qquad (5.027)$$

where

$$\alpha = \cos^2(ij\pi/2)$$
$$\beta = \cos^2(k\ell\pi/2) \qquad (5.028)$$
$$\kappa = \pi/\sqrt{12}$$

The powers α and β serve to distribute the factor κ to the thickness-strains $S_{2j}^{(0)}$ only and to multiply those strains by κ if the strain appears to the first power and by κ^2 if the strain appears to the second power.

In the abbreviated notation, (5.027) becomes

$$\overline{U} = b\kappa^{\alpha}\kappa^{\beta}c_{pq}S_p^{(0)}S_q^{(0)} + \tfrac{1}{3}b^3\overline{c}_{pq}S_p^{(1)}S_q^{(1)} \qquad (5.029)$$

where

$$\alpha = \cos^2(p\pi/2)$$
$$\beta = \cos^2(q\pi/2) \qquad (5.0210)$$
$$\kappa = \pi/\sqrt{12}$$

Then the stresses are given by

$$T_{ij}^{(0)} = \frac{\partial \overline{U}}{\partial S_{ij}^{(0)}} = 2b\kappa^{\alpha}\kappa^{\beta}c_{ijk\ell}S_{k\ell}^{(0)}$$
$$T_{ij}^{(1)} = \frac{\partial \overline{U}}{\partial S_{ij}^{(1)}} = \frac{2}{3}b^3\overline{c}_{ijk\ell}S_{k\ell}^{(1)} \qquad (5.0211)$$

or

$$T_p^{(0)} = \frac{\partial \overline{U}}{\partial S_p^{(0)}} = 2b\kappa^{\alpha}\kappa^{\beta}c_{pq}S_q^{(0)}$$
$$T_p^{(1)} = \frac{\partial \overline{U}}{\partial S_p^{(1)}} = \frac{2}{3}b^3\overline{c}_{pq}S_q^{(1)} \qquad (5.0212)$$

The stress-displacement relations become

$$T_{ij}^{(0)} = b\kappa^{\alpha}\kappa^{\beta}c_{ijk\ell}\left(u_{k,\ell}^{(0)} + u_{\ell,k}^{(0)} + \delta_{2\ell}u_k^{(1)} + \delta_{2k}u_\ell^{(1)}\right)$$
$$T_{ij}^{(1)} = \tfrac{1}{3}b^3\overline{c}_{ijk\ell}\left(u_{k,\ell}^{(1)} + u_{\ell,k}^{(1)}\right) \qquad (5.0213)$$

and the stress-equations of motion remain unchanged.

To effect the alternative adjustment (on the kinetic energy) it is only necessary to set

$$\rho_1 = \frac{\rho}{\kappa^2} = \frac{12\rho}{\pi^2} \tag{5.0214}$$

in the equations of Section 5.01.

It is apparent that the correction on the kinetic energy is the simpler one to apply. Furthermore, it has been pointed out (Benscoter, 1955) that this correction has the advantage of disappearing in the case of equations of equilibrium and also leaves the elastic constants available for separate adjustments appropriate to static problems. Accordingly, we shall adopt (5.0214) in the sequel (except for the intermediate approximation given in Section 6.04).

5.03 Uniqueness of Solutions

The establishment of sufficient conditions for unique solutions of the equations of motion follows the pattern established in Section 4.02. We consider two sets of displacements, strains, tractions and stresses in a plate and designate their respective differences by starred symbols. Then the kinetic energy-density and the strain-energy-density of the difference system are

$$\begin{aligned} \bar{K}^* &= \rho b \dot{u}_j^{(0)*} \dot{u}_j^{(0)*} + \tfrac{1}{3}\rho_1 b^3 \dot{u}_j^{(1)*} \dot{u}_j^{(1)*} \\ \bar{U}^* &= b c_{ijkl} S_{ij}^{(0)*} S_{kl}^{(0)*} + \tfrac{1}{3} b^3 \bar{c}_{ijkl} S_{ij}^{(1)*} S_{kl}^{(1)*} \end{aligned} \tag{5.031}$$

If $\bar{\mathfrak{K}}^*$ and $\bar{\mathfrak{U}}^*$ are the total kinetic energy and strain-energy of the plate, calculated from $\bar{K}^*$ and $\bar{U}^*$ and reckoned over a time interval $t - t_0$, we have

$$\bar{\mathfrak{K}}^* + \bar{\mathfrak{U}}^* = \bar{\mathfrak{K}}^*(t_0) + \bar{\mathfrak{U}}^*(t_0) + \int_{t_0}^{t} dt \int_A (\dot{\bar{K}}^* + \dot{\bar{U}}^*)\, dA \tag{5.032}$$

Now

$$\dot{K}^* = 2\rho b \ddot{u}_j^{(0)*}\dot{u}_j^{(0)*} + \tfrac{2}{3}\rho_1 b^3 \ddot{u}_j^{(1)*}\dot{u}_j^{(1)*} \tag{5.033}$$

and

$$\begin{aligned}\dot{U}^* &= \frac{\partial U^*}{\partial S_{ij}^{(0)*}}\dot{S}_{ij}^{(0)*} + \frac{\partial U^*}{\partial S_{ij}^{(1)*}} S_{ij}^{(1)*} \\ &= \tfrac{1}{2}T_{ij}^{(0)*}(\dot{u}_{i,j}^{(0)*} + \dot{u}_{j,i}^{(0)*} + \delta_{2j}\dot{u}_i^{(1)*} + \delta_{2i}\dot{u}_j^{(1)*}) + \tfrac{1}{2}T_{ij}^{(1)}(\dot{u}_{i,j}^{(1)*} + u_{j,i}^{(1)*}) \\ &= T_{ij}^{(0)*}(\dot{u}_{j,i}^{(0)*} + \delta_{2i}\dot{u}_j^{(1)*}) + T_{ij}^{(1)}\dot{u}_{j,i}^{(1)*}\end{aligned}$$

where the last step is a consequence of the symmetry of $T_{ij}^{(0)*}$ and $T_{ij}^{(1)*}$. Noting that

$$T_{ij}^{(0)*}\delta_{2i}\dot{u}_j^{(1)*} = T_{2j}^{(0)*}\dot{u}_j^{(1)*}$$

we have

$$\dot{U}^* = (T_{ij}^{(0)*}\dot{u}_j^{(0)*} + T_{ij}^{(1)*}\dot{u}_j^{(1)*})_{,i} - T_{ij,i}^{(0)*}\dot{u}_j^{(0)*} + (T_{2j}^{(0)*} - T_{ij,i}^{(1)*})\dot{u}_j^{(1)*}$$

which is converted, through the use of the equations of motion (5.0116) to

$$\begin{aligned}\dot{U}^* = (T_{ij}^{(0)*}\dot{u}_j^{(0)*} + T_{ij}^{(1)*}\dot{u}_j^{(1)*})_{,i} + F_j^{(0)*}\dot{u}_j^{(0)*} + F_j^{(1)*}\dot{u}_j^{(1)*} \\ - 2\rho b\ddot{u}_j^{(0)*}\dot{u}_j^{(0)*} - \tfrac{2}{3}\rho_1 b^3\ddot{u}_j^{(1)*}\dot{u}_j^{(1)*}\end{aligned} \tag{5.034}$$

Hence, from (5.033) and (5.034),

$$\int_A(\dot{K}^* + \dot{U}^*)\,dA = \int_A[(T_{ij}^{(0)*}\dot{u}_j^{(0)*} + T_{ij}^{(1)*}\dot{u}_j^{(1)*})_{,i} + F_j^{(0)*}\dot{u}_j^{(0)*} + F_j^{(1)*}\dot{u}_j^{(1)*}]\,dA$$

Then, noting that $(\,)_{,i} = 0$ when $i = 2$, we have

$$\int_A(T_{ij}^{(0)*}\dot{u}_j^{(0)*} + T_{ij}^{(1)*}\dot{u}_j^{(1)*})_{,i}\,dA = \oint \nu_i(T_{ij}^{(0)*}\dot{u}_j^{(0)*} + T_{ij}^{(1)*}\dot{u}_j^{(1)*})\,ds \;, \quad \nu_2 = 0$$

where s is the coordinate along the edge of the plate. The integrand of the line integral may be expressed in terms of coordinates $n, s, 2$ where n is the outward-drawn normal to the edge:

$$\nu_i\left(T_{ij}^{(0)*}\dot{u}_j^{(0)*} + T_{ij}^{(1)*}\dot{u}_j^{(1)*}\right) = \nu_h\left(T_{hg}^{(0)*}\dot{u}_g^{(0)*} + T_{hg}^{(1)*}\dot{u}_j^{(1)*}\right), \quad h,g = n,s,z$$
$$= T_{ng}^{(0)*}\dot{u}_g^{(0)*} + T_{ng}^{(1)*}\dot{u}_g^{(1)*}$$

since $\nu_n = 1$, $\nu_s = 0$, $\nu_z = 0$. Hence

$$\bar{\mathfrak{K}}^* + \mathfrak{U}^* = \mathfrak{K}^*(t_0) + \mathfrak{U}^*(t_0) + \int_{t_0}^{t} dt \oint \left(T_{ng}^{(0)*}\dot{u}_g^{(0)*} + T_{ng}^{(1)*}\dot{u}_g^{(1)*}\right) ds$$
$$+ \int_{t_0}^{t} dt \int_A \left(F_j^{(0)*}\dot{u}_j^{(0)*} + F_j^{(1)*}\dot{u}_j^{(1)*}\right) dA \qquad (5.035)$$

Accordingly, by the same argument as that employed in Section 1.05, sufficient conditions for a unique solution of the equations of motion are (see Fig. 3.062)

a. Specification of the initial displacements $u_j^{(0)}$, $u_j^{(1)}$ and initial velocities $\dot{u}_j^{(0)}$, $\dot{u}_j^{(1)}$ throughout the plate.

b. Specification of one member of each of the pairs $F_1^{(0)}u_1^{(0)}$, $F_2^{(0)}u_2^{(0)}$, $F_3^{(0)}u_3^{(0)}$, $F_1^{(1)}u_1^{(1)}$, $F_2^{(1)}u_2^{(1)}$, $F_3^{(1)}u_3^{(1)}$ throughout the plate.

c. Specification, at each point of the edge, of one member of each of the pairs $T_{nn}^{(0)}u_n^{(0)}$, $T_{ns}^{(0)}u_s^{(0)}$, $T_{nz}^{(0)}u_z^{(0)}$, $T_{nn}^{(1)}u_n^{(1)}$, $T_{ns}^{(1)}u_s^{(1)}$, $T_{nz}^{(1)}u_z^{(1)}$.

5.04 Stress-Strain Relations

The scalar expression of the stress-strain relation is obtained by expanding (5.0110) and the first of (5.018).

For the triclinic crystal, we have

$$
\begin{aligned}
T_1^{(0)} &= 2b\left[c_{11}S_1^{(0)} + c_{12}S_2^{(0)} + c_{13}S_3^{(0)} + c_{14}S_4^{(0)} + c_{15}S_5^{(0)} + c_{16}S_6^{(0)}\right] \\
T_2^{(0)} &= 2b\left[c_{21}S_1^{(0)} + c_{22}S_2^{(0)} + c_{23}S_3^{(0)} + c_{24}S_4^{(0)} + c_{25}S_5^{(0)} + c_{26}S_6^{(0)}\right] \\
T_3^{(0)} &= 2b\left[c_{31}S_1^{(0)} + c_{32}S_2^{(0)} + c_{33}S_3^{(0)} + c_{34}S_4^{(0)} + c_{35}S_5^{(0)} + c_{36}S_6^{(0)}\right] \\
T_4^{(0)} &= 2b\left[c_{41}S_1^{(0)} + c_{42}S_2^{(0)} + c_{43}S_3^{(0)} + c_{44}S_4^{(0)} + c_{45}S_5^{(0)} + c_{46}S_6^{(0)}\right] \\
T_5^{(0)} &= 2b\left[c_{51}S_1^{(0)} + c_{52}S_2^{(0)} + c_{53}S_3^{(0)} + c_{54}S_4^{(0)} + c_{55}S_5^{(0)} + c_{56}S_6^{(0)}\right] \\
T_6^{(0)} &= 2b\left[c_{61}S_1^{(0)} + c_{62}S_2^{(0)} + c_{63}S_3^{(0)} + c_{64}S_4^{(0)} + c_{65}S_5^{(0)} + c_{66}S_6^{(0)}\right]
\end{aligned}
\tag{5.041}
$$

$$
\begin{aligned}
T_1^{(1)} &= \frac{2b^3}{3}\left[\left(c_{11} - \frac{c_{12}^2}{c_{22}}\right)S_1^{(1)} + \left(c_{13} - \frac{c_{12}c_{23}}{c_{22}}\right)S_3^{(1)} + \left(c_{14} - \frac{c_{12}c_{24}}{c_{22}}\right)S_4^{(1)} + \left(c_{15} - \frac{c_{12}c_{25}}{c_{22}}\right)S_5^{(1)} + \left(c_{16} - \frac{c_{12}c_{26}}{c_{22}}\right)S_6^{(1)}\right] \\
T_3^{(1)} &= \frac{2b^3}{3}\left[\left(c_{31} - \frac{c_{32}c_{21}}{c_{22}}\right)S_1^{(1)} + \left(c_{33} - \frac{c_{23}^2}{c_{22}}\right)S_3^{(1)} + \left(c_{34} - \frac{c_{32}c_{24}}{c_{22}}\right)S_4^{(1)} + \left(c_{35} - \frac{c_{32}c_{25}}{c_{22}}\right)S_5^{(1)} + \left(c_{36} - \frac{c_{32}c_{26}}{c_{22}}\right)S_6^{(1)}\right] \\
T_4^{(1)} &= \frac{2b^3}{3}\left[\left(c_{41} - \frac{c_{42}c_{21}}{c_{22}}\right)S_1^{(1)} + \left(c_{43} - \frac{c_{42}c_{23}}{c_{22}}\right)S_3^{(1)} + \left(c_{44} - \frac{c_{24}^2}{c_{22}}\right)S_4^{(1)} + \left(c_{45} - \frac{c_{42}c_{25}}{c_{22}}\right)S_5^{(1)} + \left(c_{46} - \frac{c_{42}c_{26}}{c_{22}}\right)S_6^{(1)}\right] \\
T_5^{(1)} &= \frac{2b^3}{3}\left[\left(c_{51} - \frac{c_{52}c_{21}}{c_{22}}\right)S_1^{(1)} + \left(c_{53} - \frac{c_{52}c_{23}}{c_{22}}\right)S_3^{(1)} + \left(c_{54} - \frac{c_{52}c_{24}}{c_{22}}\right)S_4^{(1)} + \left(c_{55} - \frac{c_{25}^2}{c_{22}}\right)S_5^{(1)} + \left(c_{56} - \frac{c_{52}c_{26}}{c_{22}}\right)S_6^{(1)}\right] \\
T_6^{(1)} &= \frac{2b^3}{3}\left[\left(c_{61} - \frac{c_{62}c_{21}}{c_{22}}\right)S_1^{(1)} + \left(c_{63} - \frac{c_{62}c_{23}}{c_{22}}\right)S_3^{(1)} + \left(c_{64} - \frac{c_{62}c_{24}}{c_{22}}\right)S_4^{(1)} + \left(c_{65} - \frac{c_{62}c_{25}}{c_{22}}\right)S_5^{(1)} + \left(c_{66} - \frac{c_{26}^2}{c_{22}}\right)S_6^{(1)}\right]
\end{aligned}
$$

For the monoclinic crystal with the x_1-axis the axis of two-fold symmetry,

$$c_{15} = c_{16} = c_{25} = c_{26} = c_{35} = c_{36} = c_{45} = c_{46} = 0 \qquad (5.042)$$

Hence

$$\begin{aligned}
T_1^{(0)} &= 2b\left[c_{11}S_1^{(0)} + c_{12}S_2^{(0)} + c_{13}S_3^{(0)} + c_{14}S_4^{(0)}\right]\\
T_2^{(0)} &= 2b\left[c_{21}S_1^{(0)} + c_{22}S_2^{(0)} + c_{23}S_3^{(0)} + c_{24}S_4^{(0)}\right]\\
T_3^{(0)} &= 2b\left[c_{31}S_1^{(0)} + c_{32}S_2^{(0)} + c_{33}S_3^{(0)} + c_{34}S_4^{(0)}\right]\\
T_4^{(0)} &= 2b\left[c_{41}S_1^{(0)} + c_{42}S_2^{(0)} + c_{43}S_3^{(0)} + c_{44}S_4^{(0)}\right]\\
T_5^{(0)} &= 2b\left[c_{55}S_5^{(0)} + c_{56}S_6^{(0)}\right]\\
T_6^{(0)} &= 2b\left[c_{65}S_5^{(0)} + c_{66}S_6^{(0)}\right]
\end{aligned} \qquad (5.043)$$

$$\begin{aligned}
T_1^{(1)} &= \frac{2b^3}{3}\left[\left(c_{11} - \frac{c_{12}^2}{c_{22}}\right)S_1^{(1)} + \left(c_{13} - \frac{c_{12}c_{23}}{c_{22}}\right)S_3^{(1)} + \left(c_{14} - \frac{c_{12}c_{24}}{c_{22}}\right)S_4^{(1)}\right]\\
T_3^{(1)} &= \frac{2b^3}{3}\left[\left(c_{31} - \frac{c_{32}c_{21}}{c_{22}}\right)S_1^{(1)} + \left(c_{33} - \frac{c_{23}^2}{c_{22}}\right)S_3^{(1)} + \left(c_{34} - \frac{c_{32}c_{24}}{c_{22}}\right)S_4^{(1)}\right]\\
T_4^{(1)} &= \frac{2b^3}{3}\left[\left(c_{41} - \frac{c_{42}c_{21}}{c_{22}}\right)S_1^{(1)} + \left(c_{43} - \frac{c_{42}c_{23}}{c_{22}}\right)S_3^{(1)} + \left(c_{44} - \frac{c_{24}^2}{c_{22}}\right)S_4^{(1)}\right]\\
T_5^{(1)} &= \frac{2b^3}{3}\left[c_{55}S_5^{(1)} + c_{56}S_6^{(1)}\right.\\
T_6^{(1)} &= \frac{2b^3}{3}\left[c_{65}S_5^{(1)} + c_{66}S_6^{(1)}\right]
\end{aligned}$$

In the isotropic case we have, in addition to (5.042),

$$\begin{aligned} c_{14} &= c_{24} = c_{34} = c_{56} = 0 \\ c_{12} &= c_{13} = c_{23} \\ c_{11} &= c_{22} = c_{33} \\ c_{44} &= c_{55} = c_{66} = \tfrac{1}{2}(c_{11} - c_{12}) \end{aligned} \qquad (5.044)$$

Then

$$\begin{aligned}
T_1^{(0)} &= 2b[c_{11} S_1^{(0)} + c_{12}(S_2^{(0)} + S_3^{(0)})] = 2b[(\lambda+2\mu) S_1^{(0)} + \lambda(S_2^{(0)} + S_3^{(0)})] \\
T_2^{(0)} &= 2b[c_{11} S_2^{(0)} + c_{12}(S_3^{(0)} + S_1^{(0)})] = 2b[(\lambda+2\mu) S_2^{(0)} + \lambda(S_3^{(0)} + S_1^{(0)})] \\
T_3^{(0)} &= 2b[c_{11} S_3^{(0)} + c_{12}(S_1^{(0)} + S_2^{(0)})] = 2b[(\lambda+2\mu) S_3^{(0)} + \lambda(S_1^{(0)} + S_2^{(0)})] \\
T_4^{(0)} &= b(c_{11} - c_{12}) S_4^{(0)} = 2b\mu S_4^{(0)} \\
T_5^{(0)} &= b(c_{11} - c_{12}) S_5^{(0)} = 2b\mu S_5^{(0)} \\
T_6^{(0)} &= b(c_{11} - c_{12}) S_6^{(0)} = 2b\mu S_6^{(0)} \\
T_1^{(1)} &= \frac{2b^3}{3}\left[\left(c_{11} - \frac{c_{12}^2}{c_{11}}\right) S_1^{(1)} + \left(c_{12} - \frac{c_{12}^2}{c_{11}}\right) S_3^{(1)}\right] = \frac{2b^3 E}{3(1-\nu^2)}(S_1^{(1)} + \nu S_3^{(1)}) \\
T_3^{(1)} &= \frac{2b^3}{3}\left[\left(c_{12} - \frac{c_{12}^2}{c_{11}}\right) S_1^{(1)} + \left(c_{11} - \frac{c_{12}^2}{c_{11}}\right) S_3^{(1)}\right] = \frac{2b^3 E}{3(1-\nu^2)}(S_3^{(1)} + \nu S_1^{(1)}) \\
T_4^{(1)} &= \tfrac{1}{3} b^3 (c_{11} - c_{12}) S_4^{(1)} = \tfrac{2}{3} b^3 \mu S_4^{(1)} \\
T_5^{(1)} &= \tfrac{1}{3} b^3 (c_{11} - c_{12}) S_5^{(1)} = \tfrac{2}{3} b^3 \mu S_5^{(1)} \\
T_6^{(1)} &= \tfrac{1}{3} b^3 (c_{11} - c_{12}) S_6^{(1)} = \tfrac{2}{3} b^3 \mu S_6^{(1)}
\end{aligned} \qquad (5.045)$$

5.05 Stress-Displacement Relations

Triclinic:

$$T_1^{(0)} = 2b\left[C_{11}\frac{\partial u_1^{(0)}}{\partial x_1} + C_{12}u_2^{(1)} + C_{13}\frac{\partial u_3^{(0)}}{\partial x_3} + C_{14}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + C_{15}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + C_{16}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right]$$

$$T_2^{(0)} = 2b\left[C_{21}\frac{\partial u_1^{(0)}}{\partial x_1} + C_{22}u_2^{(1)} + C_{23}\frac{\partial u_3^{(0)}}{\partial x_3} + C_{24}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + C_{25}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + C_{26}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right]$$

$$T_3^{(0)} = 2b\left[C_{31}\frac{\partial u_1^{(0)}}{\partial x_1} + C_{32}u_2^{(1)} + C_{33}\frac{\partial u_3^{(0)}}{\partial x_3} + C_{34}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + C_{35}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + C_{36}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right]$$

$$T_4^{(0)} = 2b\left[C_{41}\frac{\partial u_1^{(0)}}{\partial x_1} + C_{42}u_2^{(1)} + C_{43}\frac{\partial u_3^{(0)}}{\partial x_3} + C_{44}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + C_{45}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + C_{46}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right]$$

$$T_5^{(0)} = 2b\left[C_{51}\frac{\partial u_1^{(0)}}{\partial x_1} + C_{52}u_2^{(1)} + C_{53}\frac{\partial u_3^{(0)}}{\partial x_3} + C_{54}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + C_{55}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + C_{56}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right]$$

$$T_6^{(0)} = 2b\left[C_{61}\frac{\partial u_1^{(0)}}{\partial x_1} + C_{62}u_2^{(1)} + C_{63}\frac{\partial u_3^{(0)}}{\partial x_3} + C_{64}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + C_{65}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + C_{66}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right]$$

(5.051)

$$T_1^{(1)} = \frac{2b^3}{3}\left[\left(C_{11} - \frac{C_{12}^2}{C_{22}}\right)\frac{\partial u_1^{(1)}}{\partial x_1} + \left(C_{13} - \frac{C_{12}C_{23}}{C_{22}}\right)\frac{\partial u_3^{(1)}}{\partial x_3} + \left(C_{14} - \frac{C_{12}C_{24}}{C_{22}}\right)\frac{\partial u_2^{(1)}}{\partial x_3} + \left(C_{15} - \frac{C_{12}C_{25}}{C_{22}}\right)\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right) + \left(C_{16} - \frac{C_{12}C_{26}}{C_{22}}\right)\frac{\partial u_2^{(1)}}{\partial x_1}\right]$$

$$T_3^{(1)} = \frac{2b^3}{3}\left[\left(C_{31} - \frac{C_{32}C_{21}}{C_{22}}\right)\frac{\partial u_1^{(1)}}{\partial x_1} + \left(C_{33} - \frac{C_{23}^2}{C_{22}}\right)\frac{\partial u_3^{(1)}}{\partial x_3} + \left(C_{34} - \frac{C_{32}C_{24}}{C_{22}}\right)\frac{\partial u_2^{(1)}}{\partial x_3} + \left(C_{35} - \frac{C_{32}C_{25}}{C_{22}}\right)\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right) + \left(C_{36} - \frac{C_{32}C_{26}}{C_{22}}\right)\frac{\partial u_2^{(1)}}{\partial x_1}\right]$$

$$T_4^{(1)} = \frac{2b^3}{3}\left[\left(C_{41} - \frac{C_{42}C_{21}}{C_{22}}\right)\frac{\partial u_1^{(1)}}{\partial x_1} + \left(C_{43} - \frac{C_{42}C_{23}}{C_{22}}\right)\frac{\partial u_3^{(1)}}{\partial x_3} + \left(C_{44} - \frac{C_{24}^2}{C_{22}}\right)\frac{\partial u_2^{(1)}}{\partial x_3} + \left(C_{45} - \frac{C_{42}C_{25}}{C_{22}}\right)\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right) + \left(C_{46} - \frac{C_{42}C_{26}}{C_{22}}\right)\frac{\partial u_2^{(1)}}{\partial x_1}\right]$$

$$T_5^{(1)} = \frac{2b^3}{3}\left[\left(C_{51} - \frac{C_{52}C_{21}}{C_{22}}\right)\frac{\partial u_1^{(1)}}{\partial x_1} + \left(C_{53} - \frac{C_{52}C_{23}}{C_{22}}\right)\frac{\partial u_3^{(1)}}{\partial x_3} + \left(C_{54} - \frac{C_{52}C_{24}}{C_{22}}\right)\frac{\partial u_2^{(1)}}{\partial x_3} + \left(C_{55} - \frac{C_{25}^2}{C_{22}}\right)\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right) + \left(C_{56} - \frac{C_{52}C_{26}}{C_{22}}\right)\frac{\partial u_2^{(1)}}{\partial x_1}\right]$$

$$T_6^{(1)} = \frac{2b^3}{3}\left[\left(C_{61} - \frac{C_{62}C_{21}}{C_{22}}\right)\frac{\partial u_1^{(1)}}{\partial x_1} + \left(C_{63} - \frac{C_{62}C_{23}}{C_{22}}\right)\frac{\partial u_2^{(1)}}{\partial x_3} + \left(C_{64} - \frac{C_{62}C_{24}}{C_{22}}\right)\frac{\partial u_2^{(1)}}{\partial x_3} + \left(C_{65} - \frac{C_{62}C_{25}}{C_{22}}\right)\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right) + \left(C_{66} - \frac{C_{26}^2}{C_{22}}\right)\frac{\partial u_3^{(1)}}{\partial x_1}\right]$$

Monoclinic:

$$T_1^{(0)} = 2b\left[c_{11}\frac{\partial u_1^{(0)}}{\partial x_1} + c_{12}u_2^{(1)} + c_{13}\frac{\partial u_3^{(0)}}{\partial x_3} + c_{14}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right)\right]$$

$$T_2^{(0)} = 2b\left[c_{21}\frac{\partial u_1^{(0)}}{\partial x_1} + c_{22}u_2^{(1)} + c_{23}\frac{\partial u_3^{(0)}}{\partial x_3} + c_{24}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right)\right]$$

$$T_3^{(0)} = 2b\left[c_{31}\frac{\partial u_1^{(0)}}{\partial x_1} + c_{32}u_2^{(1)} + c_{33}\frac{\partial u_3^{(0)}}{\partial x_3} + c_{34}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right)\right]$$

$$T_4^{(0)} = 2b\left[c_{41}\frac{\partial u_1^{(0)}}{\partial x_1} + c_{42}u_2^{(1)} + c_{43}\frac{\partial u_3^{(0)}}{\partial x_3} + c_{44}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right)\right]$$

$$T_5^{(0)} = 2b\left[c_{55}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + c_{56}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right]$$

$$T_6^{(0)} = 2b\left[c_{65}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + c_{66}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right]$$

(5.052)

$$T_1^{(1)} = \frac{2b^3}{3}\left[\left(c_{11} - \frac{c_{12}^2}{c_{22}}\right)\frac{\partial u_1^{(1)}}{\partial x_1} + \left(c_{13} - \frac{c_{12}c_{23}}{c_{22}}\right)\frac{\partial u_3^{(1)}}{\partial x_3} + \left(c_{14} - \frac{c_{12}c_{24}}{c_{22}}\right)\frac{\partial u_2^{(1)}}{\partial x_3}\right]$$

$$T_3^{(1)} = \frac{2b^3}{3}\left[\left(c_{31} - \frac{c_{32}c_{21}}{c_{22}}\right)\frac{\partial u_1^{(1)}}{\partial x_1} + \left(c_{33} - \frac{c_{23}^2}{c_{22}}\right)\frac{\partial u_3^{(1)}}{\partial x_3} + \left(c_{34} - \frac{c_{32}c_{24}}{c_{22}}\right)\frac{\partial u_2^{(1)}}{\partial x_3}\right]$$

$$T_4^{(1)} = \frac{2b^3}{3}\left[\left(c_{41} - \frac{c_{42}c_{21}}{c_{22}}\right)\frac{\partial u_1^{(1)}}{\partial x_1} + \left(c_{43} - \frac{c_{42}c_{23}}{c_{22}}\right)\frac{\partial u_3^{(1)}}{\partial x_3} + \left(c_{44} - \frac{c_{24}^2}{c_{22}}\right)\frac{\partial u_2^{(1)}}{\partial x_3}\right]$$

$$T_5^{(1)} = \frac{2b^3}{3}\left[c_{55}\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right) + c_{56}\frac{\partial u_2^{(1)}}{\partial x_3}\right]$$

$$T_6^{(1)} = \frac{2b^3}{3}\left[c_{65}\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right) + c_{66}\frac{\partial u_2^{(1)}}{\partial x_3}\right]$$

Isotropic:

$$T_1^{(0)} = 2b\left[(\lambda+2\mu)\frac{\partial u_1^{(0)}}{\partial x_1} + \lambda\left(u_2^{(1)} + \frac{\partial u_3^{(0)}}{\partial x_3}\right)\right]$$

$$T_2^{(0)} = 2b\left[(\lambda+2\mu)\,u_2^{(1)} + \lambda\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right)\right]$$

$$T_3^{(0)} = 2b\left[(\lambda+2\mu)\frac{\partial u_3^{(0)}}{\partial x_3} + \lambda\left(\frac{\partial u_1^{(0)}}{\partial x_1} + u_2^{(1)}\right)\right]$$

$$T_4^{(0)} = 2b\mu\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right)$$

$$T_5^{(0)} = 2b\mu\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right)$$

$$T_6^{(0)} = 2b\mu\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)$$

$$(5.053)$$

$$T_1^{(1)} = D\left(\frac{\partial u_1^{(1)}}{\partial x_1} + \nu\,\frac{\partial u_3^{(1)}}{\partial x_3}\right)$$

$$T_3^{(1)} = D\left(\frac{\partial u_3^{(1)}}{\partial x_3} + \nu\,\frac{\partial u_1^{(1)}}{\partial x_1}\right)$$

$$T_4^{(1)} = \frac{2}{3}b^3\mu\,\frac{\partial u_2^{(1)}}{\partial x_3}$$

$$T_5^{(1)} = \frac{D(1-\nu)}{2}\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right)$$

$$T_6^{(1)} = \frac{2}{3}b^3\mu\,\frac{\partial u_2^{(1)}}{\partial x_1}$$

where

$$D = \frac{2b^3E}{3(1-\nu^2)} \qquad (5.054)$$

5.06 Displacement-Equations of Motion

The displacement-equations of motion are obtained by substituting the stress-displacement relations (Section 5.05) in the stress-equations of motion (5.0117). The equations which follow are for plates of uniform thickness.

For the triclinic crystal, we find

$$
\begin{aligned}
&\frac{\partial}{\partial x_1}\left[c_{11}\frac{\partial u_1^{(0)}}{\partial x_1} + c_{12}u_2^{(1)} + c_{13}\frac{\partial u_3^{(0)}}{\partial x_3} + c_{14}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + c_{15}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + c_{16}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right] \\
&\quad + \frac{\partial}{\partial x_3}\left[c_{51}\frac{\partial u_1^{(0)}}{\partial x_1} + c_{52}u_2^{(1)} + c_{53}\frac{\partial u_3^{(0)}}{\partial x_3} + c_{54}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + c_{55}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + c_{56}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right] + \frac{F_1^{(0)}}{2b} = \rho\ddot{u}_1^{(0)} \\
&\frac{\partial}{\partial x_1}\left[c_{61}\frac{\partial u_1^{(0)}}{\partial x_1} + c_{62}u_2^{(1)} + c_{63}\frac{\partial u_3^{(0)}}{\partial x_3} + c_{64}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + c_{65}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + c_{66}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right] \\
&\quad + \frac{\partial}{\partial x_3}\left[c_{41}\frac{\partial u_1^{(0)}}{\partial x_1} + c_{42}u_2^{(1)} + c_{43}\frac{\partial u_3^{(0)}}{\partial x_3} + c_{44}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + c_{45}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + c_{46}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right] + \frac{F_2^{(0)}}{2b} = \rho\ddot{u}_2^{(0)} \\
&\frac{\partial}{\partial x_1}\left[c_{51}\frac{\partial u_1^{(0)}}{\partial x_1} + c_{52}u_2^{(1)} + c_{53}\frac{\partial u_3^{(0)}}{\partial x_3} + c_{54}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + c_{55}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + c_{56}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right] \\
&\quad + \frac{\partial}{\partial x_3}\left[c_{31}\frac{\partial u_1^{(0)}}{\partial x_1} + c_{32}u_2^{(1)} + c_{33}\frac{\partial u_3^{(0)}}{\partial x_3} + c_{34}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + c_{35}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + c_{36}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right] + \frac{F_3^{(0)}}{2b} = \rho\ddot{u}_3^{(0)}
\end{aligned}
\tag{5.061}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial x_1}\left[\bar{c}_{11}\frac{\partial u_1^{(1)}}{\partial x_1} + \bar{c}_{13}\frac{\partial u_3^{(1)}}{\partial x_3} + \bar{c}_{14}\frac{\partial u_2^{(1)}}{\partial x_3} + \bar{c}_{15}\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right) + \bar{c}_{16}\frac{\partial u_2^{(1)}}{\partial x_1}\right] + \frac{\partial}{\partial x_3}\left[\bar{c}_{51}\frac{\partial u_1^{(1)}}{\partial x_1} + \bar{c}_{53}\frac{\partial u_3^{(1)}}{\partial x_3} + \bar{c}_{54}\frac{\partial u_2^{(1)}}{\partial x_3} + \bar{c}_{55}\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right) + \bar{c}_{56}\frac{\partial u_2^{(1)}}{\partial x_1}\right] \\
&\quad - \frac{3}{b^2}\left[c_{61}\frac{\partial u_1^{(0)}}{\partial x_1} + c_{62}u_2^{(1)} + c_{63}\frac{\partial u_3^{(0)}}{\partial x_3} + c_{64}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + c_{65}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + c_{66}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right] + \frac{3F_1^{(1)}}{2b^3} = \rho_1\ddot{u}_1^{(1)} \\
&\frac{\partial}{\partial x_1}\left[\bar{c}_{61}\frac{\partial u_1^{(1)}}{\partial x_1} + \bar{c}_{63}\frac{\partial u_3^{(1)}}{\partial x_3} + \bar{c}_{64}\frac{\partial u_2^{(1)}}{\partial x_3} + \bar{c}_{65}\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right) + \bar{c}_{66}\frac{\partial u_2^{(1)}}{\partial x_1}\right] + \frac{\partial}{\partial x_3}\left[\bar{c}_{41}\frac{\partial u_1^{(1)}}{\partial x_1} + \bar{c}_{43}\frac{\partial u_3^{(1)}}{\partial x_3} + \bar{c}_{44}\frac{\partial u_2^{(1)}}{\partial x_3} + \bar{c}_{45}\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right) + \bar{c}_{46}\frac{\partial u_2^{(1)}}{\partial x_1}\right] \\
&\quad - \frac{3}{b^2}\left[c_{21}\frac{\partial u_1^{(0)}}{\partial x_1} + c_{22}u_2^{(1)} + c_{23}\frac{\partial u_3^{(0)}}{\partial x_3} + c_{24}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + c_{25}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + c_{26}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right] + \frac{3F_2^{(1)}}{2b^3} = \rho_1\ddot{u}_2^{(1)} \\
&\frac{\partial}{\partial x_1}\left[\bar{c}_{51}\frac{\partial u_1^{(1)}}{\partial x_1} + \bar{c}_{53}\frac{\partial u_3^{(1)}}{\partial x_3} + \bar{c}_{54}\frac{\partial u_2^{(1)}}{\partial x_3} + \bar{c}_{55}\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right) + \bar{c}_{56}\frac{\partial u_2^{(1)}}{\partial x_1}\right] + \frac{\partial}{\partial x_3}\left[\bar{c}_{31}\frac{\partial u_1^{(1)}}{\partial x_1} + \bar{c}_{33}\frac{\partial u_3^{(1)}}{\partial x_3} + \bar{c}_{34}\frac{\partial u_2^{(1)}}{\partial x_3} + \bar{c}_{35}\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right) + \bar{c}_{36}\frac{\partial u_2^{(1)}}{\partial x_1}\right] \\
&\quad - \frac{3}{b^2}\left[c_{41}\frac{\partial u_1^{(0)}}{\partial x_1} + c_{42}u_2^{(1)} + c_{43}\frac{\partial u_3^{(0)}}{\partial x_3} + c_{44}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + c_{45}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + c_{46}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right] + \frac{3F_3^{(1)}}{2b^3} = \rho_1\ddot{u}_3^{(1)}
\end{aligned}
$$

It may be seen that the extensional displacements $(u_1^{(0)}, u_3^{(0)}, u_2^{(1)})$ are coupled with the flexural displacements $(u_2^{(0)}, u_1^{(1)}, u_3^{(1)})$ in (5.061) through the constants $c_{14}, c_{24}, c_{34}, c_{54}, c_{16}, c_{26}, c_{36}$ and c_{56}. The first, third and fifth of (5.061) are essentially extensional equations of motion, but involve some flexure. The second, fourth and sixth of (5.061) are essentially flexural equations of motion, but involve some extension.

For the monoclinic case, we have

$$
\begin{aligned}
&\frac{\partial}{\partial x_1}\left[c_{11}\frac{\partial u_1^{(0)}}{\partial x_1} + c_{12}u_2^{(1)} + c_{13}\frac{\partial u_3^{(0)}}{\partial x_3} + c_{14}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right)\right] \\
&\qquad + \frac{\partial}{\partial x_3}\left[c_{55}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + c_{56}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right] + \frac{F_1^{(0)}}{2b} = \rho\ddot{u}_1^{(0)} \\
&\frac{\partial}{\partial x_1}\left[c_{65}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + c_{66}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right] \\
&\qquad + \frac{\partial}{\partial x_3}\left[c_{41}\frac{\partial u_1^{(0)}}{\partial x_1} + c_{42}u_2^{(1)} + c_{43}\frac{\partial u_3^{(0)}}{\partial x_3} + c_{44}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right)\right] + \frac{F_2^{(0)}}{2b} = \rho\ddot{u}_2^{(0)} \\
&\frac{\partial}{\partial x_1}\left[c_{55}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + c_{56}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right] \\
&\qquad + \frac{\partial}{\partial x_3}\left[c_{31}\frac{\partial u_1^{(0)}}{\partial x_1} + c_{32}u_2^{(1)} + c_{33}\frac{\partial u_3^{(0)}}{\partial x_3} + c_{34}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right)\right] + \frac{F_3^{(0)}}{2b} = \rho\ddot{u}_3^{(0)}
\end{aligned}
\tag{5.062}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial x_1}\left[\bar{c}_{11}\frac{\partial u_1^{(1)}}{\partial x_1} + \bar{c}_{13}\frac{\partial u_3^{(1)}}{\partial x_3} + \bar{c}_{14}\frac{\partial u_2^{(1)}}{\partial x_3}\right] + \frac{\partial}{\partial x_3}\left[c_{55}\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right) + c_{56}\frac{\partial u_2^{(1)}}{\partial x_1}\right] \\
&\qquad - \frac{3}{b^2}\left[c_{65}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right) + c_{66}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)\right] + \frac{3F_1^{(1)}}{2b^3} = \rho_1\ddot{u}_1^{(1)} \\
&\frac{\partial}{\partial x_1}\left[c_{65}\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right) + c_{66}\frac{\partial u_2^{(1)}}{\partial x_1}\right] + \frac{\partial}{\partial x_3}\left[\bar{c}_{41}\frac{\partial u_1^{(1)}}{\partial x_1} + \bar{c}_{43}\frac{\partial u_3^{(1)}}{\partial x_3} + \bar{c}_{44}\frac{\partial u_2^{(1)}}{\partial x_3}\right] \\
&\qquad - \frac{3}{b^2}\left[c_{21}\frac{\partial u_1^{(0)}}{\partial x_1} + c_{22}u_2^{(1)} + c_{23}\frac{\partial u_3^{(0)}}{\partial x_3} + c_{24}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right)\right] + \frac{3F_2^{(1)}}{2b^3} = \rho_1\ddot{u}_2^{(1)} \\
&\frac{\partial}{\partial x_1}\left[c_{55}\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right) + c_{56}\frac{\partial u_2^{(1)}}{\partial x_1}\right] + \frac{\partial}{\partial x_3}\left[\bar{c}_{31}\frac{\partial u_1^{(1)}}{\partial x_1} + \bar{c}_{33}\frac{\partial u_3^{(1)}}{\partial x_3} + \bar{c}_{34}\frac{\partial u_2^{(1)}}{\partial x_3}\right] \\
&\qquad - \frac{3}{b^2}\left[c_{41}\frac{\partial u_1^{(0)}}{\partial x_1} + c_{42}u_2^{(1)} + c_{43}\frac{\partial u_3^{(0)}}{\partial x_3} + c_{44}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right)\right] + \frac{3F_3^{(1)}}{2b^3} = \rho_1\ddot{u}_3^{(1)}
\end{aligned}
$$

In the isotropic case, the extensional and flexural equations are not coupled. The extensional equations are (Kane and Mindlin, 1955)

$$\begin{aligned}
&\mu\nabla_1^2 u_1^{(0)} + (\lambda+\mu)\frac{\partial e^{(0)}}{\partial x_1} + \lambda\frac{\partial u_2^{(1)}}{\partial x_1} + \frac{F_1^{(0)}}{2b} = \rho\ddot{u}_1^{(0)} \\
&\mu\nabla_1^2 u_3^{(0)} + (\lambda+\mu)\frac{\partial e^{(0)}}{\partial x_3} + \lambda\frac{\partial u_2^{(1)}}{\partial x_3} + \frac{F_3^{(0)}}{2b} = \rho\ddot{u}_3^{(0)} \qquad (5.063) \\
&\mu\nabla_1^2 u_2^{(1)} - \frac{3\lambda}{b^2}e^{(0)} - \frac{3}{b^2}(\lambda+2\mu)u_2^{(1)} + \frac{3F_2^{(1)}}{2b^3} = \rho_1\ddot{u}_2^{(1)}
\end{aligned}$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2} \quad , \qquad e^{(0)} = \frac{\partial u_1^{(0)}}{\partial x_1} + \frac{\partial u_3^{(0)}}{\partial x_3} \qquad (5.064)$$

From the uniqueness theorem (Section 5.03), sufficient conditions for unique solutions of (5.063) are (see Fig. 3.062)

a. Specification of the initial displacements $u_1^{(0)}$, $u_3^{(0)}$, $u_2^{(1)}$ and initial velocities $\dot{u}_1^{(0)}$, $\dot{u}_3^{(0)}$, $\dot{u}_2^{(1)}$ throughout the plate.

b. Specification of one member of each of the three pairs $F_1^{(0)}u_1^{(0)}$, $F_3^{(0)}u_3^{(0)}$, $F_2^{(1)}u_2^{(1)}$ throughout the plate.

c. Specification, at each point of the edge, of one member of each of the three pairs $T_{nn}^{(0)}u_n^{(0)}$, $T_{ns}^{(0)}u_s^{(0)}$, $T_{n2}^{(1)}u_2^{(1)}$.

The first two of (5.063) are essentially equations of extensional motions in the plane of the plate; i.e, "face" motions. The third of (5.063) is essentially an equation of thickness-stretch motions. The two types of motion are, however, coupled, as may be seen from the presence of $u_2^{(1)}$ in the first two equations and $u_1^{(0)}$ and $u_3^{(0)}$ in the third. The coupling of face and thickness-stretch motions may be regarded from two points of view. On the one hand, the coupling may be regarded as due to elastic interaction since it is destroyed if $\lambda=0$, i.e., if Poisson's ratio is zero. When $\nu=0$ in the first two of (5.063), these equations reduce to the equations that are obtained by setting $\nu = 0$ in the first and third of (4.043) (Poisson's equations of low-frequency extensional motions of thin plates). On the other hand, the very presence of the coupling is due to the account taken of the symmetric thickness-shear stresses $T_4^{(1)}$ and $T_6^{(1)}$, the symmetric thickness-stress $T_2^{(0)}$ and the associated thickness-acceleration $\ddot{u}_2^{(1)}$. If these components (and $F_2^{(1)}$) are set equal to zero, the third of (5.063) disappears. Then, with $T_2^{(0)} = 0$, we may use the second of the stress-displacement relations (5.053) to express $u_2^{(1)}$ in terms of $u_1^{(0)}$ and $u_3^{(0)}$. If this expression for $u_2^{(1)}$ is used in the first two of (5.073), the equations of motion reduce to Poisson's equations for all values of ν. This process is, of course, equivalent to reverting to the zero-order approximation.

The flexural equations of motion, for the isotropic plate, are (Uflyand, 1948; Mindlin, 1951)

$$
\begin{aligned}
&2b\mu\left(\nabla_1^2 u_2^{(0)} + e^{(1)}\right) + F_2^{(0)} = 2b\rho\ddot{u}_2^{(0)} \\
&\frac{D}{2}\left[(1-\nu)\nabla_1^2 u_1^{(1)} + (1+\nu)\frac{\partial e^{(1)}}{\partial x_1}\right] - 2b\mu\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right) + F_1^{(1)} = \tfrac{2}{3}b^3\rho_1\ddot{u}_1^{(1)} \\
&\frac{D}{2}\left[(1-\nu)\nabla_1^2 u_3^{(1)} + (1+\nu)\frac{\partial e^{(1)}}{\partial x_3}\right] - 2b\mu\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right) + F_3^{(1)} = \tfrac{2}{3}b^3\rho_1\ddot{u}_3^{(1)}
\end{aligned}
\qquad (5.065)
$$

where

$$
e^{(1)} = \frac{\partial u_1^{(1)}}{\partial x_1} + \frac{\partial u_3^{(1)}}{\partial x_3} \qquad (5.066)
$$

Equivalent equations of equilibrium were obtained by E. Reissner (1945, 1947). Equations (5.065) may be regarded as two-dimensional analogues of the equations of flexural vibrations of beams in which the effects of rotatory inertia and transverse shear-deformation are taken into account (Bresse, 1859; Timoshenko, 1921, 1922).

From Section 5.03, we find the conditions sufficient for unique solutions of (5.065) (see Fig. 3.062)

a. Specification of the initial displacements $u_2^{(0)}$, $u_1^{(1)}$, $u_3^{(1)}$ and initial velocities $\dot{u}_2^{(0)}$, $\dot{u}_1^{(1)}$, $\dot{u}_3^{(1)}$ throughout the plate.

b. Specification of one member of each of the three pairs $F_2^{(0)} u_2^{(0)}$, $F_1^{(1)} u_1^{(1)}$, $F_3^{(1)} u_3^{(1)}$ throughout the plate.

c. Specification, at each point of the edge, of one member of each of the three pairs $T_{nn}^{(1)} u_n^{(1)}$, $T_{ns}^{(1)} u_s^{(1)}$, $T_{n2}^{(0)} u_2^{(0)}$. (The important conclusion that three edge-conditions are required, rather than the two of the classical theory of plates, was reached by E. Reissner (1945)).

Essentially, the first of (5.065) governs the transverse displacement $u_2^{(0)}$ and the other two equations govern the thickness-shear displacements $u_1^{(1)}$ and $u_3^{(1)}$; but the three equations are coupled as the result of the appearance of the thickness-shear displacements, $u_1^{(1)}$ and $u_3^{(1)}$, in the first equation and the transverse shear strains

$$\frac{\partial u_2^{(0)}}{\partial x_1}, \quad \frac{\partial u_2^{(0)}}{\partial x_3}$$

in the second and third equations. If these are suppressed and if the rotatory inertia terms (the right hand sides of the second and third equations) are dropped, the system reduces to the zero-order flexural equations as may be seen by comparing with (4.043).

At various places in this Section, mention has been made of coupling of components of displacement. It should be recognized that coupling can arise from various sources according as the vibrations take place in an infinite body, an infinite plate or a bounded plate and according as the vibrations are free or forced.

Coupling can occur in an infinite crystal solely through the elastic constants. This type is, of course, represented as such in the two-dimensional equations of this Section. In a plate, coupling can occur as a result of reflections at the faces even in an isotropic material. The simplest case is the coupling of dilatational and equivoluminal vibrations discussed in Chapter 2. In the two-dimensional equations, this type of coupling appears as though it were through the elastic constants. This is because of the integration through the thickness. Further coupling can occur at the edges of a bounded plate. For example, in the case of cylindrical, flexural vibrations of an infinite isotropic plate, two modes of motion are permitted by the approximate equations; each mode containing two coupled components of displacement (see Section 5.07). At a free edge, these two modes couple, i.e., coupling can occur through edge-conditions. Again, coupling of flexural and extensional modes can occur, even in an isotropic plate, if the edge-conditions are neither symmetric nor antisymmetric with respect to the middle plane. Similar coupling can take place in an infinite isotropic plate if the face-conditions are neither symmetric nor antisymmetric with respect to the middle plane. In a crystal plate, coupling can occur between extension and flexure even if all face and edge-conditions are symmetric or antisymmetric. Finally, coupling occurs, in forced vibrations, between whatever modes are excited by the face- or edge-forcing terms.

5.07 Useful Range of First-Order Approximation

An estimate of the range of frequencies and wave-lengths, in which the equations of the first-order approximation are useful, may be obtained from a comparison of simple solutions, of the equations, with analogous solutions of the three-dimensional equations. Appropriate solutions of the latter are available only for the isotropic plate.

In the isotropic, flexural equations (5.065) of the first-order approximation, we consider free, cylindrical vibrations by setting

$$
\begin{aligned}
u_2^{(0)} &= A \sin \xi x_1 \, e^{i\omega t} \\
u_1^{(1)} &= B \cos \xi x_1 \, e^{i\omega t} \\
u_3^{(1)} &= 0 \\
F_1^{(1)} &= F_2^{(0)} = F_3^{(1)} = 0
\end{aligned}
\tag{5.071}
$$

and obtain

$$
\begin{aligned}
(\mu \xi^2 - \rho \omega^2) A + \mu \xi B &= 0 \\
2b\mu \xi A + (D\xi^2 + 2b\mu - \tfrac{2}{3} b^3 \rho_1 \omega^2) B &= 0
\end{aligned}
\tag{5.072}
$$

from which

$$
\xi^2 b^2 = \frac{3\kappa^2 \omega^2}{2\omega_s^2} \left\{ 1 + g \pm \left[(1-g)^2 + 4g \frac{\omega_s^2}{\omega^2} \right]^{1/2} \right\} \tag{5.073}
$$

where

$$
\omega_s = \frac{\pi}{2b}\sqrt{\frac{\mu}{\rho}} \, , \quad g = \frac{\mu(1-\nu^2)}{\kappa^2 E} = \frac{6(1-\nu)}{\pi^2} \tag{5.074}
$$

Note that ω_s is the frequency of the lowest, simple thickness-shear mode (see Fig. 2.031).

Equation (5.073) gives two relations between frequency and wave-length which are to be compared with those of the first two modes of the corresponding solution, of the three-dimensional equations, described in Section 2.11. The mode of lower frequency is essentially flexural ($|A|/|B| > 1$) and the mode of higher frequency is essentially a thickness-shear mode ($|A|/|B| < 1$) at long wave-lengths, as may be ascertained by inserting (5.073) in either of (5.072). For $\xi b \ll 1$, (5.073) reduces to

$$
\omega = \begin{cases} \dfrac{\pi}{2b}\sqrt{\dfrac{\mu}{\rho}} \\[2ex] \xi^2 b \sqrt{\dfrac{E}{3\rho(1-\nu^2)}} \end{cases} \tag{5.075}
$$

and these are precisely the limits approached by the first two modes of the exact solution. As ξb increases, the two solutions diverge slightly, as illustrated in Fig. 5.071 for $\lambda=2\mu$.

The importance of the insertion of the constant κ and the assignment of the value $\pi/\sqrt{12}$ to it are apparent from an inspection of equation (5.075) and Fig. 5.071. Without the constant κ, the limit of the frequency ratio, ω/ω_s, of the second mode would be $\sqrt{12}/\pi$, instead of unity, at $\xi b = 0$.

In general, good results may be expected from the first-order approximation if the half-wave-length is not less than the thickness of the plate and if the frequency does not exceed the thickness-shear frequency (ω_s) by more than about 20%. The latter restriction arises from the absence of next higher thickness-modes in the first-order approximation. These conclusions have been confirmed in comparisons with experiments (Mindlin, 1951 B, 1952).

Turning, now, to the extensional equations (5.063), we set

$$
\begin{aligned}
u_1^{(0)} &= A\cos\xi x_1\, e^{i\omega t} \\
u_3^{(0)} &= 0 \\
u_2^{(1)} &= B\sin\xi x_1\, e^{i\omega t} \\
F_1^{(0)} &= F_2^{(1)} = F_3^{(0)} = 0
\end{aligned}
\qquad (5.076)
$$

and obtain

$$
\begin{aligned}
&\left[(\lambda+2\mu)\xi^2-\rho\omega^2\right]A - \lambda\xi B = 0 \\
&3\lambda\xi b^{-2}A - \left[\mu\xi^2+3b^{-2}(\lambda+2\mu)-\rho_1\omega^2\right]B = 0
\end{aligned}
\qquad (5.077)
$$

from which

$$
\xi^2 b^2 = \frac{3\kappa^2}{2\beta}\left\{(\alpha+\beta)\frac{\omega^2}{\omega_t^2}-1 \pm\left[\left((\alpha+\beta)\frac{\omega^2}{\omega_t^2}-1\right)^2+4\alpha\beta\frac{\omega^2}{\omega_t^2}\left(1-\frac{\omega^2}{\omega_t^2}\right)\right]^{1/2}\right\} \qquad (5.078)
$$

where

$$
\begin{aligned}
&\alpha = \frac{v_1^2}{v_3^2}\,, \quad \beta = \frac{\kappa^2 v_2^2}{v_3^2}\,, \quad \omega_t = \frac{\pi v_1}{2b}\,, \quad \kappa^2 = \frac{\pi^2}{12} \\
&v_1^2 = \frac{\lambda+2\mu}{\rho}\,, \quad v_2^2 = \frac{\mu}{\rho}\,, \quad v_3^2 = \frac{4\mu(\lambda+\mu)}{\rho(\lambda+2\mu)} = \frac{E}{\rho(1-\nu^2)}
\end{aligned}
\qquad (5.079)
$$

We note that V_1 and V_2 are the velocities of dilatational and equivoluminal waves in an infinite body, V_3 is the "plate-velocity" and ω_t is the frequency of the lowest simple thickness-stretch mode (see Fig. 2.031).

Again we have two relations between wave-number and frequency which are to be compared with the first two modes of the exact solution described in Section 2.11. At long wave-lengths we find, from (5.078) and (5.077), that the mode of lower frequency is essentially face-extensional ($|A|/|B|>1$) and the mode of higher frequency is essentially a thickness-stretch mode ($|A|/|B|<1$). For $\xi b \ll 1$, (5.078) reduces to

$$\omega = \begin{cases} \dfrac{\pi}{2b}\sqrt{\dfrac{\lambda+2\mu}{\rho}} \\ \xi\sqrt{\dfrac{E}{\rho(1-\nu^2)}} \end{cases} \tag{5.0710}$$

These are identical with the limits for long wave-lengths of the first two modes in the symmetric case considered in Section 2.11, when $\nu < 1/3$.

As before, we see the appropriateness of the insertion of the constant κ, with its value $\pi/\sqrt{12}$.

The frequencies given by the exact and approximate solutions are plotted in Fig. 5.072, over a range of wave-lengths, for $\lambda = 2\mu$. Considering the first mode, we see that the extensional equations of the first-order approximation are limited to somewhat longer wave-lengths than those of the flexural equations. As for the second mode, the approximation is poor. However, this is partly due to the fact that the comparison is made for $\lambda = 2\mu$ (i.e., $\nu = 1/3$) which is the extreme limit of applicability of the extensional equations. For $\nu < 1/3$, the slope of the curve for the exact second mode is zero at $\xi b = 0$ (see equation (2.1136) and Fig. 2.113(a)) so that, in the range of long wave-lengths, the approximation is considerably better than that illustrated in Fig. 5.072. Nevertheless the curvature, at $\xi b = 0$, for

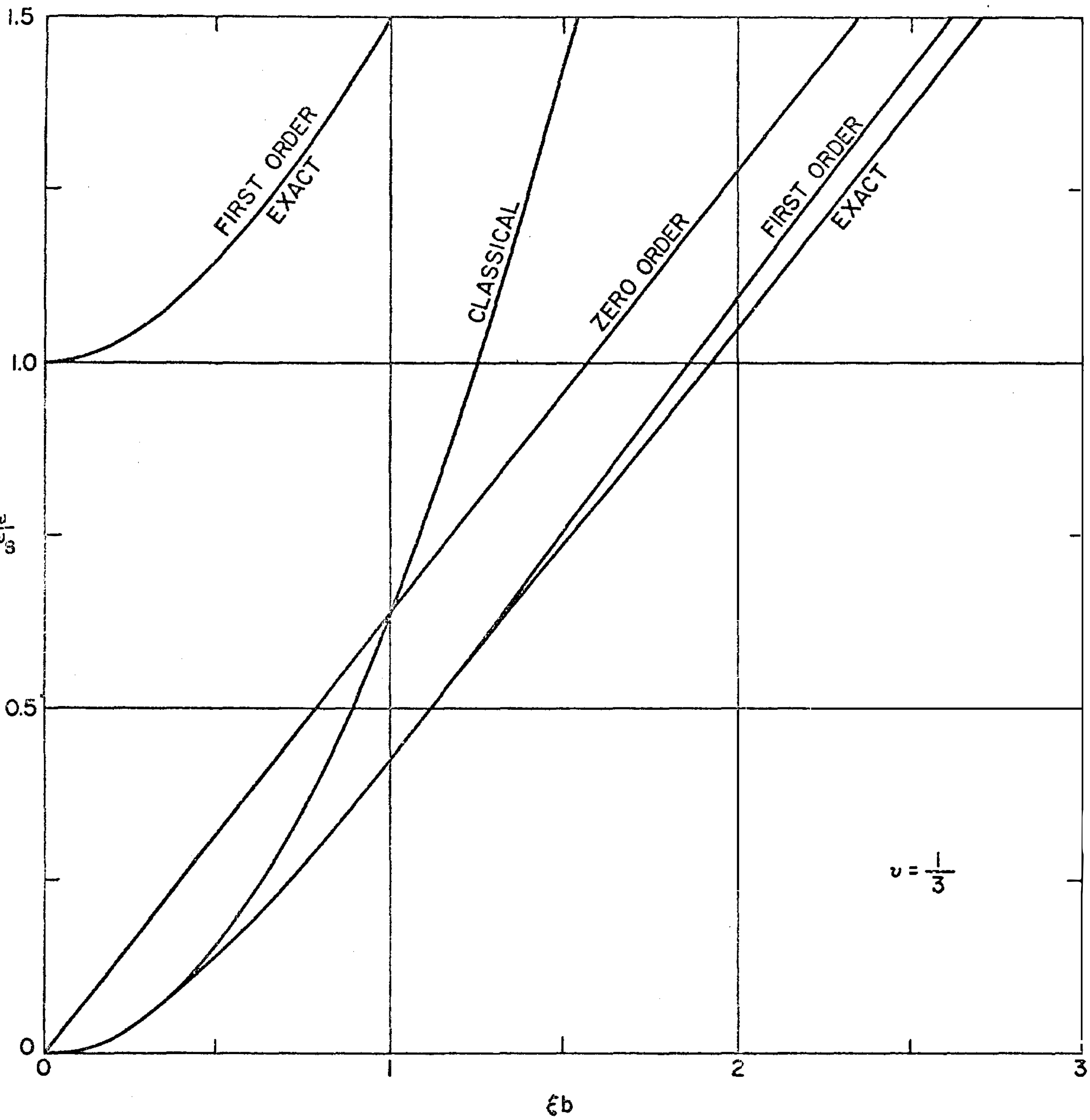
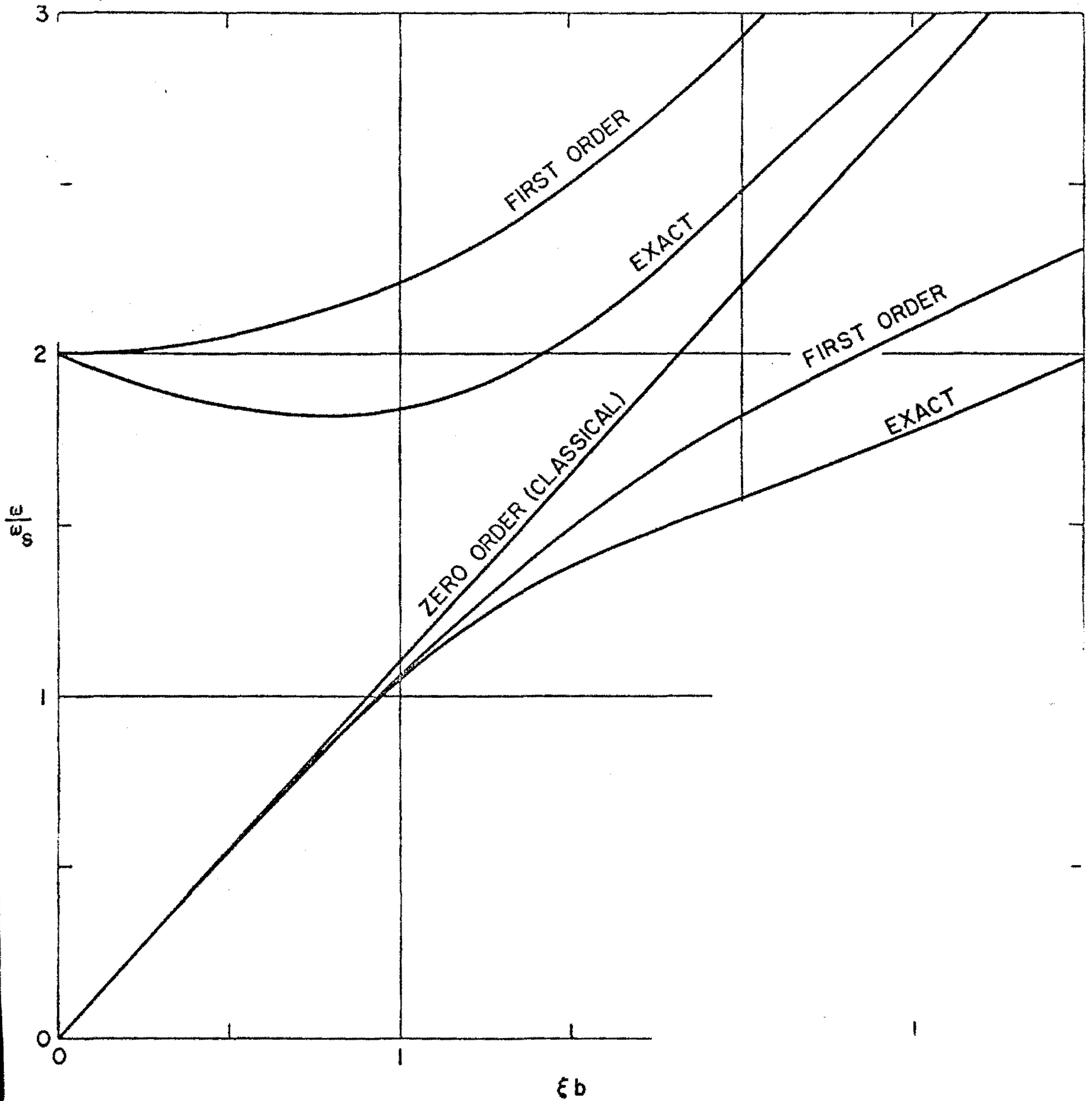


Fig. 5.071

Frequencies of first two modes of flexural vibration of an infinite, isotropic plate ($\nu = 1/3$). Comparison of zero-order and first-order approximations with solution of three-dimensional equations.

the exact second mode, is negative in the range $0<\nu<1/3$ (see (2.1137)) whereas the curvature for the second mode of the approximation is always positive. Hence, correspondence to a fraction of a percent can be expected only for very long wave-lengths of the second mode. It should be noted that frequencies of vibrations of thin plates, in the neighborhood of the frequency of the second mode, involve short wave-lengths (i.e., high overtones) of the first mode but long wave-lengths (i.e., fundamental mode or low overtones) of the second mode. For thick plates the discrepancy will depend on Poisson's ratio -- with the smaller ν yielding the better results. A comparison with experiments at $\nu=0.3$, for example, showed a maximum discrepancy of about 10%, in the frequency of the second mode, in the neighborhood of equal length and thickness (Kane and Mindlin, 1955).

It is apparent that the first-order approximation is not as good for extension as it is for flexure. To improve the extensional part, it would be necessary to take into account the influence of the lowest symmetric thickness-shear mode.


FIRST ORDER
EXACT
FIRST ORDER
EXACT
ZERO ORDER (CLASSICAL)
3
2
1
0
0
1
$\frac{\varepsilon}{\varepsilon_s}$
ξb