CHAPTER 4

ZERO-ORDER APPROXIMATION

4.01 Separation of Zero-Order Terms from Series

We proceed to extract, from the series-expansions established in the preceding chapter, a set of approximate equations in which the only displacements that appear are those of order zero. Isolation of the zero-order terms could be effected by setting $u_j^{(n)} = 0$ for $n > 0$ in all of the series-expressions, but the result would be of little practical value. This is essentially because the zero-order thickness-stretch depends only on the first-order thickness-displacement (i.e., $S_2^{(0)} = u_2^{(1)}$, see Fig. 3.031) so that, if $u_2^{(1)} = 0$, the plate is constrained to remain at constant thickness during any deformation. For example, a simple tension in the plane of the plate would not be accompanied by thickness-contraction. In addition, if we consider the possibility of variation of strain with x_1 and x_3 , we see that, if $u_j^{(n)} = 0$ for $n > 0$, the strains $S_4^{(1)}$ and $S_6^{(1)}$ would be suppressed (see Fig. 3.032). These strains are required if we are to have zero-order thickness-stretch which varies with x_1 and x_3 . The difficulty does not arise to as great an extent with $u_1^{(1)}$ and $u_3^{(1)}$ for, although they appear in the zero-order strains $S_4^{(0)}$ and $S_6^{(0)}$, they are not the sole contributors to those strains. Accordingly, we begin by setting

$$u_1^{(n)} = 0 \ , \quad u_3^{(n)} = 0 \ , \quad n > 0$$
$$u_2^{(n)} = 0 \ , \quad n > 1 \qquad (4.011)$$

These assumptions reduce the kinetic energy-density (3.0514) to

$$\overline{K} = \rho b(\dot{u}_1^{(0)}\dot{u}_1^{(0)} + \dot{u}_2^{(0)}\dot{u}_2^{(0)} + \dot{u}_3^{(0)}\dot{u}_3^{(0)}) + \tfrac{1}{3}\rho b^3 \dot{u}_2^{(1)}\dot{u}_2^{(1)} \qquad (4.012)$$

and the strain-energy-density (3.0511) to

$$\bar{U} = \frac{1}{2}\left[T_1^{(0)} \frac{\partial u_1^{(0)}}{\partial x_1} + T_3^{(0)} \frac{\partial u_3^{(0)}}{\partial x_3} + T_4^{(0)} \frac{\partial u_2^{(0)}}{\partial x_3} \right.$$

$$+ T_5^{(0)} \left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3} \right) + T_6^{(0)} \frac{\partial u_2^{(0)}}{\partial x_1} \qquad (4.013)$$

$$\left. + T_2^{(0)} u_2^{(1)} + T_4^{(1)} \frac{\partial u_2^{(1)}}{\partial x_3} + T_6^{(1)} \frac{\partial u_2^{(1)}}{\partial x_1} \right]$$

We may eliminate the contribution of $u_2^{(1)}$ to the vibrational energy, without suppressing $u_2^{(1)}$ itself, by setting

$$\dot{u}_2^{(1)} = 0 \qquad (4.014)$$

in (4.012) and

$$T_2^{(0)} = T_4^{(1)} = T_6^{(1)} = 0 \qquad (4.015)$$

in (4.013). In so doing, we eliminate the lowest order of thickness-stretch vibrations, just as we have already eliminated the higher order thickness-stretch vibrations, by setting $u_2^{(n)} = 0$ for $n > 1$, and all the thickness-shear vibrations by setting $u_1^{(n)} = u_3^{(n)} = 0$ for $n > 0$. By setting $T_2^{(0)} = T_4^{(1)} = T_6^{(1)} = 0$ in (4.013) and $\dot{u}_2^{(1)} = 0$ in (4.012), we permit the free development of not only the zero-order thickness-stretch ($S_2^{(0)} = u_2^{(1)}$) but also the portions $\partial u_2^{(1)}/\partial x_3$ and $\partial u_2^{(1)}/\partial x_1$ of the first-order shears $S_4^{(1)}$ and $S_6^{(1)}$ (see Figs. 4.011 and 3.032).

The zero-order stress-strain relation (3.043) has, at this stage, been reduced to

$$\left.\begin{aligned} T_p^{(0)} &= 2bc_{pq} S_q^{(0)} \\ T_2^{(0)} &= 2bc_{2q} S_q^{(0)} = 0 \end{aligned}\right\} \left\{ \begin{aligned} p &= 1,3,4,5,6 \\ q &= 1,\dots 6 \end{aligned}\right. \qquad (4.016)$$

Where, now,

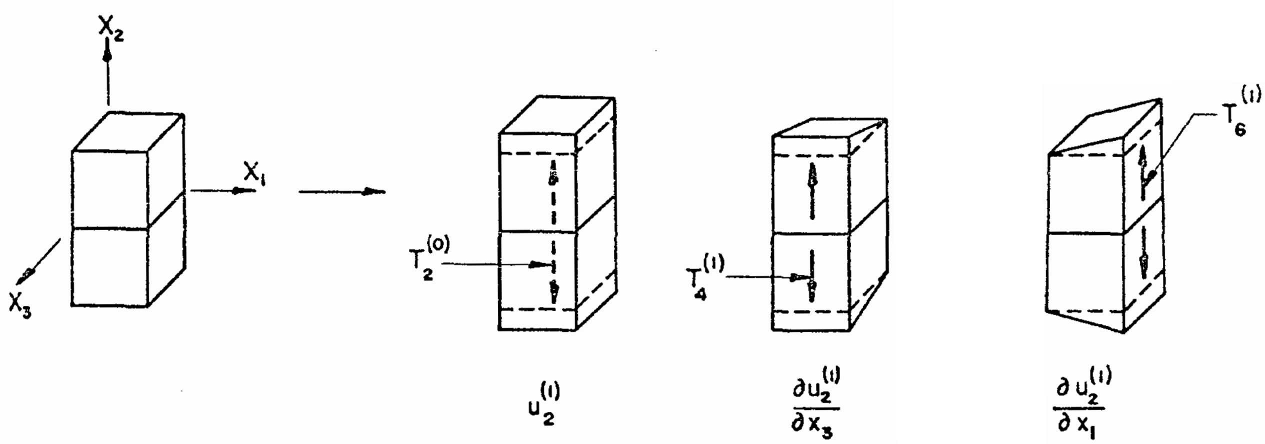

Fig. 4.011

Strains which are permitted to develop freely in the zero-order approximation.

$$S_1^{(0)} = \frac{\partial u_1^{(0)}}{\partial x_1} \qquad S_4^{(0)} = \frac{\partial u_2^{(0)}}{\partial x_3}$$

$$S_2^{(0)} = u_2^{(1)} \qquad S_5^{(0)} = \frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3} \tag{4.017}$$

$$S_3^{(0)} = \frac{\partial u_3^{(0)}}{\partial x_3} \qquad S_6^{(0)} = \frac{\partial u_2^{(0)}}{\partial x_1}$$

Equations (4.016) may be combined into a single stress-strain relation involving neither $T_2^{(0)}$ nor $S_2^{(0)}$. To this end, we separate $S_2^{(0)}$ in the first of (4.016) and solve the second equation for $S_2^{(0)}$:

$$\begin{aligned} T_p^{(0)} &= 2b(c_{pq}S_q^{(0)} - c_{p2}S_2^{(0)}) + 2b\,c_{p2}S_2^{(0)} \\ S_2^{(0)} &= -\frac{c_{2q}}{c_{22}}S_q^{(0)} + S_2^{(0)} \end{aligned} \tag{4.018}$$

The expression for $S_2^{(0)}$ in the second of (4.018) is then substituted for the $S_2^{(0)}$ outside the parentheses in the first of (4.018), with the result

$$T_p^{(0)} = 2b\,\bar{c}_{pq}S_q^{(0)}, \qquad p, q = 1, \dots 6 \tag{4.019}$$

where

$$\bar{c}_{pq} = c_{pq} - \frac{c_{p2}c_{2q}}{c_{22}}$$

This is the stress-strain relation for the zero-order approximation. It may also be written

$$T_{ij}^{(0)} = 2b\,\bar{c}_{ijkl}S_{kl}^{(0)}, \qquad i, j, k, l = 1, 2, 3 \tag{4.0110}$$

where

$$\bar{c}_{ijkl} = c_{ijkl} - \frac{c_{ij22}\,c_{22kl}}{c_{2222}} \tag{4.0111}$$

The stress-displacement relation is, from (4.0110) and (4.017)

$$T_{ij}^{(0)} = b\,\bar{c}_{ijkl}(u_{k,l}^{(0)} + u_{l,k}^{(0)}) \tag{4.0112}$$

The final ex .015),

(4.0113)

$$\bar{U} = b\bar{c}_{pq} S_p^{(0)} S_q^{(0)}$$
$$= b\bar{c}_{ijk\ell} S_{ij}^{(0)} S_{k\ell}^{(0)} \qquad (4.0114)$$

Then

$$\overline{\partial S_{ij}^{(0)}}\ , \ \left(\frac{\partial S_{ij}^{(0)}}{\partial S_{ji}^{(0)}} = 0\ ,\ i \neq j\right)$$

In v s-equations of

motion redu

(4.0116)

or

(4.0117)

$\partial T_s^{(0)}$

It then fol tions of motion are

$$\bar{c}_{ijk\ell}\left[b\left(u_{k,\ell}^{(0)} + u_{\ell,k}^{(0)}\right)\right]_{,l} + F_j^{(0)} = 2b\rho\ddot{u}_j^{(0)} \qquad (4.0118)$$

There remain stresses and equations of motion of orders higher than zero; but, as shown in the next section, they need not be taken into account.

4.02 Uniqueness of Solutions

We consider two sets of displacements, strains, tractions and stresses in a plate and designate their respective differences by starred symbols. Then the kinetic energy-density and strain-energy-density of the difference system are

$$\bar{K}^* = b\rho \dot{u}_j^{(0)*} \dot{u}_j^{(0)*} \qquad (4.021)$$

$$\bar{U}^* = b\bar{c}_{ijkl} S_{ij}^{(0)*} S_{kl}^{(0)*} \qquad (4.022)$$

If $\bar{\mathfrak{K}}^*$ and $\bar{\mathfrak{U}}^*$ are the total kinetic energy and strain-energy of the plate, calculated from $\bar{K}^*$ and $\bar{U}^*$ and reckoned over a time interval $t-t_o$, we have, as in (1.051),

$$\bar{\mathfrak{K}}^* + \bar{\mathfrak{U}}^* = \bar{\mathfrak{K}}^*(t_o) + \bar{\mathfrak{U}}^*(t_o) + \int_{t_o}^{t} dt \int_A (\dot{\bar{K}}^* + \dot{\bar{U}}^*) dA \qquad (4.023)$$

Now

$$\dot{\bar{K}}^* = 2b\rho \ddot{u}_j^{(0)*} \dot{u}_j^{(0)*}$$

and

$$\begin{aligned} \dot{\bar{U}}^* &= \frac{\partial \bar{U}^*}{\partial S_{ij}^{(0)*}} \dot{S}_{ij}^{(0)*}, \quad \left(\frac{\partial S_{ij}^{(0)*}}{\partial S_{ji}^{(0)*}} = 0, \ i \neq j\right) \\ &= \tfrac{1}{2} T_{ij}^{(0)*} (\dot{u}_{i,j}^{(0)*} + \dot{u}_{j,i}^{(0)*}) \\ &= T_{ij}^{(0)*} \dot{u}_{j,i}^{(0)*} \\ &= (T_{ij}^{(0)*} \dot{u}_j^{(0)*})_{,i} - T_{ij,i}^{(0)*} \dot{u}_j^{(0)*} \\ &= (T_{ij}^{(0)*} \dot{u}_j^{(0)*})_{,i} + F_j^{(0)*} \dot{u}_j^{(0)*} - 2b\rho \ddot{u}_j^{(0)*} \dot{u}_j^{(0)*} \end{aligned}$$

where use has been made of the fact that, since the quantities in each system satisfy the stress-equations of motion and the stress-strain-displacement relations, so do their differences. We have, then,

$$\int_A (\dot{\bar{K}}^* + \dot{\bar{U}}^*) dA = \int_A [(T_{ij}^{(0)*} \dot{u}_j^{(0)*})_{,i} + F_j^{(0)*} \dot{u}_j^{(0)*}] dA$$

Remembering that $(\ )_{,i} = 0$ when $i = 2$, we have

$$\int_A (T_{ij}^{(0)*} \dot{u}_j^{(0)*})_{,i}\, dA = \oint \nu_i T_{ij}^{(0)*} \dot{u}_j^{(0)*}\, ds \quad , \quad \nu_2 = 0$$

where s is the coordinate along the edge of the plate. The integrand of the line integral may be expressed in terms of coordinates $n, s, 2$ where n is the outward normal to the edge:

$$\begin{aligned} \nu_i T_{ij}^{(0)*} \dot{u}_j^{(0)*} &= \nu_h T_{hg}^{(0)*} \dot{u}_g^{(0)*} \quad , \quad h, g = n, s, 2 \\ &= T_{ng}^{(0)*} \dot{u}_g^{(0)*} \end{aligned}$$

since $\nu_n = 1$, $\nu_s = 0$, $\nu_2 = 0$. Hence

$$\bar{\mathfrak{K}}^* + \bar{\mathfrak{U}}^* = \bar{\mathfrak{K}}^*(t_0) + \bar{\mathfrak{U}}^*(t_0) + \int_{t_0}^{t} dt \oint T_{ng}^{(0)*} \dot{u}_g^{(0)*}\, ds + \int_{t_0}^{t} dt \int_A F_j^{(0)*} \dot{u}_j^{(0)*}\, dA \quad (4.024)$$

Accordingly, by the same argument as that employed in Section 1.05, sufficient conditions for a unique solution of the equations of motion are

a. Specification of the initial displacement $u_j^{(0)}$ and initial velocity $\dot{u}_j^{(0)}$ throughout the plate.

b. Specification of one member of each of the pairs $F_1^{(0)} u_1^{(0)}$, $F_2^{(0)} u_2^{(0)}$, $F_3^{(0)} u_3^{(0)}$ throughout the plate.

c. Specification, at each point of the edge, of one member of each of the pairs $T_{nn}^{(0)} u_n^{(0)}$, $T_{ns}^{(0)} u_s^{(0)}$, $T_{n2}^{(0)} u_2^{(0)}$.

4.03 Stress-Strain Relations

For the triclinic crystal, the stress-strain relation in the zero-order approximation is, from (4.019)

$$T_1^{(0)} = 2b\left[\left(c_{11} - \frac{c_{12}^2}{c_{22}}\right)S_1^{(0)} + \left(c_{13} - \frac{c_{12}c_{23}}{c_{22}}\right)S_3^{(0)} + \left(c_{14} - \frac{c_{12}c_{24}}{c_{22}}\right)S_4^{(0)} + \left(c_{15} - \frac{c_{12}c_{25}}{c_{22}}\right)S_5^{(0)} + \left(c_{16} - \frac{c_{12}c_{26}}{c_{22}}\right)S_6^{(0)}\right]$$

$$T_3^{(0)} = 2b\left[\left(c_{31} - \frac{c_{32}c_{21}}{c_{22}}\right)S_1^{(0)} + \left(c_{33} - \frac{c_{32}^2}{c_{22}}\right)S_3^{(0)} + \left(c_{34} - \frac{c_{32}c_{24}}{c_{22}}\right)S_4^{(0)} + \left(c_{35} - \frac{c_{32}c_{25}}{c_{22}}\right)S_5^{(0)} + \left(c_{36} - \frac{c_{32}c_{26}}{c_{22}}\right)S_6^{(0)}\right]$$

$$T_4^{(0)} = 2b\left[\left(c_{41} - \frac{c_{42}c_{21}}{c_{22}}\right)S_1^{(0)} + \left(c_{43} - \frac{c_{42}c_{23}}{c_{22}}\right)S_3^{(0)} + \left(c_{44} - \frac{c_{42}^2}{c_{22}}\right)S_4^{(0)} + \left(c_{45} - \frac{c_{42}c_{25}}{c_{22}}\right)S_5^{(0)} + \left(c_{46} - \frac{c_{42}c_{26}}{c_{22}}\right)S_6^{(0)}\right]$$

$$T_5^{(0)} = 2b\left[\left(c_{51} - \frac{c_{52}c_{21}}{c_{22}}\right)S_1^{(0)} + \left(c_{53} - \frac{c_{52}c_{23}}{c_{22}}\right)S_3^{(0)} + \left(c_{54} - \frac{c_{52}c_{24}}{c_{22}}\right)S_4^{(0)} + \left(c_{55} - \frac{c_{25}^2}{c_{22}}\right)S_5^{(0)} + \left(c_{56} - \frac{c_{52}c_{26}}{c_{22}}\right)S_6^{(0)}\right]$$

$$T_6^{(0)} = 2b\left[\left(c_{61} - \frac{c_{62}c_{21}}{c_{22}}\right)S_1^{(0)} + \left(c_{63} - \frac{c_{62}c_{23}}{c_{22}}\right)S_3^{(0)} + \left(c_{64} - \frac{c_{62}c_{24}}{c_{22}}\right)S_4^{(0)} + \left(c_{65} - \frac{c_{62}c_{25}}{c_{22}}\right)S_5^{(0)} + \left(c_{66} - \frac{c_{26}^2}{c_{22}}\right)S_6^{(0)}\right] \tag{4.031}$$

For the monoclinic crystal with the x_1-axis the axis of two-fold symmetry,

$$c_{15} = c_{16} = c_{25} = c_{26} = c_{35} = c_{36} = c_{45} = c_{46} = 0 \tag{4.032}$$

Hence

$$\begin{aligned}
T_1^{(0)} &= 2b\left[\left(c_{11} - \frac{c_{12}^2}{c_{22}}\right)S_1^{(0)} + \left(c_{13} - \frac{c_{12}c_{23}}{c_{22}}\right)S_3^{(0)} + \left(c_{14} - \frac{c_{12}c_{24}}{c_{22}}\right)S_4^{(0)}\right] \\
T_3^{(0)} &= 2b\left[\left(c_{31} - \frac{c_{32}c_{21}}{c_{22}}\right)S_1^{(0)} + \left(c_{33} - \frac{c_{32}^2}{c_{22}}\right)S_3^{(0)} + \left(c_{34} - \frac{c_{32}c_{24}}{c_{22}}\right)S_4^{(0)}\right] \\
T_4^{(0)} &= 2b\left[\left(c_{41} - \frac{c_{42}c_{21}}{c_{22}}\right)S_1^{(0)} + \left(c_{43} - \frac{c_{42}c_{23}}{c_{22}}\right)S_3^{(0)} + \left(c_{44} - \frac{c_{24}^2}{c_{22}}\right)S_4^{(0)}\right] \\
T_5^{(0)} &= 2b\left(c_{55}S_5^{(0)} + c_{56}S_6^{(0)}\right) \\
T_6^{(0)} &= 2b\left(c_{65}S_5^{(0)} + c_{66}S_6^{(0)}\right)
\end{aligned} \tag{4.033}$$

In the isotropic case, we have, in addition to (4.032),

$$\begin{gathered}
c_{14} = c_{24} = c_{34} = c_{56} = 0, \quad c_{12} = c_{13} = c_{23}, \quad c_{11} = c_{22} = c_{33} \\
c_{44} = c_{55} = c_{66} = \frac{1}{2}(c_{11} - c_{12})
\end{gathered} \tag{4.034}$$

Then

$$T_1^{(0)} = 2b\left[\left(c_{11} - \frac{c_{12}^2}{c_{22}}\right)S_1^{(0)} + \left(c_{12} - \frac{c_{12}^2}{c_{22}}\right)S_3^{(0)}\right]$$

$$T_3^{(0)} = 2b\left[\left(c_{12} - \frac{c_{12}^2}{c_{22}}\right)S_1^{(0)} + \left(c_{11} - \frac{c_{12}^2}{c_{22}}\right)S_3^{(0)}\right]$$

$$T_4^{(0)} = b(c_{11} - c_{12})S_4^{(0)} \qquad (4.035)$$

$$T_5^{(0)} = b(c_{11} - c_{12})S_5^{(0)}$$

$$T_6^{(0)} = b(c_{11} - c_{12})S_6^{(0)}$$

or (see Table 1.041)

$$T_1^{(0)} = \frac{2bE}{1-\nu^2}(S_1^{(0)} + \nu S_3^{(0)}) = \frac{4b\mu}{\lambda + 2\mu}[2(\lambda+\mu)S_1^{(0)} + \lambda S_3^{(0)}]$$

$$T_3^{(0)} = \frac{2bE}{1-\nu^2}(S_3^{(0)} + \nu S_1^{(0)}) = \frac{4b\mu}{\lambda + 2\mu}[2(\lambda+\mu)S_3^{(0)} + \lambda S_1^{(0)}]$$

$$T_4^{(0)} = \frac{bE}{1+\nu}S_4^{(0)} = 2b\mu S_4^{(0)} \qquad (4.036)$$

$$T_5^{(0)} = \frac{bE}{1+\nu}S_5^{(0)} = 2b\mu S_5^{(0)}$$

$$T_6^{(0)} = \frac{bE}{1+\nu}S_6^{(0)} = 2b\mu S_6^{(0)}$$

4.04 Displacement-Equations of Motion

The displacement-equations of motion may be obtained, in scalar form, either by expanding (4.0118) or by substituting the stress-strain relations of the preceding section in the stress-equations of motion (4.0117) and replacing the strains by the differential coefficients of displacements given in (4.017).

For the triclinic crystal, we have

$$\frac{\partial^2}{\partial x_1^2}(\bar{c}_{11}u_1^{(0)}+\bar{c}_{16}u_2^{(0)}+\bar{c}_{15}u_3^{(0)}) + \frac{\partial^2}{\partial x_3^2}(\bar{c}_{55}u_1^{(0)}+\bar{c}_{54}u_2^{(0)}+\bar{c}_{53}u_3^{(0)})$$
$$+\frac{\partial^2}{\partial x_1\partial x_3}\left[2\bar{c}_{15}u_1^{(0)}+(\bar{c}_{14}+\bar{c}_{56})u_2^{(0)}+(\bar{c}_{13}+\bar{c}_{55})u_3^{(0)}\right]+\frac{F_1^{(0)}}{2b}=\rho\frac{\partial^2 u_1^{(0)}}{\partial t^2}$$

$$\frac{\partial^2}{\partial x_1^2}(\bar{c}_{61}u_1^{(0)}+\bar{c}_{66}u_2^{(0)}+\bar{c}_{65}u_3^{(0)}) + \frac{\partial^2}{\partial x_3^2}(\bar{c}_{45}u_1^{(0)}+\bar{c}_{44}u_2^{(0)}+\bar{c}_{43}u_3^{(0)}) \tag{4.041}$$
$$+\frac{\partial^2}{\partial x_1\partial x_3}\left[(\bar{c}_{14}+\bar{c}_{56})u_1^{(0)}+2\bar{c}_{46}u_2^{(0)}+(\bar{c}_{36}+\bar{c}_{45})u_3^{(0)}\right]+\frac{F_2^{(0)}}{2b}=\rho\frac{\partial^2 u_2^{(0)}}{\partial t^2}$$

$$\frac{\partial^2}{\partial x_1^2}(\bar{c}_{51}u_1^{(0)}+\bar{c}_{56}u_2^{(0)}+\bar{c}_{55}u_3^{(0)}) + \frac{\partial^2}{\partial x_3^2}(\bar{c}_{35}u_1^{(0)}+\bar{c}_{34}u_2^{(0)}+\bar{c}_{33}u_3^{(0)})$$
$$+\frac{\partial^2}{\partial x_1\partial x_3}\left[(\bar{c}_{55}+\bar{c}_{13})u_1^{(0)}+(\bar{c}_{36}+\bar{c}_{45})u_2^{(0)}+2\bar{c}_{35}u_3^{(0)}\right]+\frac{F_3^{(0)}}{2b}=\rho\frac{\partial^2 u_3^{(0)}}{\partial t^2}$$

where the $\bar{c}_{pq}$ are given in terms of c_{pq} by (4.019).

In the monoclinic case:

$$\bar{c}_{11}\frac{\partial^2 u_1^{(0)}}{\partial x_1^2}+c_{55}\frac{\partial^2 u_1^{(0)}}{\partial x_3^2}+\frac{\partial^2}{\partial x_1\partial x_3}\left[(\bar{c}_{14}+c_{56})u_2^{(0)}+(\bar{c}_{13}+c_{55})u_3^{(0)}\right]+\frac{F_1^{(0)}}{2b}=\rho\frac{\partial^2 u_1^{(0)}}{\partial t^2}$$
$$c_{66}\frac{\partial^2 u_2^{(0)}}{\partial x_1^2}+\frac{\partial^2}{\partial x_3^2}(\bar{c}_{44}u_2^{(0)}+\bar{c}_{43}u_3^{(0)})+(\bar{c}_{14}+c_{56})\frac{\partial^2 u_1^{(0)}}{\partial x_1\partial x_3}+\frac{F_2^{(0)}}{2b}=\rho\frac{\partial^2 u_2^{(0)}}{\partial t^2} \tag{4.042}$$
$$\frac{\partial^2}{\partial x_1^2}(c_{56}u_2^{(0)}+c_{55}u_3^{(0)})+\frac{\partial^2}{\partial x_3^2}(\bar{c}_{34}u_2^{(0)}+\bar{c}_{33}u_3^{(0)})+(\bar{c}_{13}+c_{55})\frac{\partial^2 u_1^{(0)}}{\partial x_1\partial x_3}+\frac{F_3^{(0)}}{2b}=\rho\frac{\partial^2 u_3^{(0)}}{\partial t^2}$$

It is to be observed that, in (4.042), c_{55}, c_{56} and c_{66} appear rather than $\bar{c}_{55}$, $\bar{c}_{56}$ and $\bar{c}_{66}$.

For an isotropic plate:

$$\frac{\partial^2 u_1^{(0)}}{\partial x_1^2}+\frac{1-\nu}{2}\frac{\partial^2 u_1^{(0)}}{\partial x_3^2}+\frac{1+\nu}{2}\frac{\partial^2 u_3^{(0)}}{\partial x_1\partial x_3}+\frac{(1-\nu^2)F_1^{(0)}}{2bE}=\frac{\rho(1-\nu^2)}{E}\frac{\partial^2 u_1^{(0)}}{\partial t^2}$$
$$\frac{\partial^2 u_2^{(0)}}{\partial x_1^2}+\frac{\partial^2 u_2^{(0)}}{\partial x_3^2}+\frac{F_2^{(0)}}{2b\mu}=\frac{\rho}{\mu}\frac{\partial^2 u_2^{(0)}}{\partial t^2} \tag{4.043}$$
$$\frac{1-\nu}{2}\frac{\partial^2 u_3^{(0)}}{\partial x_1^2}+\frac{\partial^2 u_3^{(0)}}{\partial x_3^2}+\frac{1+\nu}{2}\frac{\partial^2 u_1^{(0)}}{\partial x_1\partial x_3}+\frac{(1-\nu^2)F_3^{(0)}}{2bE}=\frac{\rho(1-\nu^2)}{E}\frac{\partial^2 u_3^{(0)}}{\partial t^2}$$

In all three sets of equations [(4.041)-(4.043)] we have set $b = \text{const.}$

Equations equivalent to the first and third of (4.043) were obtained (also as zero-order approximations in a power-series expansion) by Poisson (1829) and Cauchy (1828) and are commonly called the equations of the classical (or elementary) theory of extensional vibrations of thin plates (Love, 1927, p. 497). If the accelerations and face-tractions are omitted from the first and third of (4.043), the two equations become identical with the corresponding equations of Filon's theory of generalized plane stress (Love, 1927, p. 138).

4.05 Useful Range of Zero-Order Approximation

The zero-order equations do not contain the simple thickness-modes and hence are limited to frequencies well below the frequencies of these modes. This is to say that the wave-length in the x_1 or x_3 direction must be large in comparison with the thickness of the plate, i.e., $\xi b \ll 1$. An estimate of how small the frequency and wave-number must be can be obtained by a comparison of solutions of the zero-order and three-dimensional equations. An appropriate solution of the three-dimensional equations is the one obtained in Section 2.11 for an isotropic plate. In Section 2.11 we found that, for $\xi b \ll 1$, the frequency of the lowest extensional mode is given by (2.1121), i.e.,

$$\omega = \xi \sqrt{\frac{E}{\rho(1-\nu^2)}} \tag{4.051}$$

and the frequency of the lowest flexural mode is given by (2.1116), i.e.,

$$\omega = \xi^2 b \sqrt{\frac{E}{3\rho(1-\nu^2)}} \tag{4.052}$$

To obtain the corresponding solutions of (4.043) we set

$$u_1^{(0)} = A \sin \xi x_1 \, e^{i\omega t}$$
$$u_2^{(0)} = B \sin \xi x_1 \, e^{i\omega t}$$
$$u_3^{(0)} = 0$$

Then we find, for the extensional vibrations,

$$\omega = \xi \sqrt{\frac{E}{\rho(1-\nu^2)}} \tag{4.053}$$

and for the flexural vibrations

$$\omega = \xi \sqrt{\frac{\mu}{\rho}} = \xi \sqrt{\frac{E}{2\rho(1+\nu)}} \tag{4.054}$$

Thus, the frequency of extensional vibration, obtained from the zero-order equations, coincides with the frequency given by the exact equations for very long waves. In the case of flexure, however, the exact and zero-order equations give quite different results.

The frequencies are plotted, along with the exact frequencies, in Figs. 5.081 and 5.082, for $\lambda = 2\mu$. The ordinate in both Figures is ω/ω_s where

$$\omega_s = \frac{\pi}{2b}\sqrt{\frac{\mu}{\rho}}$$

i.e., the frequency of the lowest, simple thickness-shear mode.

From Fig. 5.082 it may be seen that the zero-order extensional equations give satisfactory results for vibrations of the first mode in the range $\xi b < 1$, i.e., for ratios of half-wave-length to thickness greater than $\pi/2$. This is not to say that the zero-order approximation is good for finite plates whose ratio of face to thickness dimensions is as small as $\pi/2$, because the frequency is restricted to the range $\omega/\omega_s \ll 1$ due to the absence of a second mode in the zero-order approximation. This restriction confines the usefulness of the zero-order approximation to the lower modes of thin plates.

From Fig. 5.081 it may be se ; the zero-order flexural equations are of no value. Hence, in the case of crystal plates in which extension and flexure are coupled, equations (4.041) are not suitable. For appropriate equations, see Sections 6.03 and 6.04.