Appendix H

Quasistatic Theory for Fields inside a Sphere

The scattering theory derived in Sec. 3.6 uses a quasistatic approximation to determine the fields inside a sphere. Here we will show how this quasistatic theory is derived.

We take the density of the sphere to be ρ'_{m0} and the density of the surrounding material to be ρ_{m0}. We assume that the applied pressure p_i is of the form

$$p_i = A_i e^{-jkz} \tag{H.1}$$

We take the origin of the coordinates of the center of a sphere of radius a and assume that $kz << 1$. In this case we can write

$$p_i = A_i(1 - jkz) = A_i(1 - jkr\cos\theta) \tag{H.2}$$

with

$$u_r = \frac{-jkr\cos\theta}{\omega^2\rho_{m0}}A_i \tag{H.3}$$

The pressure, in general, obeys the relation

$$\nabla^2 p + k^2 p = 0 \tag{H.4}$$

In the neighborhood of the sphere, however, we expect relatively rapid changes in p. Thus if

$$\left|\frac{\partial^2 p}{\partial x^2}\right| \quad \text{or} \quad \left|\frac{\partial^2 p}{\partial y^2}\right| \quad \text{or} \quad \left|\frac{\partial^2 p}{\partial z^2}\right| >> k^2$$

we can neglect the k^2 term in Eq. (H.4). This is the case if $ka << 1$, where a is the radius of the sphere. Within the sphere, the solution of Eq. (H.4) now has a variation identical to that given by a solution of Laplace's equation. We can use the quasistatic assumption to solve for the pressure in spherical coordinates by employing Laplace's equation,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial p}{\partial\theta} = 0 \tag{H.5}$$

This has solutions of the form

$$p = \sum_n P_n \cos(\theta)\left(A_n r^n + \frac{B_n}{r^{n+1}}\right) \tag{H.6}$$

where P_n (cos θ) is the nth-order Legendre function. We will need only $P_0(\cos\theta) = 1$ and $P_1(\cos\theta) = \cos\theta$. As there are only terms independent of θ or that vary as cos θ in the exciting pressure field inside the sphere, the pressure must take the form

$$p = A_0 + A_1 r\cos\theta \tag{H.7}$$

with

$$\omega^2\rho'_{m0}u_r = A_1\cos\theta \tag{H.8}$$

while outside the sphere the pressure is

$$p = A_i(1 - jkr\cos\theta) + \frac{B_0}{r} + \frac{B_1\cos\theta}{r^2} \tag{H.9}$$

with

$$\omega^2\rho_{m0}u_r = -jkA_i\cos\theta - \frac{B_0}{r^2} - \frac{2B_1\cos\theta}{r^3} \tag{H.10}$$

The boundary conditions at the surface of the sphere $r = a$ require that u_r and p must be continuous. Thus it follows that

$$A_0 = A_i + \frac{B_0}{a} \tag{H.11}$$

$$0 = \frac{B_0}{a^2} \tag{H.12}$$

$$A_1 = -jkA_i + \frac{B_1}{a^3} \tag{H.13}$$

and

$$A_1\frac{\rho_{m0}}{\rho_{m0'}} = -\frac{jkA_i}{\omega^2\rho_{m0}} - \frac{2B_1}{a^3} \tag{H.14}$$

We conclude that

$$A_0 = A_i \tag{H.15}$$

and

$$A_1 = \frac{3jkA_i}{\omega^2 \rho_{m0}(2 + \rho_{m0}/\rho_{m0'})} \tag{H.16}$$

The internal symmetric pressure term is equal therefore to the external applied symmetric pressure term, while the internal value of u_r varies as $\cos\theta$. After dealing similarly with the u_θ component, it follows that

$$u_z = \frac{3u_{zi}}{2 + \rho_{m0}/\rho'_{m0}} \tag{H.17}$$

where u_{zi} is the external applied displacement field and u_z is the internal field.

A very similar but more complicated form of the derivation is used to solve for the exact theory for the scattering of a wave incident on a sphere. In this case, we follow the same procedure, but we employ solutions of the wave equation (H.4) instead of the quasistatic solution of Laplace's equation.