CHAPTER 3

INFINITE POWER SERIES OF TWO-DIMENSIONAL EQUATIONS

3.01 Introductory

As a preliminary to the establishment of various two-dimensional approximations, we convert the three-dimensional equations of elasticity to an infinite series of two-dimensional equations by expanding the displacement in an infinite series of powers of the thickness-coordinate of the plate and integrating through the thickness. Thus, in the integral expressions in Chapter 1, we let

$$u_j = \sum_n x_2^n u_j^{(n)} \tag{3.011}$$

and perform the indicated integrations with respect to x_2 between limits $\pm b$. In (3.011) the summation is over integral values of n from zero to infinity and the displacement of order n is a function of x_1, x_3 and t only, i.e.,

$$u_j^{(n)} = u_j^{(n)}(x_1, x_3, t) \tag{3.012}$$

Note that a superscript enclosed in parentheses is not a power, but only indicates the order of the term to which it is attached. The first three orders of $u_j^{(n)}$ are illustrated in Fig. 3.011. It is helpful to recognize that a component of displacement of order n contributes to extensional motions of the plate if $j+n$ is odd and to flexural motions if $j+n$ is even.

At a later stage we shall retain only a finite number of terms, but the infinite series expressions of displacement, strain, stress, energy and equations of motion will be of aid in deciding what to include in the various orders of approximation and in understanding the implications of what has been discarded and what retained.

The establishment of plate-equations as early terms in a power-series expansion of the three-dimensional equations of elasticity was first accomplished by Poisson (1829) and Cauchy (1828). However, that was before full use was made of energy and variational methods, which were introduced in the theory of plates by Kirchhoff (1850). The method of development of the theory of plates in this and the following two chapters is a systematic exploitation of a combination of the methods of Poisson, Cauchy and Kirchhoff.

3.02 Stress-Equations of Motion

We may write, from (1.063), (1.065) and (1.0314),

$$\int_V (T_{ij,i} - \rho \ddot{u}_j)\delta u_j \, dV = 0 \tag{3.021}$$

The integral is over the volume of the body which, in our case, is bounded by the surfaces $x_2 = \pm b$ and a right cylindrical or prismatic surface S_c which intersects $x_2 = 0$ in a curve or polygon C enclosing an area A. If the plate is multiply connected there are interior closed curves or polygons C_i and A does not include the areas within them.

Except where otherwise noted, the plate may be of slightly varying thickness. Thus

$$b = b(x_1, x_3), \quad \frac{\partial b}{\partial x_i} \ll 1, \quad i = 1, 3. \tag{3.022}$$

On substituting (3.011) into (3.021), we obtain

$$\int_{-b}^{b} \int_A \Big(T_{ij,i} - \rho \sum_m x_2^m \ddot{u}_j^{(m)}\Big) \sum_n x_2^n \delta u_j^{(n)} \, dx_2 \, dA = 0 \tag{3.023}$$

Now, for $i = 1$ or 3, we write

$$\int_{-b}^{b} T_{ij,i} \sum_n x_2^n \delta u_j^{(n)} \, dx_2 = \sum_n T_{ij,i}^{(n)} \delta u_j^{(n)} \tag{3.024}$$

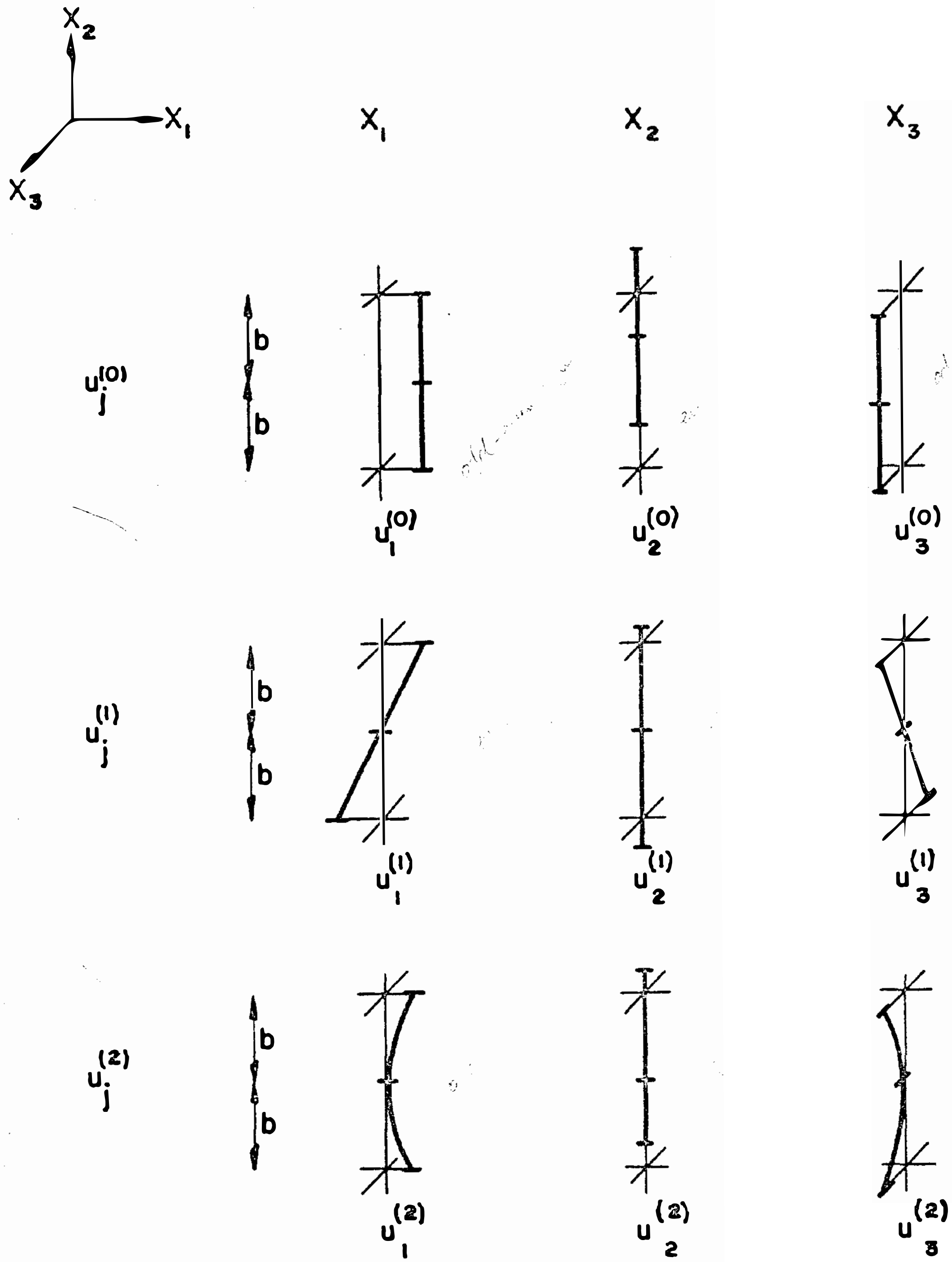


Fig. 3.011

Components of displacement of orders zero, one and two.

where

$$T_{ij}^{(n)} = \int_{-b}^{b} T_{ij} x_2^n dx_2 \qquad (3.025)$$

are defined as the n^{th} order components of stress. For $i = 2$,

$$\int_{-b}^{b} T_{2j,2} \sum_n x_2^n \delta u_j^{(n)} dx_2 = \sum_n \{[x_2^n T_{2j}]_{-b}^{b} - n\int_{-b}^{b} T_{2j}\, x_2^{(n-1)} dx_2\} \delta u_j^{(n)}$$
$$= \sum_n (F_j^{(n)} - n\, T_{2j}^{(n-1)}) \delta u_j^{(n)} \qquad (3.026)$$

where

$$F_j^{(n)} = [x_2^n T_{2j}]_{-b}^{b} \qquad (3.027)$$

are defined as the n^{th} order components of face-traction. (The first two orders of stress and face-traction are depicted in Fig. 3.021.) Finally,

$$\int_{-b}^{b} (\sum_m x_2^m \ddot{u}_j^{(m)})(\sum_n x_2^n \delta u_j^{(n)}) dx_2 = \int_{-b}^{b} \sum_m \sum_n x_2^{m+n} \ddot{u}_j^{(m)} \delta u_j^{(n)} dx_2$$
$$= \sum_m \sum_n B_{mn}\, \ddot{u}_j^{(m)} \delta u_j^{(n)} \qquad (3.028)$$

where

$$B_{mn} = \begin{cases} 2b^{m+n+1}(m+n+1)^{-1}, & m+n \text{ even} \\ 0, & m+n \text{ odd} \end{cases} \qquad (3.029)$$

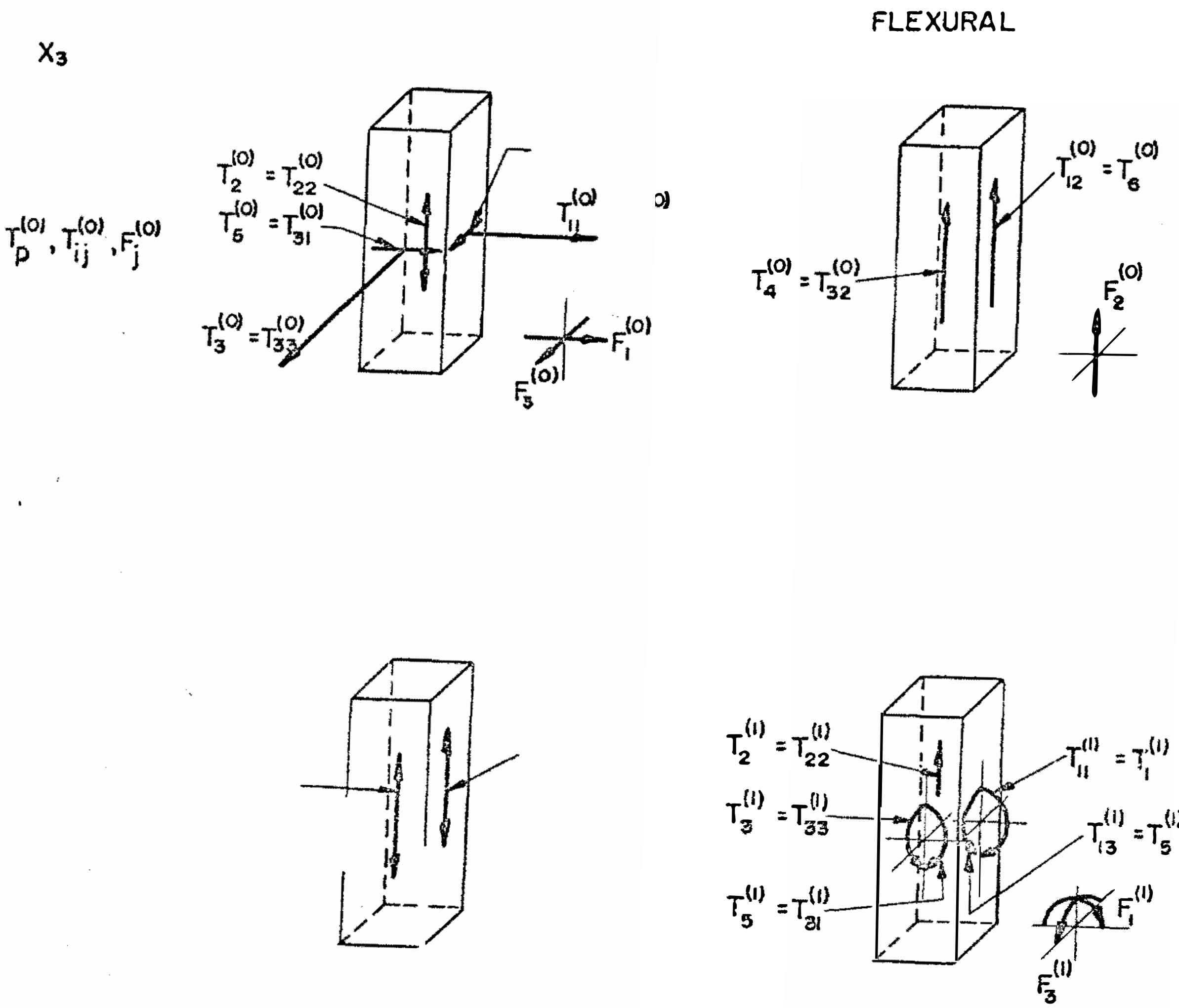
Assembling these results, (3.021) becomes

$$\int_A \sum_n (T_{ij,i}^{(n)} - nT_{2j}^{(n-1)} + F_j^{(n)} - \rho \sum_m B_{mn} \ddot{u}_j^{(m)}) \delta u_j^{(n)} dA = 0 \qquad (3.0210)$$

in which we note that, for $i = 2$, $T_{ij,i}^{(n)} = 0$ because $T_{ij}^{(n)}$ is independent of x_2. Since the coefficients of the $\delta u_j^{(n)}$ must vanish separately, we have

$$T_{ij,i}^{(n)} - nT_{2j}^{(n-1)} + F_j^{(n)} = \rho \sum_m B_{mn} \ddot{u}_j^{(m)} \qquad (3.0211)$$

which are the stress-equations of motion of order n. For example, the first four orders are


FLEXURAL
X_3
$T_p^{(0)}, T_{ij}^{(0)}, F_j^{(0)}$
$T_2^{(0)} = T_{22}^{(0)}$
$T_5^{(0)} = T_{31}^{(0)}$
$T_{11}^{(0)}$
$T_3^{(0)} = T_{33}^{(0)}$
$F_1^{(0)}$
$F_3^{(0)}$
$T_{12}^{(0)} = T_6^{(0)}$
$T_4^{(0)} = T_{32}^{(0)}$
$F_2^{(0)}$
$T_2^{(1)} = T_{22}^{(1)}$
$T_{11}^{(1)} = T_1^{(1)}$
$T_3^{(1)} = T_{33}^{(1)}$
$T_{13}^{(1)} = T_5^{(1)}$
$T_5^{(1)} = T_{31}^{(1)}$
$F_1^{(1)}$
$F_3^{(1)}$

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$$
\begin{aligned}
T_{ij,i}^{(0)} + F_j^{(0)} &= \rho\left(2b\ddot{u}_j^{(0)} + \frac{2b^3}{3}\ddot{u}_j^{(2)} + \cdots\right) \\
T_{ij,i}^{(1)} - T_{2j}^{(0)} + F_j^{(1)} &= \rho\left(\frac{2b^3}{3}\ddot{u}_j^{(1)} + \frac{2b^5}{5}\ddot{u}_j^{(3)} + \cdots\right) \\
T_{ij,i}^{(2)} - 2T_{2j}^{(1)} + F_j^{(2)} &= \rho\left(\frac{2b^3}{3}\ddot{u}_j^{(0)} + \frac{2b^5}{5}\ddot{u}_j^{(2)} + \cdots\right) \\
T_{ij,i}^{(3)} - 3T_{2j}^{(2)} + F_j^{(3)} &= \rho\left(\frac{2b^5}{5}\ddot{u}_j^{(1)} + \frac{2b^7}{7}\ddot{u}_j^{(2)} + \cdots\right)
\end{aligned}
\qquad (3.0212)
$$

In scalar form the first two of these are

$$
\begin{aligned}
\frac{\partial T_{11}^{(0)}}{\partial x_1} + \frac{\partial T_{31}^{(0)}}{\partial x_3} + F_1^{(0)} &= \rho\left(2b\ddot{u}_1^{(0)} + \frac{2b^3}{3}\ddot{u}_1^{(2)} + \cdots\right) \\
\frac{\partial T_{12}^{(0)}}{\partial x_1} + \frac{\partial T_{32}^{(0)}}{\partial x_3} + F_2^{(0)} &= \rho\left(2b\ddot{u}_2^{(0)} + \frac{2b^3}{3}\ddot{u}_2^{(2)} + \cdots\right) \\
\frac{\partial T_{13}^{(0)}}{\partial x_1} + \frac{\partial T_{33}^{(0)}}{\partial x_3} + F_3^{(0)} &= \rho\left(2b\ddot{u}_3^{(0)} + \frac{2b^3}{3}\ddot{u}_3^{(2)} + \cdots\right)
\end{aligned}
\qquad (3.0213)
$$

$$
\begin{aligned}
\frac{\partial T_{11}^{(1)}}{\partial x_1} + \frac{\partial T_{31}^{(1)}}{\partial x_3} - T_{21}^{(0)} + F_1^{(1)} &= \rho\left(\frac{2b^3}{3}\ddot{u}_1^{(1)} + \frac{2b^5}{5}\ddot{u}_1^{(3)} + \cdots\right) \\
\frac{\partial T_{12}^{(1)}}{\partial x_1} + \frac{\partial T_{32}^{(1)}}{\partial x_3} - T_{22}^{(0)} + F_2^{(1)} &= \rho\left(\frac{2b^3}{3}\ddot{u}_2^{(1)} + \frac{2b^5}{5}\ddot{u}_2^{(3)} + \cdots\right) \\
\frac{\partial T_{13}^{(1)}}{\partial x_1} + \frac{\partial T_{33}^{(1)}}{\partial x_3} - T_{23}^{(0)} + F_3^{(1)} &= \rho\left(\frac{2b^3}{3}\ddot{u}_3^{(1)} + \frac{2b^5}{5}\ddot{u}_3^{(3)} + \cdots\right)
\end{aligned}
\qquad (3.0214)
$$

The left hand sides of the first and third of (3.0213) are the usual ones of Poisson's theory of a thin plate stretched in its plane (Poisson, 1829; Cauchy, 1828; Timoshenko, 1940, p. 301) or of Filon's theory of generalized plane stress (Love, 1927, p. 138). The left hand sides of the second of (3.0213) and the first and third of (3.0214) are the usual ones of the Lagrange theory of flexure of thin plates (Timoshenko, 1940, pp. 86,87).

Referring to Fig. 3.021 we see that components of face-traction $F_j^{(n)}$ are associated with extension when $j+n$ is odd and with flexure when $j+n$ is even, just as in the case of displacement $u_j^{(n)}$. On the other hand, the stress

$T_{ij}^{(n)}$ is associated with extension when $i+j+n$ is even and with flexure when $i+j+n$ is odd.

In the abbreviated notation we may write, in place of (3.025),

$ij \to p$

$$T_p^{(n)} = \int_{-b}^{b} T_p x_2^n \, dx_2 \qquad (3.0215)$$

Then (3.0213) and (3.0214) become

$$\begin{aligned}
\frac{\partial T_1^{(0)}}{\partial x_1} + \frac{\partial T_5^{(0)}}{\partial x_3} + F_1^{(0)} &= \rho\left(2b\ddot{u}_1^{(0)} + \frac{2b^3}{3}\ddot{u}_1^{(2)} + \cdots\right) \\
\frac{\partial T_6^{(0)}}{\partial x_1} + \frac{\partial T_4^{(0)}}{\partial x_3} + F_2^{(0)} &= \rho\left(2b\ddot{u}_2^{(0)} + \frac{2b^3}{3}\ddot{u}_2^{(2)} + \cdots\right) \\
\frac{\partial T_5^{(0)}}{\partial x_1} + \frac{\partial T_3^{(0)}}{\partial x_3} + F_3^{(0)} &= \rho\left(2b\ddot{u}_3^{(0)} + \frac{2b^3}{3}\ddot{u}_3^{(2)} + \cdots\right)
\end{aligned} \qquad (3.0216)$$

$$\begin{aligned}
\frac{\partial T_1^{(1)}}{\partial x_1} + \frac{\partial T_5^{(1)}}{\partial x_3} - T_6^{(0)} + F_1^{(1)} &= \rho\left(\frac{2b^3}{3}\ddot{u}_1^{(1)} + \frac{2b^5}{5}\ddot{u}_1^{(3)} + \cdots\right) \\
\frac{\partial T_6^{(1)}}{\partial x_1} + \frac{\partial T_4^{(1)}}{\partial x_3} - T_2^{(0)} + F_2^{(1)} &= \rho\left(\frac{2b^3}{3}\ddot{u}_2^{(1)} + \frac{2b^5}{5}\ddot{u}_2^{(3)} + \cdots\right) \\
\frac{\partial T_5^{(1)}}{\partial x_1} + \frac{\partial T_3^{(1)}}{\partial x_3} - T_4^{(0)} + F_3^{(1)} &= \rho\left(\frac{2b^3}{3}\ddot{u}_3^{(1)} + \frac{2b^5}{5}\ddot{u}_3^{(3)} + \cdots\right)
\end{aligned} \qquad (3.0217)$$

In the notation employed by Timoshenko (1940) (except for the use of subscripts 1,2,3 in place of x,y,z and the adoption of the x_2-axis, rather than the z-axis, as the normal to the faces) we should write

$$\begin{array}{lll}
T_1^{(0)} = T_{11}^{(0)} = N_1 & \qquad & T_1^{(1)} = T_{11}^{(1)} = M_1 \\
T_2^{(0)} = T_{22}^{(0)} = - & & T_2^{(1)} = T_{22}^{(1)} = - \\
T_3^{(0)} = T_{33}^{(0)} = N_3 & & T_3^{(1)} = T_{33}^{(1)} = M_3 \\
T_4^{(0)} = T_{32}^{(0)} = Q_3 & & T_4^{(1)} = T_{32}^{(1)} = - \\
T_5^{(0)} = T_{13}^{(0)} = N_{13} & & T_5^{(1)} = T_{13}^{(1)} = M_{13} \\
T_6^{(0)} = T_{12}^{(0)} = Q_1 & & T_6^{(1)} = T_{12}^{(1)} = -
\end{array}$$

$$F_2^{(0)} = q$$

3.03 Strain

The expression (1.012) of the strain in terms of displacement, i.e.,

$$S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{3.031}$$

becomes, on substitution of the series-expansion (3.011),

$$S_{ij} = \frac{1}{2}\left[\sum_n (x_2^n u_{i,j}^{(n)} + n x_2^{n-1} x_{2,j} u_i^{(n)}) + \sum_n (x_2^n u_{j,i}^{(n)} + n x_2^{n-1} x_{2,i} u_j^{(n)})\right] \tag{3.032}$$

We write $x_{2,j} = \delta_{2j}$ and $x_{2,i} = \delta_{2i}$ where, for $k = i$ or j,

$$\delta_{2k} = \begin{cases} 1, & k = 2 \\ 0, & k \neq 2 \end{cases} \tag{3.033}$$

Using this notation and rearranging terms, we have

$$S_{ij} = \frac{1}{2}\sum_n \left[x_2^n (u_{i,j}^{(n)} + u_{j,i}^{(n)}) + n x_2^{n-1} (\delta_{2j} u_i^{(n)} + \delta_{2i} u_j^{(n)})\right] \tag{3.034}$$

In order to define a strain of order n it is necessary that it appear with a factor x_2^n in the above expression. To this end, we shift terms in the series to obtain

$$S_{ij} = \frac{1}{2}\sum_n x_2^n \left[u_{i,j}^{(n)} + u_{j,i}^{(n)} + (n+1)(\delta_{2j} u_i^{(n+1)} + \delta_{2i} u_j^{(n+1)})\right] \tag{3.035}$$

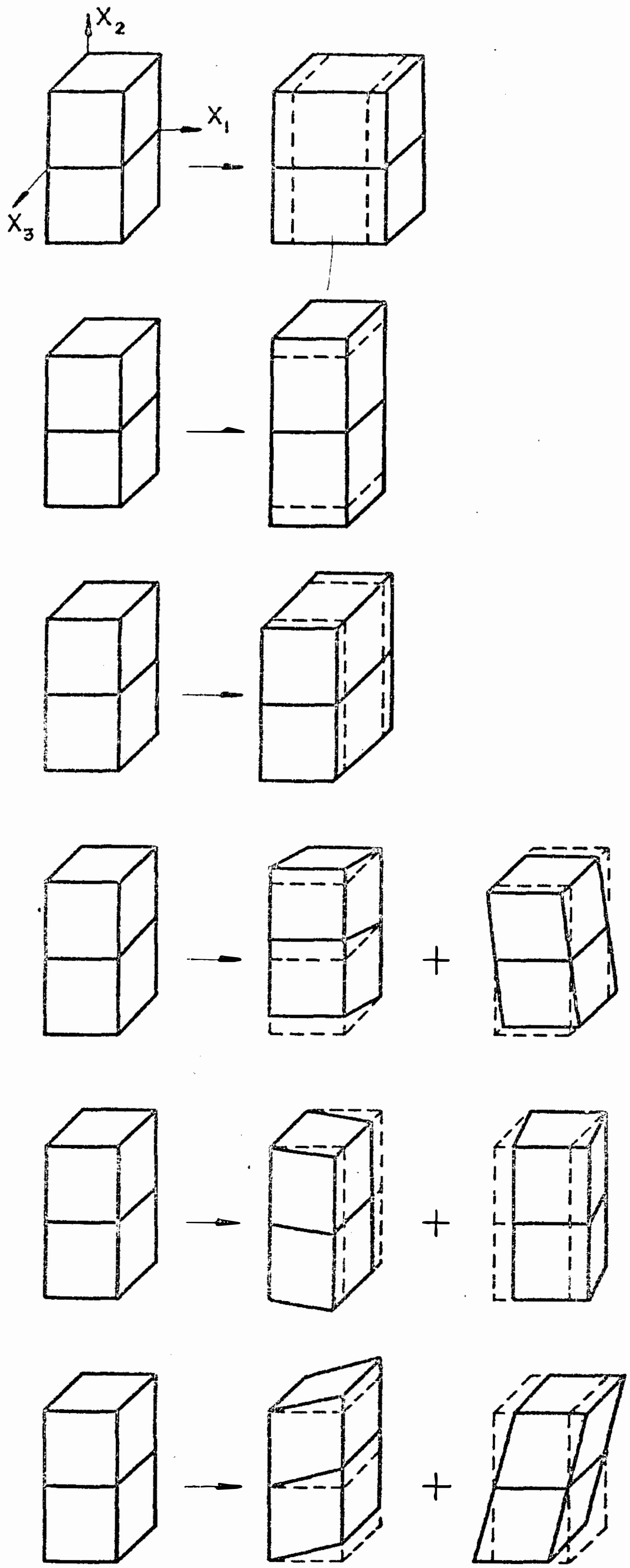
or

$$S_{ij} = \sum_n x_2^n S_{ij}^{(n)} \tag{3.036}$$

where

$$S_{ij}^{(n)} = \frac{1}{2}\left[u_{i,j}^{(n)} + u_{j,i}^{(n)} + (n+1)(\delta_{2j} u_i^{(n+1)} + \delta_{2i} u_j^{(n+1)})\right] \tag{3.037}$$

are the components of strain of order n. These are illustrated, for orders zero and unity, in Figs. 3.031 and 3.032, respectively. It will be observed



$$S_1^{(0)} = S_{11}^{(0)} = \frac{\partial u_1^{(0)}}{\partial x_1}$$

$$S_2^{(0)} = S_{22}^{(0)} = u_2^{(1)}$$

$$S_3^{(0)} = S_{33}^{(0)} = \frac{\partial u_3^{(0)}}{\partial x_3}$$

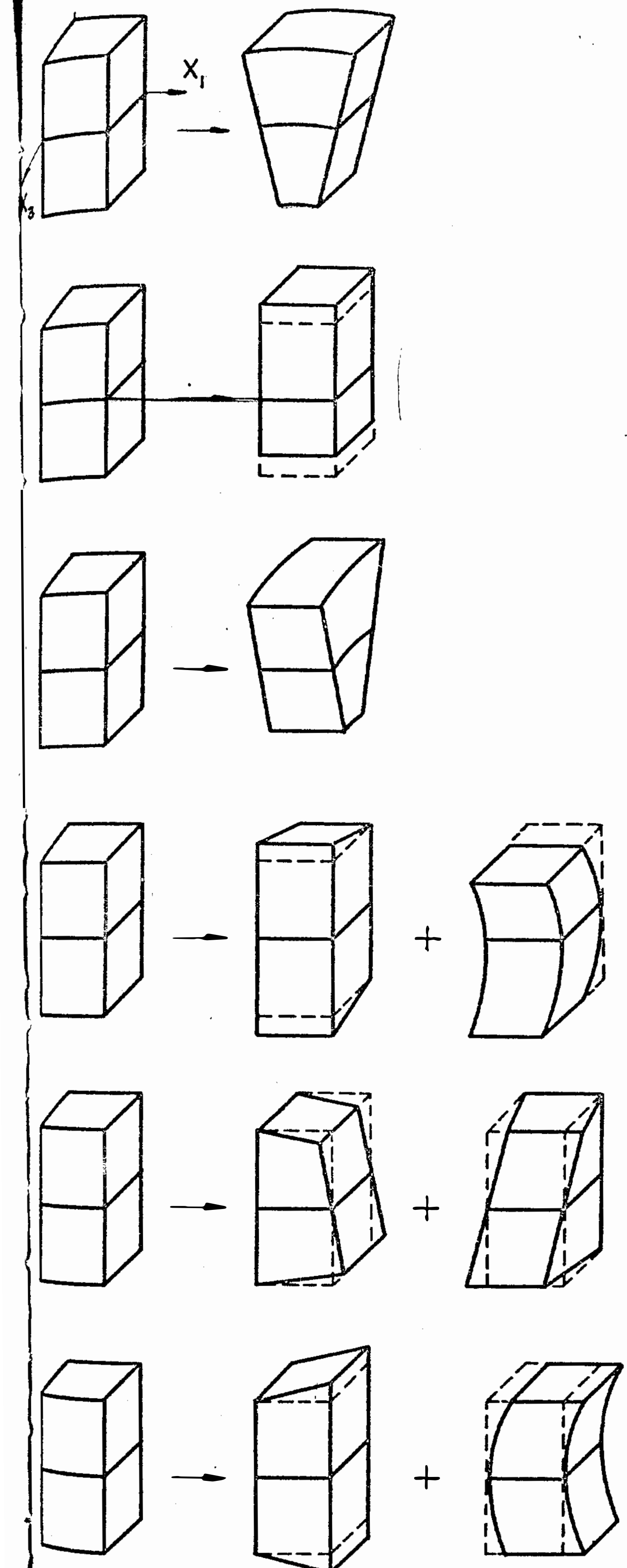
$$\frac{1}{2}\, S_4^{(0)} = S_{23}^{(0)} = S_{32}^{(0)} = \frac{1}{2}\left(\frac{\partial u_2^{(0)}}{\partial x_3} + u_3^{(1)}\right)$$

$$\frac{1}{2}\, S_5^{(0)} = S_{31}^{(0)} = S_{13}^{(0)} = \frac{1}{2}\left(\frac{\partial u_3^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_3}\right)$$

$$\frac{1}{2}\, S_6^{(0)} = S_{12}^{(0)} = S_{21}^{(0)} = \frac{1}{2}\left(\frac{\partial u_2^{(0)}}{\partial x_1} + u_1^{(1)}\right)$$

Fig. 3.031

Components of strain of order zero.



$$S_1^{(1)} = S_{11}^{(1)} = \frac{\partial u_1^{(1)}}{\partial x_1}$$

$$S_2^{(1)} = S_{22}^{(1)} = 2u_2^{(2)}$$

$$S_3^{(1)} = S_{33}^{(1)} = \frac{\partial u_3^{(1)}}{\partial x_3}$$

$$\frac{1}{2} S_4^{(1)} = S_{23}^{(1)} = S_{32}^{(1)} = \frac{1}{2}\left(\frac{\partial u_2^{(1)}}{\partial x_3} + 2u_3^{(2)}\right)$$

$$\frac{1}{2} S_6^{(1)} = S_{31}^{(1)} = S_{13}^{(1)} = \frac{1}{2}\left(\frac{\partial u_3^{(1)}}{\partial x_1} + \frac{\partial u_1^{(1)}}{\partial x_3}\right)$$

$$\frac{1}{2} S_6^{(1)} = S_{12}^{(1)} = S_{21}^{(1)} = \frac{1}{2}\left(\frac{\partial u_2^{(1)}}{\partial x_1} + 2u_1^{(2)}\right)$$

Fig. 3.032

Components of strain of order one.

that, as with stress, a component of strain $S_{ij}^{(n)}$ is associated with extension of the plate when $i+j+n$ is even and with flexure of the plate when $i+j+n$ is odd.

In the abbreviated notation $(S_p^{(n)})$, (3.037) becomes

$$
\begin{aligned}
S_{11}^{(n)} &= S_1^{(n)} = \frac{\partial u_1^{(n)}}{\partial x_1} & 2S_{23}^{(n)} &= 2S_{32}^{(n)} = S_4^{(n)} = \frac{\partial u_2^{(n)}}{\partial x_3} + (n+1)u_3^{(n+1)} \\
S_{22}^{(n)} &= S_2^{(n)} = (n+1)u_2^{(n+1)} & 2S_{31}^{(n)} &= 2S_{13}^{(n)} = S_5^{(n)} = \frac{\partial u_3^{(n)}}{\partial x_1} + \frac{\partial u_1^{(n)}}{\partial x_3} \\
S_{33}^{(n)} &= S_3^{(n)} = \frac{\partial u_3^{(n)}}{\partial x_3} & 2S_{12}^{(n)} &= 2S_{21}^{(n)} = S_6^{(n)} = \frac{\partial u_2^{(n)}}{\partial x_1} + (n+1)u_1^{(n+1)}
\end{aligned}
\qquad (3.038)
$$

and (3.036) may be written as

$$S_p = \sum_n x_2^n S_p^{(n)}, \quad p = 1, \dots 6. \qquad (3.039)$$

3.04 Stress-Strain Relations

The stress-strain relations

$$T_p = c_{pq} S_q \qquad (3.041)$$

become, on using (3.039),

$$T_p = c_{pq} \sum_n x_2^n S_q^{(n)} \qquad (3.042)$$

Then, from (3.0215),

$$T_p^{(n)} = \int_{-b}^{b} T_p x_2^n \, dx_2 = c_{pq} \sum_m S_q^{(m)} \int_{-b}^{b} x_2^{m+n} \, dx_2$$

or

$$T_p^{(n)} = c_{pq} \sum_m B_{mn} S_q^{(m)} \qquad (3.043)$$

where B_{mn} is given by (3.029). Thus, the n^{th} order stress depends on all the strains of even order if n is even and on all the strains of odd order if n is odd. For example

$$
\begin{aligned}
T_p^{(0)} &= 2bc_{pq}S_q^{(0)} + \frac{2b^3}{3}c_{pq}S_q^{(2)} + \frac{2b^5}{5}c_{pq}S_q^{(4)} + \cdots \\
T_p^{(1)} &= \frac{2b^3}{3}c_{pq}S_q^{(1)} + \frac{2b^5}{5}c_{pq}S_q^{(3)} + \cdots \\
T_p^{(2)} &= \frac{2b^3}{3}c_{pq}S_q^{(0)} + \frac{2b^5}{5}c_{pq}S_q^{(2)} + \frac{2b^7}{7}c_{pq}S_q^{(4)} + \cdots \\
T_p^{(3)} &= \frac{2b^5}{5}c_{pq}S_q^{(1)} + \frac{2b^7}{7}c_{pq}S_q^{(3)} + \cdots
\end{aligned}
\tag{3.044}
$$

In the unabbreviated notation, (3.043) is written as

$$T_{ij}^{(n)} = c_{ijkl}\sum_m B_{mn} S_{kl}^{(m)} \tag{3.045}$$

3.05 <u>Strain-Energy and Kinetic Energy</u>

The strain-energy-density

$$U = \frac{1}{2} c_{pq} S_p S_q$$

becomes

$$U = \frac{1}{2} c_{pq} \sum_m \sum_n x_2^{m+n} S_p^{(m)} S_q^{(n)}$$

We define a plate-strain-energy-density

$$\bar{U} = \int_{-b}^{b} U\, dx_2 = \frac{1}{2} c_{pq} \sum_m \sum_n B_{mn} S_p^{(m)} S_q^{(n)} \tag{3.051}$$

Then, for $r = 1, \ldots 6$, $l = 0, \ldots \infty$,

$$\frac{\partial \bar{U}}{\partial S_r^{(l)}} = \frac{1}{2} c_{pq} \sum_m \sum_n B_{mn} \left(\frac{\partial S_p^{(m)}}{\partial S_r^{(l)}} S_q^{(n)} + \frac{\partial S_q^{(n)}}{\partial S_r^{(l)}} S_p^{(m)} \right) \tag{3.052}$$

Now

$$\frac{\partial S_p^{(m)}}{\partial S_r^{(l)}} = \begin{cases} 0, & r \neq p \\ 0, & l \neq m \\ 1, & r = p,\ l = m \end{cases}$$

$$\frac{\partial S_q^{(n)}}{\partial S_r^{(l)}} = \begin{cases} 0, & r \neq q \\ 0, & l \neq n \\ 1, & r = q,\ l = n \end{cases} \tag{3.053}$$

so that (3.052) reduces to

$$\frac{\partial \overline{U}}{\partial S_r^{(l)}} = \tfrac{1}{2} c_{rq} \sum_n B_{ln} S_q^{(n)} + \tfrac{1}{2} c_{pr} \sum_m B_{ml} S_p^{(m)} \tag{3.054}$$

Since p and n are dummy indices they may be changed to q and m. Then (3.054) becomes

$$\frac{\partial \overline{U}}{\partial S_r^{(l)}} = \tfrac{1}{2} \sum_m (c_{rq} B_{lm} + c_{qr} B_{ml}) S_q^{(m)} \tag{3.055}$$

but $c_{rq} = c_{qr}$ and $B_{lm} = B_{ml}$; hence

$$\frac{\partial \overline{U}}{\partial S_r^{(l)}} = \tfrac{1}{2} c_{rq} \sum_m B_{ml} S_q^{(m)} \tag{3.056}$$

Comparing (3.056) with (3.043), we see that

$$T_p^{(n)} = \frac{\partial \overline{U}}{\partial S_p^{(n)}} \tag{3.057}$$

We may also write

$$\overline{U} = \tfrac{1}{2} c_{ijkl} \sum_m \sum_n B_{mn} S_{ij}^{(m)} S_{kl}^{(n)} \tag{3.058}$$

$$T_{ij}^{(n)} = \frac{\partial \overline{U}}{\partial S_{ij}^{(n)}} \tag{3.059}$$

provided we adopt the convention (see Section 1.03)

$$\frac{\partial S_{ij}^{(n)}}{\partial S_{jl}^{(n)}} = 0 \ , \ i \neq j \qquad (3.0510)$$

Finally, from (3.043) and (3.051), we have

$$\bar{U} = \frac{1}{2} \sum_n T_p^{(n)} S_p^{(n)} \qquad (3.0511)$$

and similarly, from (3.044) and (3.058),

$$\bar{U} = \frac{1}{2} \sum_n T_{ij}^{(n)} S_{ij}^{(n)} \qquad (3.0512)$$

The kinetic energy-density K is treated similarly. Thus, we define a kinetic energy-density of the plate as

$$\bar{K} = \int_{-b}^{b} K \, dx_2 = \frac{1}{2} \int_{-b}^{b} \rho \dot{u}_j \dot{u}_j \, dx_2 \qquad (3.0513)$$

Replacing u_j by its power-series expansion (3.012) we have, as in (3.028),

$$\bar{K} = \frac{1}{2} \sum_m \sum_n \rho B_{mn} \dot{u}_j^{(m)} \dot{u}_j^{(n)} \qquad (3.0514)$$

or

$$\begin{aligned}
\rho^{-1} \bar{K} = \ & b \dot{u}_j^{(0)} \dot{u}_j^{(0)} + \tfrac{1}{3} b^3 \dot{u}_j^{(2)} \dot{u}_j^{(0)} + \tfrac{1}{5} b^5 \dot{u}_j^{(4)} \dot{u}_j^{(0)} + \cdots \\
& + \tfrac{1}{3} b^3 \dot{u}_j^{(1)} \dot{u}_j^{(1)} + \tfrac{1}{5} b^5 \dot{u}_j^{(3)} \dot{u}_j^{(1)} + \cdots \\
& + \tfrac{1}{3} b^3 \dot{u}_j^{(0)} \dot{u}_j^{(2)} + \tfrac{1}{5} b^5 \dot{u}_j^{(2)} \dot{u}_j^{(2)} + \tfrac{1}{7} b^7 \dot{u}_j^{(4)} \dot{u}_j^{(2)} + \cdots \qquad (3.0515) \\
& + \tfrac{1}{5} b^5 \dot{u}_j^{(1)} \dot{u}_j^{(3)} + \tfrac{1}{7} b^7 \dot{u}_j^{(3)} \dot{u}_j^{(3)} + \cdots \\
& + \tfrac{1}{5} b^5 \dot{u}_j^{(0)} \dot{u}_j^{(4)} + \tfrac{1}{7} b^7 \dot{u}_j^{(2)} \dot{u}_j^{(4)} + \tfrac{1}{9} b^9 \dot{u}_j^{(4)} \dot{u}_j^{(4)} + \cdots \\
& \vdots
\end{aligned}$$

3.06 Uniqueness of Solutions

The uniqueness theorem for the three-dimensional theory (Section 1.05) is based on the expression (1.053) of the total energy in terms of the initial energy and the work done by the external forces:

$$\mathfrak{K}^* + \mathfrak{U}^* = \mathfrak{K}^*(t_0) + \mathfrak{U}^*(t_0) + \int_{t_0}^{t} \int_S t_j^* \dot{u}_j^* \, dS \, dt \qquad (3.061)$$

where t_j^* and u_j^* are the differences between the two surface-tractions and the two surface-velocities associated with two systems of stress and displacement satisfying the stress-equations of motion and the stress-strain-displacement relations. To adapt the uniqueness theorem to the infinite series of plate-equations, it is only necessary to specify the surfaces over which the surface integration in (3.061) is to be performed, replace the integrand by its series-expansion and perform the integration with respect to the thickness-coordinate.

The plate is bounded by the faces $x_2 = \pm b$ and the cylindrical surface (or surfaces) S' whose generators are normal to the middle plane $x_2 = 0$ and intersect that plane in a curve (or curves) C. Components of stress and displacement on $x_2 = \pm b$ are referred to coordinates x_1, x_2, x_3 while components on S' are referred to coordinates n, s, x_2 where n is the outward normal to S' and n, s and x_2 form a right-handed orthogonal coordinate system in the order named (see Fig. 3.061). Then, in (3.061),

$$t_j^* \dot{u}_j^* = \nu_i T_{ij}^* \dot{u}_j^* = \nu_h T_{hg}^* \dot{u}_g^* \qquad (3.062)$$

where

$$h, g = \begin{cases} i, j = 1, 2, 3 & \text{on } x_2 = \pm b \\ n, s, 2 & \text{on } S' \end{cases} \qquad (3.063)$$

On $x_2 = \pm b$,

$$\nu_h T_{hg}^* \dot{u}_g^* = (\nu_1 T_{1j}^* + \nu_2 T_{2j}^* + \nu_3 T_{3j}^*) \dot{u}_j^* \qquad (3.064)$$

where

$$\begin{aligned} \nu_1 = 0 \;, \quad \nu_2 = 1 \;, \quad \nu_3 = 0 \quad \text{on} \quad x_2 = b \\ \nu_1 = 0 \quad \nu_2 = -1 \;, \quad \nu_3 = 0 \quad \text{on} \quad x_2 = -b \end{aligned} \qquad (3.065)$$

Hence

$$\nu_h T^*_{hg} \dot{u}^*_g = \pm T^*_{2j} \dot{u}^*_j = \pm \left(T^*_{21} \dot{u}^*_1 + T^*_{22} \dot{u}^*_2 + T^*_{23} \dot{u}^*_3 \right) \quad \text{on} \quad x_2 = \pm b \tag{3.066}$$

On the cylindrical surface

$$\nu_h T^*_{hg} \dot{u}^*_g = \left(\nu_n T^*_{ng} + \nu_s T^*_{sg} + \nu_2 T^*_{2g} \right) \dot{u}^*_g \tag{3.067}$$

where

$$\nu_n = 1 \,, \quad \nu_s = 0 \,, \quad \nu_2 = 0 \tag{3.068}$$

Hence

$$\nu_h T^*_{hg} \dot{u}^*_g = T^*_{ng} \dot{u}^*_g = T^*_{nn} \dot{u}^*_n + T^*_{ns} \dot{u}^*_s + T^*_{n2} \dot{u}^*_2 \quad \text{on } S' \tag{3.069}$$

Using (3.062), (3.066) and (3.069), we have

$$\int_S t^*_j \dot{u}^*_j \, dS = \int_A \left[T^*_{2j} \dot{u}^*_j \right]^b_{-b} dA + \int_{S'} T^*_{ng} \dot{u}^*_g \, dS' \tag{3.0610}$$

where A is the area within C or between C and internal boundaries if the plate is multiply connected.

When $\dot{u}^*_j$ and $\dot{u}^*_g$ are replaced by their series-expansions, (3.0610) becomes

$$\int_S t^*_j \dot{u}^*_j \, dS = \int_A \sum_n \left[T^*_{2j} x_2^n \right]^b_{-b} \dot{u}^{*(n)}_j \, dA$$

$$+ \oint \int_{-b}^{b} \sum_n T^*_{ng} x_2^n \dot{u}^{*(n)}_g \, dx_2 \, ds \tag{3.0611}$$

Now, from (3.025) and (3.027)

$$\begin{aligned} \int_{-b}^{b} T^*_{ng} x_2^n \, dx_2 &= T^{*(n)}_{ng} \\ \left[T^*_{2j} x_2^n \right]^b_{-b} &= F^{*(n)}_j \end{aligned} \tag{3.0612}$$

Hence, the surface-integral in (3.061) becomes

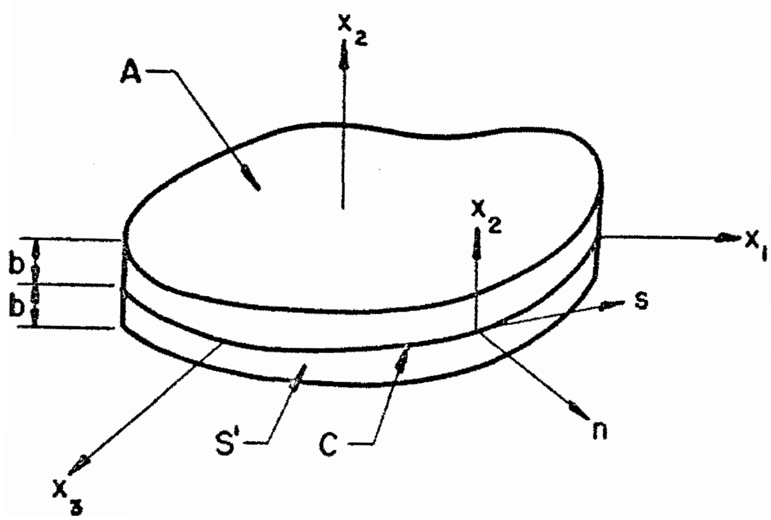


Fig. 3.061

Coordinates and boundaries of three-dimensional plate.

$$\int_S t_j^* \dot{u}_j^* dS = \int_A \sum_n F_j^{*(n)} \dot{u}_j^{*(n)} dA + \oint_C \sum_n T_{ng}^{*(n)} \dot{u}_g^{*(n)} ds$$

$$= \int_A \sum_n \left(F_\alpha^{*(n)} \dot{u}_\alpha^{*(n)} + F_2^{*(n)} \dot{u}_2^{*(n)} + F_\gamma^{*(n)} \dot{u}_\gamma^{*(n)} \right) dA$$

$$+ \oint_C \sum_n \left(T_{nn}^{*(n)} \dot{u}_n^{*(n)} + T_{ns}^{*(n)} \dot{u}_s^{*(n)} + T_{n2}^{*(n)} \dot{u}_2^{*(n)} \right) ds$$

where α and γ are orthogonal directions in the x_1-x_3-plane.

The initial values of the kinetic and potential energies in (3.061) are also expressible in terms of their series-expansions, through the use of (3.051) and (3.0514). Then, sufficient conditions for a unique solution of (3.0211), (3.045) and (3.037) are found by the same procedure as in Section 1.05, leading, in this case, to

a. Specification, for each and every order n, of the initial displacement $u_j^{(n)}$ and initial velocity $\dot{u}_j^{(n)}$ throughout the plate.

b. Specification, for each and every order n and at each point on the edge of the plate, of any one of the eight combinations formed by choosing one member of each of the three products $T_{nn}^{(n)} u_n^{(n)}$, $T_{ns}^{(n)} u_s^{(n)}$, $T_{n2}^{(n)} u_2^{(n)}$.

c. Specification at each point in the interior of the plate, for each and every order n, of any one of the eight combinations formed by choosing one member of each of the three products $F_\alpha^{(n)} u_\alpha^{(n)}$, $F_2^{(n)} u_2^{(n)}$, $F_\gamma^{(n)} u_\gamma^{(n)}$.

These components are illustrated in Fig. 3.062. The conditions for uniqueness are, of course, subject to the same limitations and extensions as those in Section 1.05.

3.07 Plane Tensors

It may be observed that $u_i^{(n)}, F_i^{(n)}, T_{ij}^{(n)}$ and $S_{ij}^{(n)}$ are components of plane tensors; i.e., the tensors are invariant with respect to rotation about the normal to the x_1-x_3-plane. If the axes x_1, x_2, x_3 are rotated about x_2 to x_1', x_2, x_3'

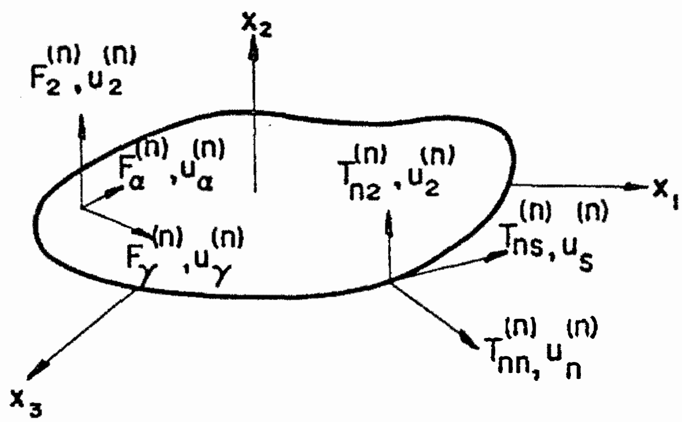


Fig. 3.062

Components of face-traction, face-displacement, edge-traction and edge-displacement of two-dimensional plate.

according to the scheme of direction cosines

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ x_1' & \ell_{11} & 0 & \ell_{13} \\ x_2 & 0 & 1 & 0 \\ x_3' & \ell_{31} & 0 & \ell_{33} \end{array}$$

the usual transformation formulas apply. Thus, if primed symbols refer to rotated axes and $r, s, t, u = 1,2,3$:

$$u_r^{(n)\prime} = \ell_{ri}\, u_i^{(n)}$$
$$u_i^{(n)} = \ell_{ri}\, u_r^{(n)\prime}$$

$$T_{rs}^{(n)\prime} = \ell_{ri}\,\ell_{sj}\, T_{ij}^{(n)}$$
$$T_{ij}^{(n)} = \ell_{ri}\,\ell_{sj}\, T_{rs}^{(n)\prime}$$

$$S_{rs}^{(n)\prime} = \ell_{ri}\,\ell_{sj}\, S_{ij}^{(n)}$$
$$S_{ij}^{(n)} = \ell_{ri}\,\ell_{sj}\, S_{rs}^{(n)\prime}$$

$$T_{rs}^{(n)\prime} = \ell_{ri}\,\ell_{sj}\, T_{ij}^{(n)} = \ell_{ri}\,\ell_{sj}\, c_{ijk\ell}\, S_{k\ell}^{(n)} = \ell_{ri}\,\ell_{sj}\, c_{ijk\ell}\,\ell_{tk}\,\ell_{u\ell}\, S_{tu}^{(n)\prime}$$

$$\text{or} \quad T_{rs}^{(n)\prime} = c'_{rstu}\, S_{tu}^{(n)\prime}$$

$$\text{where} \quad c'_{rstu} = \ell_{ri}\,\ell_{sj}\,\ell_{tk}\,\ell_{u\ell}\, c_{ijk\ell}$$