Appendix G

Method of Stationary Phase

The method of minimum phase is a technique for evaluating exponential integrals in regions where the phase is changing rapidly. Suppose that we wish to evaluate

$$S = \int F(x)e^{j\phi(x)}\, dx \tag{G.1}$$

When the phase term $\phi(x)$ is changing rapidly with x, the exponential yields contributions that cancel each other out. Thus it is apparent that the main contribution to the integral is from the region where ϕ is changing least rapidly, that is, from the region around x_0 where

$$\left.\frac{\partial\phi}{\partial x}\right|_{x=x_0} = 0 \tag{G.2}$$

We may therefore write a Taylor expansion in the form

$$\phi(x) = \phi_0 + A(x - x_0)^2 \tag{G.3}$$

where

$$A = \left.\frac{1}{2}\frac{\partial^2\phi}{\partial x^2}\right|_{x=x_0} \tag{G.4}$$

Thus we write

$$S \approx F(x_0)e^{j\phi_0}\int_{-\infty}^{\infty} e^{jA(x-x_0)^2}\, dx \tag{G.5}$$

We have taken the limits of the integral to be $-\infty$ and ∞, because the contributions from the regions where $A(x - x_0)^2 >> 1$ are negligible. By substituting $z^2 = A(x - x_0)^2$, Eq. (G.5) becomes

$$S \approx \frac{F(x_0)e^{j\phi_0}}{\sqrt{A}} \int_{-\infty}^{\infty} e^{jz^2}\, dz \tag{G.6}$$

Equation (G.6) is in the form of a standard Fresnel integral, where

$$\sqrt{\frac{\pi}{2}} = \int_{-\infty}^{\infty} \cos x^2\, dx \qquad \sqrt{\frac{\pi}{2}} = \int_{-\infty}^{\infty} \sin x^2\, dx \tag{G.7}$$

Thus

$$S \approx (1 + j)F(x_0)e^{j\phi_0}\sqrt{\frac{\pi}{2A}} \tag{G.8}$$

with

$$A = \frac{1}{2}\frac{\partial^2\phi}{\partial x^2}\bigg|_{x=x_0} \tag{G.9}$$

and

$$\frac{\partial\phi}{\partial x} = 0 \quad \text{at} \quad x = x_0 \tag{G.10}$$