Appendix F

Transducer Admittance Matrix

We follow the method of analysis given by Smith et al. [1]. We obtain the three-port admittance matrix of the transducer by finding the matrix of one interdigital period and then applying a cascading formalism. The admittance matrix for one section can easily be found by standard circuit analysis of the circuit shown in Fig. 2.4.7. Because this circuit is symmetric, the appropriate form of the admittance matrix is

$$[y] = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{12} & y_{11} & -y_{13} \\ y_{13} & -y_{13} & y_{33} \end{bmatrix} \tag{F.1}$$

where

$$\begin{bmatrix} i_{n-1} \\ -i_n \\ i_{3n} \end{bmatrix} = [y] \begin{bmatrix} e_{n-1} \\ e_n \\ e_{3n} \end{bmatrix} \tag{F.2}$$

The values of the four independent elements differ accordingly as to whether the negative capacitor in Fig. 2.4.7 is short-circuited or included in the circuit. These two cases represent the *crossed-field* or *in-line* equivalent circuits, respectively (see Prob. 1.5.1).

1. The admittance parameters for the crossed-field model are

$$
\begin{aligned}
y_{11} &= -jG_0 \cot \theta \\
y_{12} &= jG_0 \csc \theta \\
y_{13} &= -jG_0 \tan\frac{\theta}{4} \\
y_{33} &= j\left(4G_0 \tan \frac{\theta}{4} + \omega C_s\right)
\end{aligned}
\tag{F.3}
$$

2. The admittance parameters for the in-line model are

$$
\begin{aligned}
y_{11} &= -jG_0 \cot \frac{\theta}{4} \left(x - \cot \frac{\theta}{2}\right)2 - \frac{[P - \csc (\theta/2)]^2}{[P - \cot (\theta/2)]^2} \\
y_{12} &= jG_0 \frac{\cot \theta/4\, [P - \csc (\theta/2)]^2}{2[2P - \cot (\theta/4)][P - \cot (\theta/2)]} \\
y_{13} &= -jG_0 \frac{\tan (\theta/4)}{1 - 2P \tan (\theta/4)} \\
y_{33} &= \frac{j\omega C_s}{1 - 2P \tan (\theta/4)}
\end{aligned}
\tag{F.4}
$$

where $G_0 = R_0$, $P = 2G_0/\omega C_s$, and $\theta = 2\pi\omega/\omega_0$.

The three-port matrix for the entire transducer is found by connecting the N periodic sections in cascade acoustically and in parallel electrically, as shown in Fig. 2.4.8.

The total transducer current is the sum of the currents flowing into the N sections. With the help of Eq. (F.1), we find that the total current into the electrical port is

$$
I_3 = \sum_{n=1}^{N} i_{3n} = y_{13}(e_0 = e_N) + y_{33}\sum_{n=1}^{N} e_{3n} \tag{F.5}
$$

We apply the boundary conditions ($e_0 = V_1$, $e_N = V_2$, and $e_{3n} = V_3$) and network symmetry to obtain the following results for the complete transducer, as illustrated in Fig. 2.4.8:

$$
\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{12} & -Y_{11} & -Y_{13} \\ Y_{13} & -Y_{13} & Y_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \tag{F.6}
$$

with

$$
\begin{aligned}
Y_{13} &= Y_{31} = y_{13} \\
Y_{23} &= Y_{32} = -y_{13} \\
Y_{33} &= Ny_{33}
\end{aligned}
\tag{F.7}
$$

From Eq. (F.2), the following recursion relation may also be found:

$$\begin{bmatrix} e_n \\ i_n \end{bmatrix} = [R]\begin{bmatrix} e_{n-1} \\ i_{n-1} \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} V_3 \tag{F.8}$$

where

$$[R] = \frac{1}{y_{12}}\begin{bmatrix} -y_{11} & 1 \\ y_{11}^2 - y_{12}^2 & -y_{11} \end{bmatrix} \tag{F.9}$$

The parameters d_1 and d_2 are also functions of y_{ij}, but will not be needed here. Applying Eq. (F.8) N times gives

$$\begin{bmatrix} e_N \\ i_N \end{bmatrix} = [R]^N\begin{bmatrix} e_0 \\ i_0 \end{bmatrix} + \sum_{n=0}^{N-1} [R]^n \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} V_3 \tag{F.10}$$

Solving Eq. (F.10) for i_N and i_0, and again applying the boundary conditions, gives the result

$$\begin{aligned} Y_{11} &= Y_{22} = \frac{-R_{N11}}{R_{N12}} \\ Y_{12} &= Y_{21} = \frac{1}{R_{N12}} \end{aligned} \tag{F.11}$$

where $[R_N] = [R]^N$. Equations (F.9) and (F.11) summarize the admittance parameters for the entire transducer. For the crossed-field model, the recursion matrix becomes

$$[R] = \begin{bmatrix} \cos\theta & -jR_0 \sin\theta \\ -jG_0 \sin\theta & \cos\theta \end{bmatrix} \tag{F.12}$$

This is the familiar circuit matrix for a transmission line of impedance R_0. Matrix $[S]$ is then obtained simply by replacing the θ in $[R]$ with $N\theta$.

In-Line Model

For frequencies near acoustic synchronism ($\theta = 2\pi\omega/\omega_0 \approx 2\pi$), the admittance matrix for both in-line and crossed-field models can be reduced to a much simpler form. In particular, for the in-line model at $\theta = 2\pi$, the matrix of one periodic section becomes

$$[y] = \frac{j\omega_0 C_s}{16}\begin{bmatrix} -1 & 1 & 4 \\ 1 & -1 & -4 \\ 4 & -4 & 0 \end{bmatrix} \tag{F.13}$$

It can be seen from Eq. (F.12) that $[R]$ is in canonical form, so that the transducer matrix is

$$[Y] = \frac{j\omega_0 C_s}{16}\begin{bmatrix} -\frac{1}{N} & \frac{1}{N} & 4 \\ \frac{1}{N} & -\frac{1}{N} & -4 \\ 4 & -4 & 0 \end{bmatrix} \tag{F.14}$$

Crossed-Field Model

For the crossed-field model, the matrix elements are infinite at $\theta = 2\pi$. However, the matrix can be simplified by writing $\theta = 2\pi + \delta$ and expanding the elements to first order in δ. The result is given in Eq. (2.4.24).

REFERENCE

1. W. R. Smith, H. M. Gerard, J. H. Collins, T. M. Reeder, and H. J. Shaw, "Analysis of Interdigital Surface Wave Transducers by Use of an Equivalent Circuit Model," *IEEE Trans. on Microwave Theory and Techniques*, MTT-17, No. 11 (Nov. 1969), 856–64.