Appendix C

Poynting's Theorem in Piezoelectric Media

We derive Poynting's theorem for piezoelectric media by generalizing the derivation of Eq. (1.1.21), the one-dimensional Poynting's theorem for nonpiezoelectric media. We shall show that Poynting's theorem can be generalized by adding the electromagnetic power flow in the medium to the mechanical power flow.

Appendix A shows that the three-dimensional equation of motion for an acoustic wave, in which all components vary as exp $(j\omega t)$, is, in symbolic notation,

$$\nabla \cdot \mathbf{T} = j\omega\rho_{m0}\mathbf{v} \tag{C.1}$$

or, in tensor notation,

$$\frac{\partial T_{ij}}{\partial x_j} = j\omega\rho_{m0}v_i \tag{C.2}$$

Similarly, the relation between $\mathbf{v}$ and $\mathbf{S}$ is, in symbolic notation,

$$\nabla_s\mathbf{v} = j\omega\mathbf{S} \tag{C.3}$$

or, in tensor notation,

$$\frac{\partial v_i}{\partial x_j} = j\omega S_{ij} \tag{C.4}$$

It is convenient to consider the expression related to the Poynting vector

$$\frac{\partial}{\partial x_j}(v_i^* T_{ij}) = T_{ij}\frac{\partial v_i^*}{\partial x_j} + v_i^*\frac{\partial T_{ij}}{\partial x_j} \tag{C.5}$$

Substituting from Eq. (C.2) and the complex conjugate form of Eq. (C.4) into Eq. (C.5), it follows that

$$\frac{\partial}{\partial x_j}(v_i^* T_{ij}) = j\omega(\rho_{m0} v_i v_i^* - T_{ij} S_{ij}^*) \tag{C.6}$$

In reduced symbolic notation, Eq. (C.6) becomes

$$\nabla \cdot (\mathbf{v}^* \cdot \mathbf{T}) = j\omega(\rho_{m0}\mathbf{v} \cdot \mathbf{v}^* - \mathbf{T} : \mathbf{S}^*) \tag{C.7}$$

To keep the notation simple, it will be more convenient from this point to use the symbolic tensor notation. Thus, from Eq. (C.7) and the piezoelectric constitutive relation

$$\mathbf{T} = \mathbf{c}^E : \mathbf{S} - \mathbf{e} \cdot \mathbf{E} \tag{C.8}$$

we see that

$$\nabla \cdot (\mathbf{v}^* \cdot \mathbf{T}) = j\omega\rho_{m0}\mathbf{v} \cdot \mathbf{v}^* - j\omega\mathbf{S}^* : \mathbf{c}^E : \mathbf{S} + j\omega\mathbf{S}^* : \mathbf{e} \cdot \mathbf{E} \tag{C.9}$$

It follows from symmetry that $\mathbf{S}^* : \mathbf{e} \cdot \mathbf{E} = \mathbf{E} \cdot \mathbf{e} : \mathbf{S}^*$. Hence we can substitute the piezoelectric constitutive relation

$$\mathbf{D} = \boldsymbol{\varepsilon}^S \cdot \mathbf{E} + \mathbf{e} : \mathbf{S} \tag{C.10}$$

into Eq. (C.9), to write it in the form

$$\begin{aligned}\nabla \cdot (\mathbf{v}^* \cdot \mathbf{T}) = j\omega\rho_{m0}\mathbf{v} \cdot \mathbf{v}^* - j\omega\mathbf{S}^* : \mathbf{c}^E : \mathbf{S} \\ - j\omega\mathbf{E}^* \cdot \boldsymbol{\varepsilon}^s \cdot \mathbf{E} + j\omega\mathbf{D}^* \cdot \mathbf{E}\end{aligned} \tag{C.11}$$

Let us now consider the last term in Eq. (C.11). We use the complex conjugate form of Maxwell's equation,

$$\nabla \times \mathbf{H}^* = -j\omega\mathbf{D}^* + \mathbf{i}_c^* \tag{C.12}$$

where $\mathbf{i}_c$ is the conduction current in the medium. After taking the dot product of Eq. (C.12) with $\mathbf{E}$, it follows that

$$j\omega\mathbf{D}^* \cdot \mathbf{E} = \mathbf{E} \cdot \mathbf{i}_c^* - \mathbf{E} \cdot (\nabla \times \mathbf{H}^*) \tag{C.13}$$

We now use the vector relation

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \tag{C.14}$$

with Maxwell's equation,

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{C.15}$$

and it follows that Eq. (C.13) can be written in the form

$$\begin{aligned} j\omega\mathbf{D}^* \cdot \mathbf{E} &= \mathbf{E} \cdot \mathbf{i}_c^* + \nabla \cdot (\mathbf{E} \times \mathbf{H}^*) - \mathbf{H}^* \cdot (\nabla \times \mathbf{E}) \\ &= \mathbf{E} \cdot \mathbf{i}_c^* + \nabla \cdot (\mathbf{E} \times \mathbf{H}^*) + j\omega\mu\mathbf{H} \cdot \mathbf{H}^* \end{aligned} \tag{C.16}$$

Finally, substituting Eq. (C.16) into Eq. (C.11) leads to the following relation:

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^* - \mathbf{v}^* \cdot \mathbf{T}) = j\omega \mathbf{S}^* : \mathbf{c}^E : \mathrm{S} - j\omega\rho_{m0}\mathbf{v} \cdot \mathbf{v}^*$$
$$+ j\omega\mathbf{E}^* \cdot \boldsymbol{\varepsilon}^s \cdot E - j\omega\mu\mathbf{H}^* \cdot \mathbf{H} - i_c^* \cdot \mathbf{E} \qquad \text{(C.17)}$$

Assuming that $\mathbf{c}^E$, ρ_{m0}, $\boldsymbol{\varepsilon}^s$, and μ are real, and adding Eq. (C.17) to its complex conjugate, it follows that

$$\mathrm{Re}\,[\nabla \cdot (\mathbf{E} \times \mathbf{H}^* - \mathbf{v}^* \cdot \mathbf{T})] = -\mathrm{Re}\,(\mathbf{i}_c^* \cdot \mathbf{E}) \qquad \text{(C.18)}$$

Suppose that we now integrate this relation over a region of volume V enclosed by a surface s. After using Gauss's theorem, we find that

$$\frac{1}{2}\,\mathrm{Re} \int_s (\mathbf{E} \times \mathbf{H}^* - \mathbf{v}^* \cdot \mathbf{T}) \cdot \mathbf{n}\, ds = -\frac{1}{2}\,\mathrm{Re} \int (\mathbf{i}_c^* \cdot \mathbf{E}\, dV) \qquad \text{(C.19)}$$

where $\mathbf{n}$ is the outward normal to the surface.

The left-hand side of this equation is the total power emitted from the volume V by the piezoelectric material. Therefore, the generalization of both the electromagnetic and mechanical forms of Poynting's theorem agree with intuition. We merely add the two contributions to the total power.