CHAPTER 1

ELEMENTS OF THE LINEAR THEORY OF ELASTICITY

1.01 Notation

Rectangular coordinates are designated by x_1, x_2, x_3 or x_i, $(i = 1,2,3)$ and components of displacement, referred to these coordinates, by u_1, u_2, u_3 or u_i, $(i = 1,2,3)$.

The symbols for components of traction, stress and strain are t_i, T_{ij} and S_{ij}, respectively, where $i = 1,2,3$ and $j = 1,2,3$.

The summation convention for repeated indices is employed. Thus, if t_j are the components of traction across a surface, at a point where the components of the outward-drawn unit normal are ν_i, the relations between the t_j and the components of stress at that point are

$$t_j = \nu_i T_{ij} \equiv \nu_1 T_{1j} + \nu_2 T_{2j} + \nu_3 T_{3j} \qquad (1.011)$$

i.e., the terms are summed over the repeated index i.

Differentiation with respect to space coordinates is indicated by a comma followed by an index:

$$,i \equiv \frac{\partial}{\partial x_i}$$

Thus, the relations between components of strain and displacement are expressed as

$$S_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \qquad (1.012)$$

where it is to be understood that the indices i and j range over $1,2,3$, i.e., (1.012) represents the nine relations

$$S_{11} = \frac{\partial u_1}{\partial x_1} \qquad S_{23} = S_{32} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right)$$

$$S_{22} = \frac{\partial u_2}{\partial x_2} \qquad S_{31} = S_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \tag{1.013}$$

$$S_{33} = \frac{\partial u_3}{\partial x_3} \qquad S_{12} = S_{21} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right)$$

Similarly, the components of rotation, ω_{ij}, are given by

$$\omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) \tag{1.014}$$

Differentiation with respect to time is indicated by a dot over a symbol. Thus

$$\dot{u}_i \equiv \frac{\partial u_i}{\partial t}$$

are components of velocity and

$$\ddot{u}_i \equiv \frac{\partial^2 u_i}{\partial t^2}$$

are components of acceleration.

The comma and dot notations and the summation convention are all employed in writing the stress-equations of motion in the form

$$T_{ij,i} = \rho \ddot{u}_j \tag{1.015}$$

where ρ is the density and the body-force has been omitted. The scalar expansion of (1.015) is

$$\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{21}}{\partial x_2} + \frac{\partial T_{31}}{\partial x_3} = \rho \frac{\partial^2 u_1}{\partial t^2}$$

$$\frac{\partial T_{12}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{32}}{\partial x_3} = \rho \frac{\partial^2 u_2}{\partial t^2} \tag{1.016}$$

$$\frac{\partial T_{13}}{\partial x_1} + \frac{\partial T_{23}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} = \rho \frac{\partial^2 u_3}{\partial t^2}$$

An abbreviated, indicial notation is also employed. In this, a pair of indices ranging over the integers $1,2,3$ is replaced by one index ranging over the integers $1,2,3,4,5,6$; according to the scheme given in Table 1.011.

Table 1.011

23	31	12
32	13	21
4	5	6

In the case

$$T_{ij} \equiv \begin{cases} T_p , \\ T_p , \quad i \neq j , \quad p = 4,5,6 \end{cases} \tag{1.017}$$

$$\begin{aligned} T_{11} &= T_1 & \quad {}_3 = T_{32} &= T_4 \\ T_{22} &= T_2 & \quad T_{31} = T_{13} &= T_5 \\ T_{33} &= T_3 & \quad T_{12} = T_{21} &= T_6 \end{aligned} \tag{1.018}$$

Accordingly, in the abbreviated notation, the scalar stress-equations of motion are

$$\begin{aligned} \frac{\partial T_1}{\partial x_1} + \frac{\partial T_6}{\partial x_2} + \frac{\partial T_5}{\partial x_3} &= \rho \frac{\partial^2 u_1}{\partial t^2} \\ \frac{\partial T_6}{\partial x_1} + \frac{\partial T_2}{\partial x_2} + \frac{\partial T_4}{\partial x_3} &= \rho \frac{\partial^2 u_2}{\partial t^2} \\ \frac{\partial T_5}{\partial x_1} + \frac{\partial T_4}{\partial x_2} + \frac{\partial T_3}{\partial x_3} &= \rho \frac{\partial^2 u_3}{\partial t^2} \end{aligned} \tag{1.019}$$

In the case of strain, the single-index shears are twice as large as the double-index shears, but the single- and double-index extensions are the same:

$$S_{ij} \equiv \begin{cases} S_p, & i=j, \; p=1,2,3 \\ \frac{1}{2}S_p, & i\neq j, \; p=4,5,6 \end{cases} \qquad (1.0110)$$

or

$$\begin{aligned} S_{11} &= S_1 = \frac{\partial u_1}{\partial x_1} & 2S_{23} &= 2S_{32} = S_4 = \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} \\ S_{22} &= S_2 = \frac{\partial u_2}{\partial x_2} & 2S_{31} &= 2S_{13} = S_5 = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \\ S_{33} &= S_3 = \frac{\partial u_3}{\partial x_3} & 2S_{12} &= 2S_{21} = S_6 = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \end{aligned} \qquad (1.0111)$$

The notation of the text, for the three-dimensional theory, may be compared with those employed by Love (1927)*, Timoshenko and Goodier (1951), Sokolnikoff (1946) and the Institute of Radio Engineers (1949) by referring to Table 1.012. Notations for the two-dimensional theory of plates are introduced in Chapter 3.

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* A name followed by the year - Jones (1950) or (Jones, 1950) - indicates an entry in the list of references at the end of the monograph.

Table 1.012

Notations Employed in the Three-Dimensional Theory

Text	Love	Timoshenko and Goodier	Sokolnikoff	I.R.E.
x_1, x_2, x_3	x, y, z	x, y, z	x_1, x_2, x_3	X_1, X_2, X_3
u_1, u_2, u_3	u, v, w	u, v, w	u_1, u_2, u_3	u_1, u_2, u_3
S_{11}, S_{22}, S_{33}	e_{xx}, e_{yy}, e_{zz}	$\epsilon_x, \epsilon_y, \epsilon_z$	e_{11}, e_{22}, e_{33}	S_{11}, S_{22}, S_{33}
S_{23}, S_{32}	$\frac{1}{2}e_{yz}, \frac{1}{2}e_{zy}$	$\frac{1}{2}\gamma_{yz}, \frac{1}{2}\gamma_{zy}$	e_{23}, e_{32}	S_{23}, S_{32}
S_{31}, S_{13}	$\frac{1}{2}e_{zx}, \frac{1}{2}e_{xz}$	$\frac{1}{2}\gamma_{zx}, \frac{1}{2}\gamma_{xz}$	e_{31}, e_{13}	S_{31}, S_{13}
S_{12}, S_{21}	$\frac{1}{2}e_{xy}, \frac{1}{2}e_{yx}$	$\frac{1}{2}\gamma_{xy}, \frac{1}{2}\gamma_{yx}$	e_{12}, e_{21}	S_{12}, S_{21}
$\omega_{32}, \omega_{13}, \omega_{21}$	$\varpi_x, \varpi_y, \varpi_z$	$\omega_x, \omega_y, \omega_z$	$\omega_{32}, \omega_{13}, \omega_{21}$	—
t_1, t_2, t_3	X_ν, Y_ν, Z_ν	$\overline{X}, \overline{Y}, \overline{Z}$	$\overset{\nu}{T}_1, \overset{\nu}{T}_2, \overset{\nu}{T}_3$	—
ν_1, ν_2, ν_3	l, m, n	l, m, n	ν_1, ν_2, ν_3	—
T_{11}, T_{22}, T_{33}	X_x, Y_y, Z_z	$\sigma_x, \sigma_y, \sigma_z$	$\tau_{11}, \tau_{22}, \tau_{33}$	T_{11}, T_{22}, T_{33}
T_{23}, T_{32}	Z_y, Y_z	τ_{yz}, τ_{zy}	τ_{23}, τ_{32}	T_{23}, T_{32}
T_{31}, T_{13}	X_z, Z_x	τ_{zx}, τ_{xz}	τ_{31}, τ_{13}	T_{31}, T_{13}
T_{12}, T_{21}	Y_x, X_y	τ_{xy}, τ_{yx}	τ_{12}, τ_{21}	T_{12}, T_{21}
S_1, S_2, S_3	e_{xx}, e_{yy}, e_{zz}	$\epsilon_x, \epsilon_y, \epsilon_z$	e_1, e_2, e_3	S_1, S_2, S_3
S_4	e_{yz}, e_{zy}	γ_{yz}, γ_{zy}	e_4	S_4
S_5	e_{zx}, e_{xz}	γ_{zx}, γ_{xz}	e_5	S_5
S_6	e_{xy}, e_{yx}	γ_{xy}, γ_{yx}	e_6	S_6
T_1, T_2, T_3	X_x, Y_y, Z_z	$\sigma_x, \sigma_y, \sigma_z$	τ_1, τ_2, τ_3	T_1, T_2, T_3
T_4	Z_y, Y_z	τ_{yz}, τ_{zy}	τ_4	T_4
T_5	X_z, Z_x	τ_{zx}, τ_{xz}	τ_5	T_5
T_6	Y_x, X_y	τ_{xy}, τ_{yx}	τ_6	T_6
$\omega_1, \omega_2, \omega_3$	$\varpi_x, \varpi_y, \varpi_z$	$\omega_x, \omega_y, \omega_z$	—	—

1.02 Principle of Conservation of Energy

If body-forces and thermal and electromagnetic effects are d, the principle of conservation of energy states that the rate of i f energy of a body is equal to the rate at which work is done by su tractions. In the linear theory of elasticity the statement take m

$$\int_V (\dot{K} + \dot{U})\, dV = \int_S t_j \dot{u}_j \, dS$$

where K and U are the kinetic and internal energy-densities, resp , The volume-integral in (1.021) is taken over the volume, V, of th d the surface-integral over its surface, S.

Now, the kinetic energy-density is, by assumption,

$$K = \tfrac{1}{2} \rho \dot{u}_j \dot{u}_j \tag{1.022}$$

where ρ is the density of the medium. Hence

$$\dot{K} = \rho \ddot{u}_j \dot{u}_j \tag{1.023}$$

If we make use of the stress-equations of motion (1.015) (which a d from the principle of conservation of linear momentum) (1.023) ma ten as

$$\dot{K} = T_{ij,i} \dot{u}_j = (T_{ij} \dot{u}_j)_{,i} - T_{ij} \dot{u}_{j,i} \tag{1.024}$$

Now, by the divergence theorem,

$$\begin{aligned} \int_V (T_{ij} \dot{u}_j)_{,i}\, dV &= \int_S \nu_i T_{ij} \dot{u}_j \, dS \\ &= \int_S t_j \dot{u}_j \, dS \end{aligned} \tag{1.025}$$

where the last step is based on the definition of stress (1.011).

Inserting (1.022)-(1.025) in (1.021), we find

$$\int_V (\dot{U} - T_{ij} \dot{u}_{j,i})\, dV = 0 \tag{1.026}$$

This relation must hold f of V; hence

$$\dot{U} = T_{ij}\,\dot{u}_{j,i} \tag{1.027}$$

Now, from (1.012) and (1.

$$\dot{\omega}_{ij} \tag{1.028}$$

Also, from the principle of angular momentum,

$$T_{ij} = T_{ji} \tag{1.029}$$

Thus, T_{ij} is symmetric but etric; hence

(1.0210)

Accordingly, the principa on of energy takes the form

$$\dot{U} = T_{ij}\,\dot{S}_{ij} \tag{1.0211}$$

1.03 Hooke's Law

We assume that the internal energy is a function of the six components of strain:

$$U = U(S_p)\,, \quad p = 1, \ldots 6 \tag{1.031}$$

so that U may now be called the strain-energy-density.

The components of strain are functions of the time; hence

$$\dot{U} = \frac{\partial U}{\partial S_p}\,\dot{S}_p \tag{1.032}$$

In the abbreviated notation (1.0211) takes the form

$$\dot{U} = T_p \dot{S}_p \tag{1.033}$$

Now, the six components of strain are independent. If, in addition, we assume that the stress is independent of the strain-rate, the coefficients of $\dot{S}_p$ in (1.032) and (1.033) may be equated, with the result

$$T_p = \frac{\partial U}{\partial S_p} \tag{1.034}$$

We now assume that the strain-energy is a homogeneous quadratic function of the six components of strain:

$$U = \tfrac{1}{2} c_{pq} S_p S_q \,, \qquad c_{pq} = c_{qp} \tag{1.035}$$

where the c_{pq} are the constants of elasticity. Then, from (1.034) we have, for $r = 1, \ldots, 6$,

$$T_r = \frac{\partial U}{\partial S_r} = \tfrac{1}{2}\left(c_{pq} \frac{\partial S_p}{\partial S_r} S_q + c_{pq} \frac{\partial S_q}{\partial S_r} S_p\right) \tag{1.036}$$

Now

$$\frac{\partial S_p}{\partial S_r} = \begin{cases} 0, & p \neq r \\ 1, & p = r \end{cases} \qquad \frac{\partial S_q}{\partial S_r} = \begin{cases} 0, & q \neq r \\ 1, & q = r \end{cases} \tag{1.037}$$

Hence

$$T_r = \tfrac{1}{2}\left(c_{rq} S_q + c_{pr} S_p\right) \tag{1.038}$$

But p is a dummy index, so it may be changed to q , giving

$$\begin{aligned} T_r &= \tfrac{1}{2}\left(c_{rq} + c_{qr}\right) S_q \\ &= c_{rq} S_q \end{aligned} \tag{1.039}$$

Finally, replacing r with p, we have Hooke's Law in the familiar form

$$T_p = c_{pq} S_q \tag{1.0310}$$

Also, from (1.035) and (1.0310),

$$U = \frac{1}{2} T_p S_p \tag{1.0311}$$

Hooke's Law (1.0310) may be written, in the unabbreviated, indicial notation, as

$$T_{ij} = c_{ijkl} S_{kl} \tag{1.0312}$$

$$(c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij} = c_{jilk} = c_{lkij} = c_{klji} = c_{lkji})$$

where the c_{ijkl} are the same constants as the c_{pq} when the pairs of indices ij and kl are replaced by p and q, respectively, in accordance with Table 1.011. The form (1.0312) can be obtained from the strain-energy-function

$$U = \frac{1}{2} c_{ijkl} S_{ij} S_{kl} \tag{1.0313}$$

through the relation

$$T_{ij} = \frac{\partial U}{\partial S_{ij}} \tag{1.0314}$$

if, in performing the differentiations, it is assumed that

$$\frac{\partial S_{ij}}{\partial S_{ji}} = 0, \quad i \neq j \tag{1.0315}$$

This because, in unabbreviated notation, the analogue of (1.033) is

$$\dot{U} = T_{ij} \dot{S}_{ij} = T_{11} \dot{S}_{11} + T_{22} \dot{S}_{22} + T_{33} \dot{S}_{33} + 2(T_{23} \dot{S}_{23} + T_{31} \dot{S}_{31} + T_{12} \dot{S}_{12}) \tag{1.0316}$$

whereas the analogue of (1.032) is

$$\dot{U} = \frac{\partial U}{\partial S_{ij}} \dot{S}_{ij} = \frac{\partial U}{\partial S_{11}} \dot{S}_{11} + \frac{\partial U}{\partial S_{22}} \dot{S}_{22} + \frac{\partial U}{\partial S_{33}} \dot{S}_{33} + \frac{\partial U}{\partial S_{23}} \dot{S}_{23} + \frac{\partial U}{\partial S_{31}} \dot{S}_{31} + \frac{\partial U}{\partial S_{12}} \dot{S}_{12} \qquad (1.0317)$$

Hence, equating coefficients:

$$T_{ij} = \begin{cases} \dfrac{\partial U}{\partial S_{ij}}, & i = j \\ \dfrac{1}{2} \dfrac{\partial U}{\partial S_{ij}}, & i \neq j \end{cases} \qquad (1.0318)$$

Equations (1.0318) may be written in the more convenient form (1.0314) if the convention (1.0315) is adopted.

1.04 Constants of Elasticity

We shall have occasion to consider only four types of elastic materials: triclinic, monoclinic , trigonal and isotropic (Mason, 1950, p. 44).

The elastic properties of a triclinic material are represented by the full array of 21 constants :

$$\begin{aligned}
T_1 &= c_{11} S_1 + c_{12} S_2 + c_{13} S_3 + c_{14} S_4 + c_{15} S_5 + c_{16} S_6 \\
T_2 &= c_{21} S_1 + c_{22} S_2 + c_{23} S_3 + c_{24} S_4 + c_{25} S_5 + c_{26} S_6 \\
T_3 &= c_{31} S_1 + c_{32} S_2 + c_{33} S_3 + c_{34} S_4 + c_{35} S_5 + c_{36} S_6 \\
T_4 &= c_{41} S_1 + c_{42} S_2 + c_{43} S_3 + c_{44} S_4 + c_{45} S_5 + c_{46} S_6 \\
T_5 &= c_{51} S_1 + c_{52} S_2 + c_{53} S_3 + c_{54} S_4 + c_{55} S_5 + c_{56} S_6 \\
T_6 &= c_{61} S_1 + c_{62} S_2 + c_{63} S_3 + c_{64} S_4 + c_{65} S_5 + c_{66} S_6
\end{aligned} \qquad (1.041)$$

The stress-strain relation of a monoclinic crystal involves 13 independent constants of elasticity. When the x_1-axis is the axis of two-fold symmetry,

$$c_{15} = c_{16} = c_{25} = c_{26} = c_{35} = c_{36} = c_{45} = c_{46} = 0 \qquad (1.042)$$

Then

$$
\begin{aligned}
T_1 &= c_{11}S_1 + c_{12}S_2 + c_{13}S_3 + c_{14}S_4 \\
T_2 &= c_{21}S_1 + c_{22}S_2 + c_{23}S_3 + c_{24}S_4 \\
T_3 &= c_{31}S_1 + c_{32}S_2 + c_{33}S_3 + c_{34}S_4 \\
T_4 &= c_{41}S_1 + c_{42}S_2 + c_{43}S_3 + c_{44}S_4 \\
T_5 &= \qquad\qquad\qquad\qquad\qquad c_{55}S_5 + c_{56}S_6 \\
T_6 &= \qquad\qquad\qquad\qquad\qquad c_{65}S_5 + c_{66}S_6
\end{aligned}
\tag{1.043}
$$

In the trigonal system, if x_3 is a trigonal axis and x_1 a binary axis, we have, in addition to (1.042),

$$
\begin{gathered}
c_{34} = 0, \quad c_{24} = -c_{14}, \quad c_{11} = c_{22}, \quad c_{23} = c_{13} \\
c_{44} = c_{55}, \quad c_{56} = c_{14}, \quad c_{66} = \tfrac{1}{2}(c_{11} - c_{12})
\end{gathered}
\tag{1.044}
$$

so that there are six independent constants. Then

$$
\begin{aligned}
T_1 &= c_{11}S_1 + c_{12}S_2 + c_{13}S_3 + c_{14}S_4 \\
T_2 &= c_{21}S_1 + c_{11}S_2 + c_{13}S_3 - c_{14}S_4 \\
T_3 &= c_{31}S_1 + c_{31}S_2 + c_{33}S_3 \\
T_4 &= c_{41}S_1 - c_{14}S_2 \qquad\quad + c_{44}S_4 \\
T_5 &= \qquad\qquad\qquad\qquad\qquad c_{44}S_5 + c_{14}S_6 \\
T_6 &= \qquad\qquad\qquad\qquad\qquad c_{14}S_5 + \tfrac{1}{2}(c_{11} - c_{12})S_6
\end{aligned}
\tag{1.045}
$$

In the case of quartz, which belongs to the trigonal system, x_1, x_2, x_3 are the electrical, mechanical and optical axes, respectively (Mason, 1946, p. 28). Let us designate these axes as $x_1^\circ, x_2^\circ, x_3^\circ$, respectively and the six constants in (1.045) as c_{pq}°. Then, if the stress-strain relation is referred to a system of axes x_1, x_2, x_3 in which x_1 coincides with x_1° and the positive x_3-axis lies in the positive x_2°-x_3° quadrant, making an angle θ with the positive

x_3^o-axis (see Fig. 1.041), the stress-strain relation has the same form as (1.043), i.e., the monoclinic case, but the thirteen constants are not independent. They are related to the c^o_{pq} as follows (Sykes, 1946, p. 247):

$$
\begin{aligned}
c_{11} &= c^o_{11} \\
c_{22} &= c^o_{11} c^4 + c^o_{33} s^4 + 2(2c^o_{44} + c^o_{13}) s^2 c^2 + 4 c^o_{14} s c^3 \\
c_{33} &= c^o_{11} s^4 + c^o_{33} c^4 + 2(2c^o_{44} + c^o_{13}) s^2 c^2 - 4 c^o_{14} s^3 c \\
c_{44} &= c^o_{44} + (c^o_{11} + c^o_{33} - 4c^o_{44} - 2c^o_{13}) s^2 c^2 - 2c^o_{14}(c^2 - s^2) sc \\
c_{55} &= c^o_{44} c^2 + c^o_{66} s^2 + 2c^o_{14} sc \\
c_{66} &= c^o_{44} s^2 + c^o_{66} c^2 - 2c^o_{14} sc \\
c_{12} &= c^o_{12} c^2 + c^o_{13} s^2 - 2c^o_{14} sc \\
c_{13} &= c^o_{12} s^2 + c^o_{13} c^2 + 2c^o_{14} sc \\
c_{14} &= c^o_{14}(c^2 - s^2) + (c^o_{12} - c^o_{13}) sc \\
c_{23} &= c^o_{13}(c^4 + s^4) + (c^o_{11} + c^o_{33} - 4c^o_{44}) s^2 c^2 - 2c^o_{14}(c^2 - s^2) sc \\
c_{24} &= c^o_{14}(4s^2 - 1) c^2 + [c^o_{11} c^2 - c^o_{33} s^2 - (2c^o_{44} + c^o_{13})(c^2 - s^2)] sc \\
c_{34} &= -c^o_{14}(4c^2 - 1) s^2 + [c^o_{11} s^2 - c^o_{33} c^2 + (2c^o_{44} + c^o_{13})(c^2 - s^2)] sc \\
c_{56} &= c^o_{14}(c^2 - s^2) + (c^o_{66} - c^o_{44}) sc
\end{aligned}
\qquad (1.046)
$$

where $s = \sin\theta$, $c = \cos\theta$.

The values of c^o_{pq} for quartz (Mason, 1950, p. 84) are, in uni .0[10] dyne/cm^2,

$$
\begin{aligned}
c^o_{11} &= 86.05 & c^o_{14} &= 18.25 \\
c^o_{12} &= 4.85 & c^o_{33} &= 107.1 \\
c^o_{13} &= 10.45 & c^o_{44} &= 58.65 \\
& & c_{66} &= 40.60
\end{aligned}
\qquad (
$$

In the case of an isotropic material (2 constants) the stres n relation reduces to

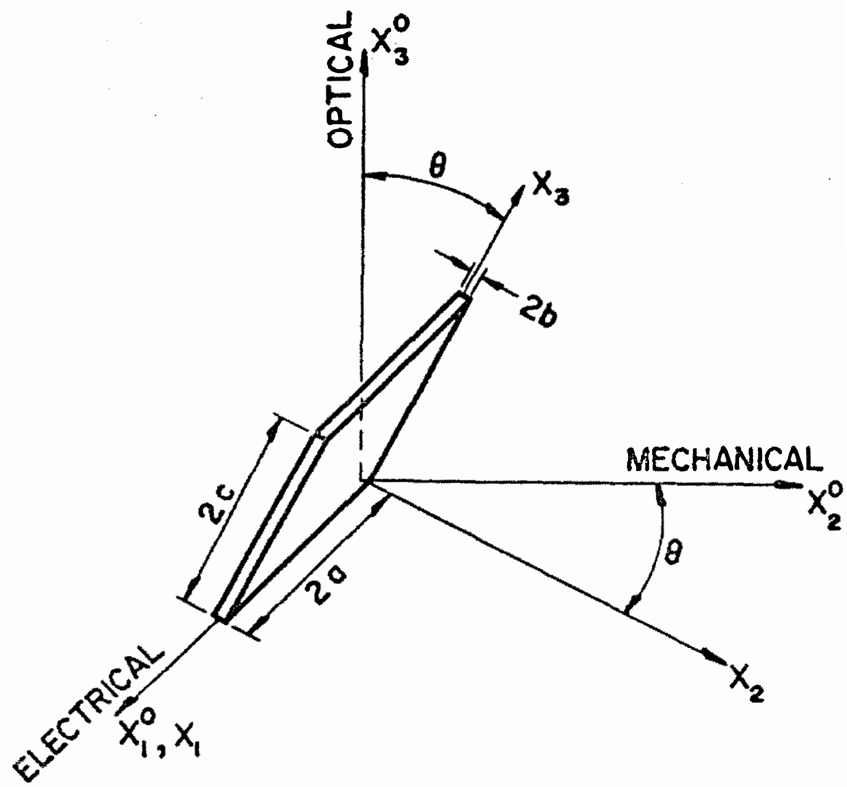

Fig. 1.041

Rotated X_1-cut of quartz.

$$
\begin{aligned}
T_1 &= c_{11} S_1 + c_{12} S_2 + c_{12} S_3 \\
T_2 &= c_{21} S_1 + c_{11} S_2 + c_{12} S_3 \\
T_3 &= c_{21} S_1 + c_{21} S_2 + c_{11} S_3 \\
T_4 &= \tfrac{1}{2}(c_{11} - c_{12}) S_4 \\
T_5 &= \tfrac{1}{2}(c_{11} - c_{12}) S_5 \\
T_6 &= \tfrac{1}{2}(c_{11} - c_{12}) S_6
\end{aligned}
\tag{1.048}
$$

The relations of the constants c_{11} and c_{12} to Lamé's constants (λ, μ) and to Young's modulus, E, and Poisson's ratio, ν, are given in Table 1.041.

Table 1.041 Elastic Constants of Isotropic Materials

	c_{11}, c_{12}	λ, μ	E, ν
c_{11}	c_{11}	$\lambda + 2\mu$	$\dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)}$
c_{12}	c_{12}	λ	$\dfrac{E\nu}{(1+\nu)(1-2\nu)}$
λ	c_{12}	λ	$\dfrac{E\nu}{(1+\nu)(1-2\nu)}$
μ	$\dfrac{c_{11}-c_{12}}{2}$	μ	$\dfrac{E}{2(1+\nu)}$
E	$\dfrac{(c_{11}+2c_{12})(c_{11}-c_{12})}{c_{11}+c_{12}}$	$\dfrac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	E
ν	$\dfrac{c_{12}}{c_{11}+c_{12}}$	$\dfrac{\lambda}{2(\lambda+\mu)}$	ν

1.05 Uniqueness of Solutions

Subject to certain restrictions, initial and boundary conditions, sufficient to assure a unique solution of the stress-equations of motion, may be obtained from Neumann's theorem (Love, 1927, p. 176).

In a body occupying a volume V, bounded by a surface S, consider two sets of displacements, strains, tractions and stresses and let their respective differences be designated by starred symbols. Also, let K^* and U^* be the kinetic and strain-energy-densities calculated from the difference-displacements and difference-strains and let $\mathfrak{K}^*$ and $\mathfrak{U}^*$ be the total kinetic and strain-energies of the body, calculated from the difference-energies and reckoned from an initial time t_0 to a later time t. Then, the total energy in the body at time t is

$$\mathfrak{K}^* + \mathfrak{U}^* \equiv \mathfrak{K}^*(t_0) + \mathfrak{U}^*(t_0) + \int_{t_0}^{t} dt \int_V (\dot{K}^* + \dot{U}^*) dV \qquad (1.051)$$

where $\mathfrak{K}^*(t_0)$ and $\mathfrak{U}^*(t_0)$ are the initial values of $\mathfrak{K}^*$ and $\mathfrak{U}^*$.

Now

$$\dot{U}^* = \frac{\partial U^*}{\partial S_p^*} \dot{S}_p^* = T_p^* \dot{S}_p^* = T_{ij}^* \dot{u}_{j,i}^*$$

$$= (T_{ij}^* \dot{u}_j^*)_{,i} - T_{ij,i}^* \dot{u}_j^*$$

and, by the divergence theorem,

$$\int_V (T_{ij}^* \dot{u}_j^*)_{,i} dV = \int_S \nu_i T_{ij}^* \dot{u}_j^* dS = \int_S t_j^* \dot{u}_j^* dS \qquad (1.052)$$

Also,

$$\dot{K}^* = \tfrac{1}{2}\rho \frac{\partial}{\partial t}(\dot{u}_j^* \dot{u}_j^*) = \rho \ddot{u}_j^* \dot{u}_j^*$$

Hence

$$\int_V (\dot{K}^* + \dot{U}^*) dV = \int_V (\rho \ddot{u}_j^* - T_{ij,i}^*) \dot{u}_j^* dV + \int_S t_j^* \dot{u}_j^* dS$$

If each of the two systems satisfies the stress-equations of motion, the difference-system does also, since the equations are linear. Thus

$$\rho \ddot{u}_j^* - T_{ij,i}^* = 0$$

and (1.051) reduces to

$$\mathfrak{K}^* + \mathfrak{U}^* = \mathfrak{K}^*(t_0) + \mathfrak{U}^*(t_0) + \int_{t_0}^{t} dt \int_S t_j^* \dot{u}_j^* \, dS \qquad (1.053)$$

that is, the total energy of the difference-system is equal to the initial energy plus the work done by the difference-tractions acting through the difference-displacements, on the surface S, during the interval $t - t_0$.

The argument proceeds in three steps: (1) it is shown that, if $\mathfrak{K}^* + \mathfrak{U}^* = 0$ the two systems must be identical except for a rigid-body-displacement; (2) conditions sufficient to make $\mathfrak{K}^* + \mathfrak{U}^* = 0$ are established; (3) these results are converted to conditions for the uniqueness of a solution.

(1) If $\mathfrak{K}^* + \mathfrak{U}^* = 0$, $\mathfrak{K}^*$ and $\mathfrak{U}^*$ must vanish separately, since both are positive.

If $\mathfrak{K}^*$ vanishes, K^* vanishes, since it is positive, and hence the $\dot{u}_i^*$ vanish since K^* is proportional to the sum of the squares of the $\dot{u}_i^*$.

Now, U^* is a homogeneous, quadratic function (of the difference-strains) which must be positive to secure the stability of the body (Love, 1927, p. 99). Hence if $\mathfrak{U}^*$ vanishes, U^* must vanish and, with it, the difference-strains. If the latter vanish, so must the difference-stresses (through Hooke's Law) and the difference-displacements, except for the displacement possible in a rigid body.

Hence, if $\mathfrak{K}^* + \mathfrak{U}^* = 0$, the two systems must be identical except, possibly, for a rigid-body-displacement (independent of the time since the difference-velocities vanish) and the latter can be eliminated by requiring the initial displacements to be the same.

(2) $\mathfrak{K}^*+\mathfrak{U}^*$ will vanish under conditions sufficient to make the right hand side of (1.053) vanish. If the initial displacements of the two systems are the same and the initial velocities are the same, then $\mathfrak{K}^*(t_o)$ and $\mathfrak{U}^*(t_o)$ vanish. This leaves only the integral in (1.053) which will vanish if one member of each of the three terms of the product $t_j^* u_j^*$ vanishes at each point of the surface. Thus, if n, s, t are orthogonal directions at any point of the surface, it is sufficient that T_{nn}^* or u_n^* and T_{ns}^* or u_s^* and T_{nt}^* or u_t^* vanish at each point.

(3) Returning, now, to a single system, sufficient conditions for a unique solution of the stress-equations of motion (subject to the limitations noted below) are

(a) Specification of the initial displacement and velocity throughout the body.

(b) Specification, at each and every point of the surface, of any one of the eight combinations formed by choosing one member of each of the three products $T_{nn}u_n$, $T_{ns}u_s$, $T_{nt}u_t$.

[Because of the use of a restricted form of the divergence theorem in passing from (1.051) to (1.053), these conditions are subject to the limitations as to continuity, single-valuedness, singularities and behavior at infinity associated with this form of the divergence theorem (Kellogg, 1929, Chapter IV). Some of the limitations may be removed.

For example, conditions which assure single-valuedness of displacements may be included in the uniqueness theorem by taking into account the possibility of surfaces of displacement-discontinuity. In that case (1.052) would have additional terms of the form $\int_{S'} t_j^* \Delta u_j^* dS$ where the integrations are over surfaces S' across which the displacements have discontinuities Δu_j^*. Then the condition $\Delta u_j^* \equiv \oint_C du_j^* = 0$, where the integration is around any closed curve C, leads, by Cesaro's and Weingarten's theorems, (Love, 1927, p. 222) to (1) the necessity of the equations of compatibility; (2) their sufficiency in the case of simply connected bodies and (3) the requirement of six additional conditions for each degree of multiple connectivity. Alternatively, the possibility $\Delta u_j^* \neq 0$ leads to the necessity of specifying (1) the incompatibility tensor throughout the body; (2) at each point of each surface of discontinuity, one of the eight combinations formed by choosing one member of each of the three products $T_{nn}\Delta u_n, T_{ns}\Delta u_s, T_{nt}\Delta u_t$; or, (3) both (1) and (2).

Another example of an extension of the scope of the uniqueness theorem is given by Sternberg and Eubanks (1955) who included the singularity corresponding to a concentrated force.]

It should be noted that (subject to the limitations mentioned) the uniqueness conditions are sufficient, rather than necessary, so that there may be other conditions sufficient for a unique solution. For example, one or more of the three components of surface-traction may be a function of the corresponding component of displacement, as in the case of an elastic support.

It should also be noted that the uniqueness theorem applies to an isolated body. In the case of a pair of bodies in juxtaposition, conditions relating to the continuity of all six components ($T_{nn}, T_{ns}, T_{nt}, u_n, u_s, u_t$) must be specified at each point of the interface.

1.06 Variational Equation of Motion

When the form of the strain-energy-function U is known, the displacement-equations of motion may be deduced from Hamilton's principle (Love, 1927, p. 166).

Let

$$\mathfrak{K} = \int_V K \, dV$$
$$\mathfrak{U} = \int_V U \, dV$$

be, respectively, the total kinetic and potential energies of the body and let

$$\delta W = \int_S t_j \, \delta u_j \, dS \tag{1.061}$$

be the work done by the surface-tractions when the displacement undergoes a variation δu_j between fixed values at an initial time t_0 and a final time t_1 . Then, by Hamilton's principle,

$$\delta \int_{t_0}^{t_1} (\mathfrak{K} - \mathfrak{U}) dt \; + \int_{t_0}^{t_1} \delta W \, dt = 0 \tag{1.062}$$

Now,

$$\delta \int_{t_0}^{t_1} \mathfrak{K} \, dt = \int_{t_0}^{t_1} dt \int_V \tfrac{1}{2} \rho \, \delta(\dot{u}_j \dot{u}_j) \, dV = \int_{t_0}^{t_1} dt \int_V \rho \dot{u}_j \frac{\partial}{\partial t}(\delta u_j) \, dV$$
$$= \int_V \rho \dot{u}_j \, \delta u_j \Big]_{t_0}^{t_1} dV - \int_{t_0}^{t_1} dt \int_V \rho \ddot{u}_j \, \delta u_j \, dV$$

Since δu_j vanishes at t_0 and t_1 , the first term on the right vanishes. Equation (1.062) then becomes the variational equation of motion

$$\int_V (\rho \ddot{u}_j \, \delta u_j + \delta U) \, dV = \int_S t_j \, \delta u_j \, dS \tag{1.063}$$

(Note that, if the variations in (1.063) are replaced by time derivatives, the variational equation of motion is converted to (1.021), i.e., the principle of conservation of energy.)

Again, with due regard to (1.0315),

$$\delta U = \frac{\partial U}{\partial S_{ij}} \delta S_{ij} = \frac{1}{2} \frac{\partial U}{\partial S_{ij}} \delta(u_{i,j} + u_{j,i}) = \frac{\partial U}{\partial S_{ij}} (\delta u_j)_{,i}$$
$$= \left(\frac{\partial U}{\partial S_{ij}} \delta u_j \right)_{,i} - \left(\frac{\partial U}{\partial S_{ij}} \right)_{,i} \delta u_j$$

and

$$\int_V \left(\frac{\partial U}{\partial S_{ij}} \delta u_j \right)_{,i} dV = \int_S \nu_i \frac{\partial U}{\partial S_{ij}} \delta u_j \, dS$$

Hence, the variational equation of motion becomes

$$\int_V \left[\left(\frac{\partial U}{\partial S_{ij}} \right)_{,i} - \rho \ddot{u}_j \right] \delta u_j \, dV + \int_S \left(t_j - \nu_i \frac{\partial U}{\partial S_{ij}} \right) \delta u_j \, dS = 0 \qquad (1.063)$$

The coefficients of the variations δu_j must vanish separately. Accordingly

$$\left(\frac{\partial U}{\partial S_{ij}} \right)_{,i} = \rho \ddot{u}_j \qquad (1.064)$$

throughout the body and, on the surface,

$$\nu_i \frac{\partial U}{\partial S_{ij}} = t_j \qquad (1.065)$$

1.07 Displacement-Equations

ubstituted in the stress-

equati

$$c_{ijkl} S_{kl,i} = \rho \ddot{u}_j \qquad (1.071)$$

The st ese to the displacement-

equati

$$c_{ijkl} \left(u_{k,li} + u_{l,ki} \right) = 2 \rho \ddot{u}_j \qquad (1.072)$$

reduces to

$$\mu u_{j,ii} + (\lambda + \mu) u_{i,ij} = \rho \ddot{u}_j$$

or

$$\mu \nabla^2 u_j + (\lambda+\mu) \frac{\partial \Delta}{\partial x_j} = \rho \ddot{u}_j \tag{1.074}$$

where ∇^2 is Laplace's operator and Δ is the dilatation:

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \tag{1.075}$$

$$\Delta = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \tag{1.076}$$

The displacement may be expressed in terms of displacement-potentials φ and H_i according to

$$\begin{aligned} u_1 &= \frac{\partial \varphi}{\partial x_1} + \frac{\partial H_3}{\partial x_2} - \frac{\partial H_2}{\partial x_3} \\ u_2 &= \frac{\partial \varphi}{\partial x_2} + \frac{\partial H_1}{\partial x_3} - \frac{\partial H_3}{\partial x_1} \\ u_3 &= \frac{\partial \varphi}{\partial x_3} + \frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial x_2} \end{aligned} \tag{1.077}$$

provided

$$\frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial x_2} + \frac{\partial H_3}{\partial x_3} = 0 \tag{1.078}$$

Then

$$\begin{aligned} \nabla^2 \varphi &= \Delta \\ \nabla^2 H_j &= -2\omega_j \end{aligned} \tag{1.079}$$

Thus, φ is the potential which gives rise to the dilatation and the H_j are the potentials which give rise to the components of rotation. In an isotropic material, the displacement-equations of motion are satisfied if the four potentials satisfy the equations (Love, 1927, p. 304; Poisson, 1829, p. 623)

$$\begin{aligned} V_1^2 \nabla^2 \varphi &= \ddot{\varphi} \\ V_2^2 \nabla^2 H_j &= \ddot{H}_j \end{aligned} \tag{1.0710}$$

where

$$V_1 = \sqrt{(} \qquad\qquad (1.0711)$$

$$V_2 =$$

are the velocities of the dilat otation, respectively.

Equations of the simple 10) do not appear to be available for anisotropic materials excep l cases of high elastic symmetry (Carrier, 1946).