Appendix E

A Rigorous Derivation of Normal-Mode Theory

Here we will carry out a derivation of the normal-mode theory that is parallel to that of Sec. 2.5.2, using a rigorous approach based directly on the constitutive relations for the waves. To keep the technique as simple as possible, we will carry out the derivation for a nonpiezoelectric medium. This can be easily generalized to cover the case of a piezoelectric medium, yielding a result similar to Eq. (2.5.25).

We first obtain an orthogonality theorem for acoustic waves. This is needed to determine how an acoustic wave is excited by an external source or by a perturbation (i.e., to determine the coefficient α). It is shown in Appendix A that the three-dimensional equation of motion for an acoustic wave, in which all components vary as $\exp(j\omega t)$, is

$$\nabla \cdot \hat{\mathbf{T}} = j\omega\rho_{m0}\hat{\mathbf{v}} \tag{E.1}$$

or

$$\frac{\partial \hat{T}_{ij}}{\partial x_i} = j\omega\rho_{m0}\hat{v}_j \tag{E.2}$$

where we have used the sign $\hat{}$ to indicate parameters that vary with z. The relationship between $\hat{\mathbf{v}}$ and $\hat{\mathbf{S}}$ is, in reduced notation,

$$\nabla_s\hat{\mathbf{v}} = j\omega\hat{\mathbf{S}} \tag{E.3}$$

or, in tensor notation,

$$\frac{1}{2}\left(\frac{\partial \hat{v}_i}{\partial x_j} + \frac{\partial \hat{v}_j}{\partial x_i}\right) = j\omega\hat{S}_{ij} \tag{E.4}$$

Now, suppose that there are two modes of the system denoted by the subscripts m and n, such that

$$\begin{aligned} \hat{\mathbf{T}}_n(x, y, z) &= \mathbf{T}_n(x, y)e^{-jk_nz} \\ \hat{\mathbf{T}}_m(x, y, z) &= \mathbf{T}_m(x, y)e^{-jk_mz} \end{aligned} \tag{E.5}$$

Similar definitions can be given for the other fields associated with these waves. By writing the appropriate equations [Eqs. (E.1) and (E.2)] for each mode, as well as their complex conjugates (i.e., reversing the sign of $j\omega$), we obtain the following relation:

$$\hat{\mathbf{v}}_m^* \cdot \nabla \cdot \hat{\mathbf{T}}_n + \hat{\mathbf{T}}_n : \nabla_s \hat{\mathbf{v}}_m^* = j\omega\rho_{m0}\hat{\mathbf{v}}_m^* \cdot \hat{\mathbf{v}}_n - j\omega\hat{\mathbf{S}}_m^* : \hat{\mathbf{T}}_n \tag{E.6}$$

or

$$\hat{v}_{mi}^* \frac{\partial \hat{T}_{nij}}{\partial x_j} + \hat{T}_{nij} \frac{\partial \hat{v}_{mi}^*}{\partial x_j} = j\omega\rho_{m0}\hat{v}_{mi}^*\hat{v}_{ni} - j\omega\hat{S}_{mij}^*\hat{T}_{nij} \tag{E.7}$$

Using the tensor relation $\mathbf{a} \cdot \nabla \cdot \mathbf{B} + \mathbf{B} : \nabla_s\mathbf{a} = \nabla \cdot (\mathbf{a} \cdot \mathbf{B})$, Eq. (E.6) can be written in the form

$$\nabla \cdot (\hat{\mathbf{v}}_m^* \cdot \hat{\mathbf{T}}_n) = j\omega\rho_{m0}\hat{\mathbf{v}}_m^* \cdot \hat{\mathbf{v}}_n - j\omega\hat{\mathbf{S}}_m^* : \hat{\mathbf{T}}_n \tag{E.8}$$

or in the form

$$\frac{\partial}{\partial x_j} (\hat{v}_{mi}^*\hat{T}_{nij}) = j\omega\rho_{m0}\hat{v}_{mi}^*\hat{v}_{ni} - j\omega\hat{S}_{mij}^*\hat{T}_{nij} \tag{E.9}$$

It follows similarly that

$$\nabla \cdot (\hat{\mathbf{v}}_n \cdot \hat{\mathbf{T}}_m^*) = -j\omega\rho_{m0}\hat{\mathbf{v}}_m^* \cdot \hat{\mathbf{v}}_n + j\omega\hat{\mathbf{T}}_m^* : \hat{\mathbf{S}}_n \tag{E.10}$$

By using the constitutive relations

$$\begin{aligned} \hat{\mathbf{T}}_m &= \mathbf{c} : \hat{\mathbf{S}}_m \\ \hat{\mathbf{T}}_n &= \mathbf{c} : \hat{\mathbf{S}}_n \end{aligned} \tag{E.11}$$

and the symmetry of the tensor **c**, we can add Eqs. (E.8) and (E.10) to show that

$$\nabla \cdot (\hat{\mathbf{v}}_m^* \cdot \mathbf{T}_n + \hat{\mathbf{v}}_n \cdot \hat{\mathbf{T}}_m) = 0 \tag{E.12}$$

or

$$\frac{\partial}{\partial x_j} (\hat{v}_{mi}^*\hat{T}_{nij} + \hat{v}_{ni}\hat{T}_{mij}^*) = 0 \tag{E.13}$$

We can now take advantage of Gauss's theorem, which is commonly employed in electromagnetic (EM) theory, and use it for this acoustic problem. With the reduced notation, we can write, for a vector **B**, $\int \nabla \cdot \mathbf{B}\, dV = \int \mathbf{B} \cdot \mathbf{n}\, ds$, where the surface integral is taken around the enclosing surface of the volume V and **n** is the outward vector normal to the enclosing surface.

We consider a region of length dz in a cylindrical system uniform in the z

direction, as illustrated in Fig. 2.5.1, and integrate through its volume. This yields the two-dimensional form of Gauss's theorem,

$$\int_s \nabla \cdot \mathbf{B}\, ds = \int_s \frac{\partial B_z}{\partial z}\, ds + \oint_l \mathbf{B} \cdot \mathbf{n}\, dl \tag{E.14}$$

where s is the cross section of the system and the line integral is taken around the circumference l of this area.

By using the relation in Eq. (E.14), it follows that

$$\frac{\partial}{\partial z} \int_s (\hat{\mathbf{v}}_m^* \cdot \hat{\mathbf{T}}_n + \hat{\mathbf{v}}_n \cdot \hat{\mathbf{T}}_m^*)_z\, ds + \oint_l (\hat{\mathbf{v}}_m^* \cdot \hat{\mathbf{T}}_n + \hat{\mathbf{v}}_n \cdot \hat{\mathbf{T}}_m^*) \cdot \mathbf{n}\, dl = 0 \tag{E.15}$$

We take the boundary condition at the surface of the cylinder to be either: (1) $\hat{\mathbf{T}} \cdot \mathbf{n} = 0$ (i.e., a stress-free boundary); or (2) $\mathbf{v} = 0$ (i.e., a rigid boundary). In this case, the line integral is zero and we find that

$$\frac{\partial}{\partial z} \int (\hat{\mathbf{v}}_m^* \cdot \hat{\mathbf{T}}_n + \hat{\mathbf{v}}_n \cdot \hat{\mathbf{T}}_m^*)_z\, ds = 0 \tag{E.16}$$

It then follows from Eq. (E.16) that

$$(k_m^* - k_n) \int (\hat{\mathbf{v}}_m^* \cdot \mathbf{T}_n + \mathbf{v}_n \cdot \mathbf{T}_m^*)_z\, ds = 0 \tag{E.17}$$

From this equation, either $k_n = k_m^*$ or

$$\int (\hat{\mathbf{v}}_m^* \cdot \mathbf{T}_n + \mathbf{v}_n \cdot \mathbf{T}_m^*)_z\, ds = 0 \tag{E.18}$$

which is known as the *orthogonality condition*. Note that $k_m^* = k_m$ for a propagating wave, but for a nonpropagating wave or cutoff mode, $k_m^* = -k_m$.

Normal-mode expansion. We now consider how to express the fields at any plane z of the system. We suppose that at any cross section, the total field is the sum of the fields of the individual modes of the system. Thus we write

$$\hat{\mathbf{T}}(x, y, z) = \sum_n A_n(z)\, \mathbf{T}_n(x, y) \tag{E.19}$$

and

$$\hat{\mathbf{v}}(x, y, z) = \sum_n A_n(z) \mathbf{v}_n(x, y) \tag{E.20}$$

Because of the orthogonality condition, it follows that the total power flow at the plane z is

$$p = -\tfrac{1}{2} \operatorname{Re} \int (\hat{\mathbf{T}} \cdot \hat{\mathbf{v}}^*)_z\, ds = -\tfrac{1}{4} \int (\hat{\mathbf{T}} \cdot \hat{\mathbf{v}}^* + \mathbf{T}^* \cdot \hat{\mathbf{v}})_z\, ds \tag{E.21}$$

Substituting from Eqs. (E.19) and (E.20), we see that

$$P = -\tfrac{1}{4} \sum A_n A_n^* \int (\mathbf{T}_n \cdot \mathbf{v}_n^* + \mathbf{T}_n^* \cdot \mathbf{v}_n)_z\, ds \tag{E.22}$$

or

$$P = \sum_n A_n A_n^* P_n \tag{E.23}$$

We shall define the amplitudes of T_n and v_n so that they correspond to unit power flow with $P_n = 1$ for forward waves and $P_n = -1$ for backward waves, that is, for a forward wave,

$$\tfrac{1}{4} \int_s (\mathbf{T}_n \cdot \mathbf{v}_n^* + \mathbf{T}_n^* \cdot \mathbf{v}_n)_z \, ds = -1 \tag{E.24}$$

It follows that the total power P is

$$P = \sum_n A_n A_n^* - B_n B_n^* \tag{E.25}$$

where B_n is the amplitude of the nth backward wave. Note that Eq. (E.24) is the generalization of Eq. (2.5.15).

The amplitude of the nth mode at the plane Z can be found by multiplying Eq. (E.19) by $v_n^*(x, y)$ and Eq. (E.20) by $T_n^*(x, y)$, adding the results, and integrating over the cross section. Because of the orthogonality condition, it follows that

$$A_n = -\tfrac{1}{4} \int (\hat{\mathbf{T}} \cdot \mathbf{v}_n^* + \mathbf{T}_n^* \cdot \hat{\mathbf{v}})_z \, ds \tag{E.26}$$

Thus if we know T and v at $z = 0$, we know T and v at all planes z.

We now consider how a wave is excited by an incident signal at the enclosing surface, or a perturbation on the enclosing surface. We take the unperturbed boundary condition to be $\mathbf{T}_n \cdot \mathbf{n} = 0$ at the enclosing surface (stress-free boundary conditions). The fields T and v obey Eqs. (E.1) and (E.2) within the volume of the system. However, $\mathbf{T}$ does not necessarily obey stress-free boundary conditions at the surface.

Following the same type of derivation that we made for Eq. (E.15), it follows that

$$\frac{\partial}{\partial z} \int (\hat{\mathbf{v}} \cdot \hat{\mathbf{T}}_n^* + \hat{\mathbf{v}}_n^* \cdot \hat{\mathbf{T}})_z \, ds = -\oint (\hat{\mathbf{v}} \cdot \hat{\mathbf{T}}_n^* + \hat{\mathbf{v}}_n^* \cdot \hat{\mathbf{T}}) \cdot \mathbf{n} \, dl \tag{E.27}$$

Substituting from Eq. (E.5), it follows that

$$\frac{\partial}{\partial z} e^{jk_n z} \int (\hat{\mathbf{v}} \cdot \mathbf{T}_n^* + \mathbf{v}_n^* \cdot \mathbf{T})_z \, ds = e^{jk_n z} \oint (\hat{\mathbf{v}} \cdot \mathbf{T}_n^* + \mathbf{v}_n^* \cdot \hat{\mathbf{T}}) \cdot \mathbf{n} \, dl \tag{E.28}$$

A further substitution from Eq. (E.26) and the use of the boundary condition $\mathbf{T}_n^* \cdot \mathbf{n} = 0$ leads to the result

$$\frac{dA_n}{dz} + jk_n A_n = \tfrac{1}{4} \oint (\mathbf{v}_n^* \cdot \hat{\mathbf{T}}) \cdot \mathbf{n} \, dl \tag{E.29}$$

Thus we have found a rigorously correct expression for the excitation of the nth mode by a stress field at the surface. This is the generalization for a nonpiezoelectric medium. With it, we can determine, for instance, how a perturbation at the surface of a substrate on which a Rayleigh wave can propagate will affect the amplitude of the wave.

A similar expression for the amplitude of the backward wave is

$$\frac{dB_n}{dz} - jk_nB_n = -\tfrac{1}{4}\int (\mathbf{v}_{-n}^* \cdot \hat{\mathbf{T}}) \cdot \mathbf{n}\, dl \tag{E.30}$$

where $\mathbf{v}_{-n}$ is the velocity field associated with the backward wave. Note that for any particular component in a planar system with the surface at $y = 0$, it follows by symmetry that if

$$v_{-nz} = -v_{nz} \tag{E.31}$$

then

$$v_{-ny} = v_{ny} \tag{E.32}$$

It follows, for example, that if the only term of importance at the surface $y = 0$ is T_2, then only the v_{ny} term matters, and Eq. (E.30) becomes

$$\frac{dB_n}{dz} - jk_nB_n = -\tfrac{1}{4}\int v_{ny}^* \hat{T}_2\, dx \tag{E.33}$$

Similar relations can also be derived for the cutoff modes. Such modes are required to form a complete set.

The same type of analysis can be carried through for acoustic waves in piezoelectric materials. Suppose that the potential associated with the nth wave is ϕ_n and the boundary condition at the surface is that of continuity of displacement density $\mathbf{D}_n \cdot \mathbf{n}$. Then, if a surface charge $\hat{\rho}_s = \hat{D}^+ - \hat{D}^-$ is introduced, as would be the case if electrodes were placed on the substrate, the equivalent of Eq. (E.29) is

$$\frac{dA_n}{dz} + jk_nA_n = \frac{j\omega}{4}\oint \hat{\rho}_s\phi_n^*\, dl \tag{E.34}$$

Alternatively, suppose that the boundary condition, initially, is $\phi_n = 0$ (short-circuit boundary condition) and the potential is changed to make ϕ finite, which occurs if a gap is cut in a metal film placed on the substrate. In such a case, the expression for the amplitude variation of the nth mode is

$$\frac{dA_n}{dz} + jk_nA_n = -\frac{j\omega}{4}\oint (\hat{\phi}\mathbf{D}_n^*) \cdot \mathbf{n}\, dl \tag{E.35}$$