

SAN FRANCISCO



2011 JOINT CONFERENCE OF THE
IEEE INTERNATIONAL FREQUENCY CONTROL SYMPOSIUM
& EUROPEAN FREQUENCY AND TIME FORUM

May 1-5, 2011 | San Francisco, California, USA

Variance Measurements

Practical Use - Statistics - Long Term Prediction

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Outline

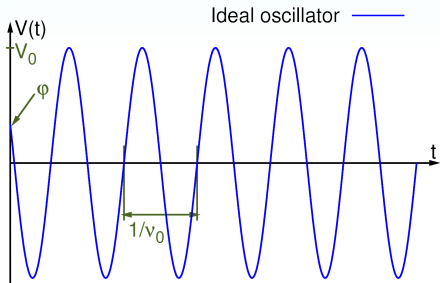
- 1 Introduction
- 2 Practical use of the Allan variance
- 3 Statistics of the Allan variance and the Allan deviation
- 4 Prediction of very long term time stability

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- 1 Introduction
 - Notations in the time domain
 - Notations in the frequency domain
 - Noise model
- 2 Practical use of the Allan variance
- Statistics of the Allan variance and the Allan deviation
- Prediction of very long term time stability

Introduction

Notations in the time domain



$$V(t) = V_0 \sin [2\pi\nu_0 t + \varphi]$$

where $\varphi(t)$ is the phase “noise”

- Time error $x(t)$:

$$V(t) = V_0 \sin [2\pi\nu_0 (t + x(t))]$$

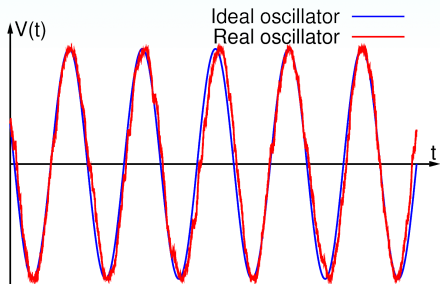
$$\text{with } x(t) = \frac{\varphi(t)}{2\pi\nu_0} \quad [\text{s}]$$

“My watch is 39 seconds late”:

- $t_{\text{watch}} = 10 \text{ h } 10 \text{ min } 37 \text{ s}$
- $t_{\text{ref}} = 10 \text{ h } 11 \text{ min } 16 \text{ s}$
- $\Rightarrow x(t) = -39 \text{ s}$

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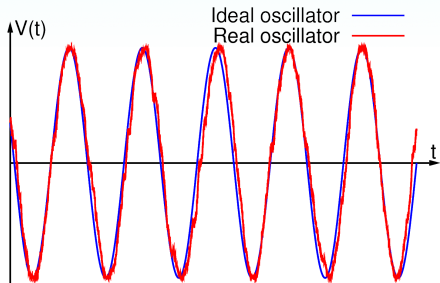
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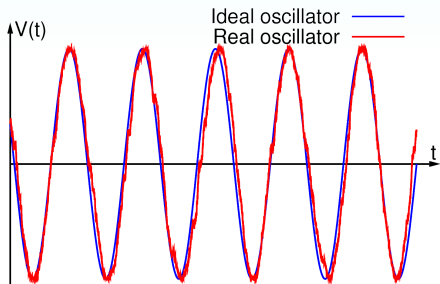
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Frequency noise

$$V(t) = V_0 \sin [2\pi\nu_0 t + \varphi(t)]$$

- Instantaneous frequency $\nu(t)$:

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with
$$\nu(t) = \frac{1}{2\pi} \frac{d[2\pi\nu_0 t + \varphi(t)]}{dt} = \nu_0 + \frac{1}{2\pi} \frac{d\varphi(t)}{dt} \quad [Hz]$$

- Frequency noise $\Delta\nu(t)$:

$$\Delta\nu(t) = \frac{1}{2\pi} \frac{d\varphi(t)}{dt} \quad [Hz]$$

- Frequency deviation $y(t)$:

$$y(t) = \frac{\Delta\nu(t)}{\nu_0} = \frac{1}{2\pi\nu_0} \frac{d\varphi(t)}{dt} \quad [dimensionless]$$

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Frequency noise vs Phase noise

Phase and frequency noise: 2 representations of 1 phenomenon

$$\left. \begin{aligned} x(t) &= \frac{\varphi(t)}{2\pi\nu_0} \\ y(t) &= \frac{1}{2\pi\nu_0} \frac{d\varphi(t)}{dt} \end{aligned} \right\} \Rightarrow y(t) = \frac{dx(t)}{dt}$$

A fundamental difference:

- $\varphi(t)$ and $x(t)$ are **instantaneous**
- $\Delta\nu(t)$ and $y(t)$ have to be **averaged**

$$\bar{y}_k = \frac{1}{\tau} \int_{t_k}^{t_k+\tau} y(t) dt = \frac{x(t_k + \tau) - x(t_k)}{\tau}$$

Frequency noise vs Phase noise

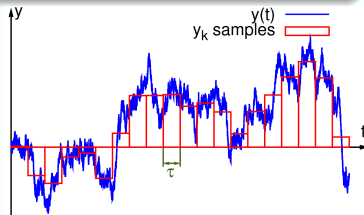
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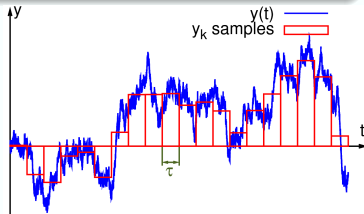
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Notations in the frequency domain

Power Spectral Densities (PSD)

- Fourier Transform (finite energy):

$$\Phi(f) = \int_{-\infty}^{+\infty} \varphi(t) e^{-j2\pi ft} dt \quad [s]$$

- Energy Spectral Density (finite energy):

$$|\Phi(f)|^2 = \left| \int_{-\infty}^{+\infty} \varphi(t) e^{-j2\pi ft} dt \right|^2 \quad [s^2]$$

- Power Spectral Density (finite power):

$$S_{\varphi}(f) = \left\langle \lim_{T \rightarrow \infty} \left[\frac{1}{T} \left| \int_{-\infty}^{+\infty} \varphi(t) e^{-j2\pi ft} dt \right|^2 \right] \right\rangle \quad [s] \equiv [Hz^{-1}]$$

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Relationships between PSD

Time error PSD: $S_x(f)$

- $x(t) = \frac{\varphi(t)}{2\pi\nu_0} \quad \Rightarrow \quad S_x(f) = \frac{1}{4\pi^2\nu_0^2} S_\varphi(f)$
- Dimension: $[s^3] \equiv [Hz^{-3}]$

Frequency deviation PSD: $S_y(f)$

- $y(t) = \frac{1}{2\pi\nu_0} \frac{d\varphi(t)}{dt} \quad \Rightarrow \quad S_y(f) = \frac{f^2}{\nu_0^2} S_\varphi(f)$
- $y(t) = \frac{dx(t)}{dt} \quad \Rightarrow \quad S_y(f) = 4\pi^2 f^2 S_x(f)$
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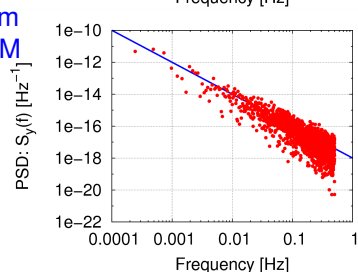
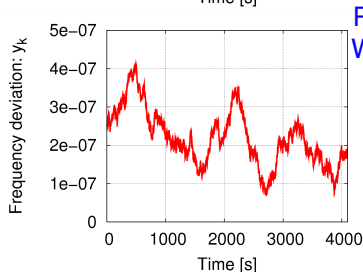
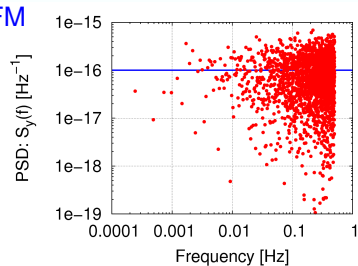
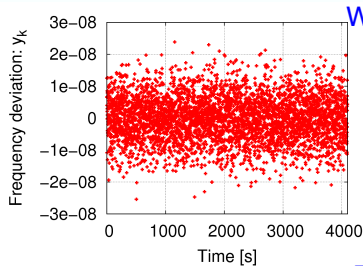
Noise model

The power law noise model

$$S_y(f) = \sum_{\alpha=-2}^{+2} h_{\alpha} f^{\alpha} \quad \alpha \text{ integer}$$

$S_y(f)$	$S_{\varphi}(f)$	Noise type	Origin
$h_{-2}f^{-2}$	$b_{-4}f^{-4}$	Random Walk Freq. Mod.	Environment
$h_{-1}f^{-1}$	$b_{-3}f^{-3}$	Flicker F.M.	Resonator
h_0	$b_{-2}f^{-2}$	White F.M.	Thermal noise
h_1f	$b_{-1}f^{-1}$	Flicker Phase Mod.	Electronic noise
h_2f^2	b_0	White P.M.	External white noise

White FM vs Random Walk FM



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- 1 Introduction
- 2 **Practical use of the Allan variance**
 - A statistical estimator as well as a spectral analysis tool
 - Practical calculation of the Allan variance
 - Allan variance versus Allan deviation
- 3 Statistics of the Allan variance and the Allan deviation
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A statistical estimator

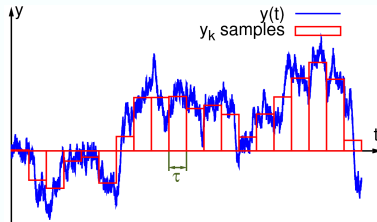
as well as a spectral analysis tool

- Definition of the true variance:

$$I^2(\tau) = \langle (\bar{y}_k - \langle \bar{y}_k \rangle)^2 \rangle$$

- Estimation of the true variance:

$$\sigma^2(N, \tau) = \frac{1}{N-1} \sum_{i=1}^N \left(\bar{y}_i - \frac{1}{N} \sum_{j=1}^N \bar{y}_j \right)^2$$



- The Allan variance (2-sample variance):**

$$\sigma_y^2(\tau) = \langle \sigma^2(2, \tau) \rangle = \sum_{i=1}^2 \left(\bar{y}_i - \frac{1}{2} \sum_{j=1}^2 \bar{y}_j \right)^2$$

$$\sigma_y^2(\tau) = \frac{1}{2} \langle (\bar{y}_2 - \bar{y}_1)^2 \rangle = \text{AVAR}(\tau)$$

$\langle \rangle$ stands for:

- ensemble average
- time average
- \equiv convolution...

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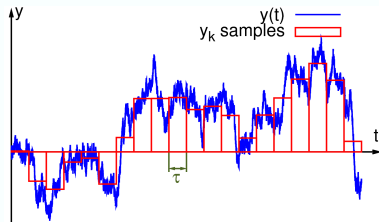
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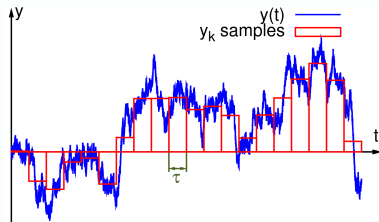
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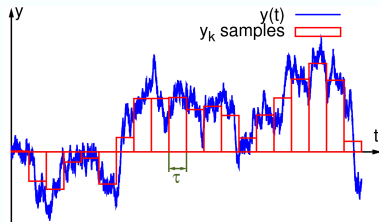
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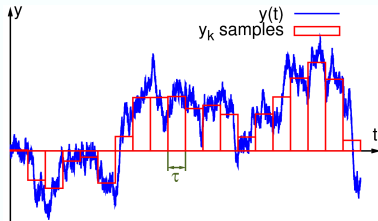
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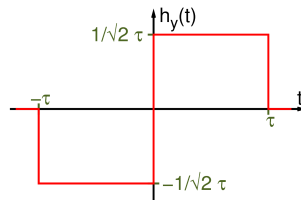
A spectral analysis tool

as well as a statistical estimator

Convolution in the time domain. . .

$$\sigma_y^2(\tau) = \left[\int_{-\infty}^{+\infty} y(t) h_y(t_k - t) dt \right]^2$$

$$\text{with } \begin{cases} h_y(t) = \frac{-1}{\sqrt{2\tau}} & \text{if } -\tau \geq t < 0 \\ h_y(t) = \frac{+1}{\sqrt{2\tau}} & \text{if } 0 \leq t < \tau \\ h_y(t) = 0 & \text{else} \end{cases}$$



... filtering in the frequency domain

$$\sigma_y^2(\tau) = \int_0^{\infty} S_y(f) |H_y(f)|^2 df$$

$$\text{with } |H_y(f)|^2 = |\text{FT}[h_y(t)]|^2 = 2 \frac{\sin^4(\pi \tau f)}{(\pi \tau f)^2}$$

A spectral analysis tool

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Convolution in the time domain. . .

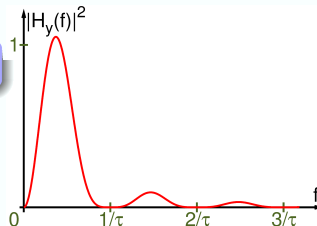
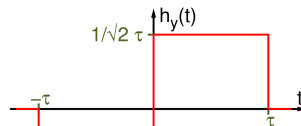
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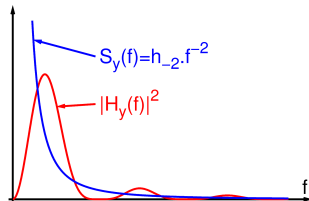


Convergence criterion: the moment condition

Convergence for drift

$\sigma_y^2(\tau)$ is a first-order difference (derivative):

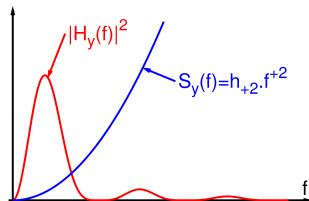
- not sensitive to constant (lin. ph. drift)
- sensitive to linear frequency drift



Convergence for power-law noise

$$\sigma_y^2(\tau) = \int_0^{\infty} h_{\alpha} f^{\alpha} |H_y(f)|^2 df$$

- converges for f^{-2} , f^{-1} and white FM
- does not converge for f^1 and f^2 FM



The moment condition

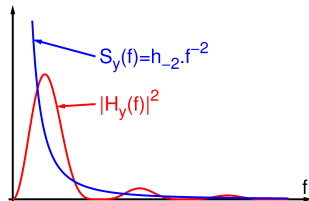
$$\int_{-\infty}^{+\infty} |H_y(f)|^2 f^{\alpha} df \text{ converges} \Leftrightarrow \int_{-\infty}^{+\infty} h_y(t) t^q dt = 0 \text{ for } 0 \leq q \leq \frac{-\alpha - 1}{2}$$

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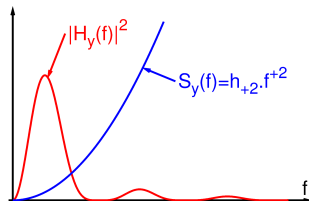
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Convergence for power-law noise

$$\sigma_y^2(\tau) = \int_0^{\infty} h_{\alpha} f^{\alpha} |H_y(f)|^2 df$$

- converges for f^{-2} , f^{-1} and white FM
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The moment condition

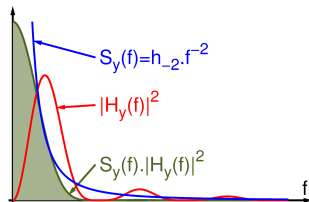
$$\int_{-\infty}^{+\infty} |H_y(f)|^2 f^{\alpha} df \text{ converges} \Leftrightarrow \int_{-\infty}^{+\infty} h_y(t) t^q dt = 0 \text{ for } 0 \leq q \leq \frac{-\alpha - 1}{2}$$

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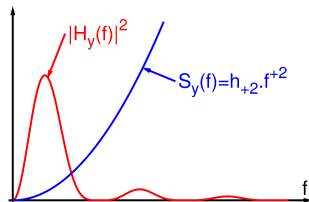
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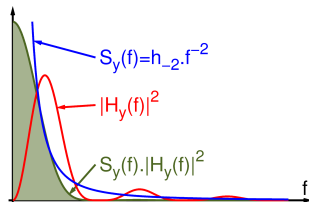
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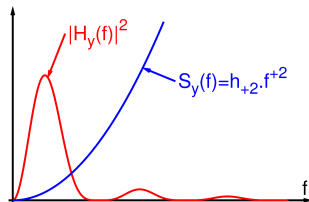
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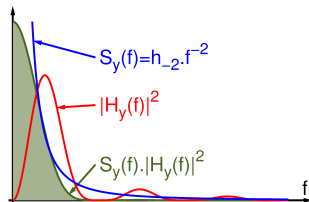
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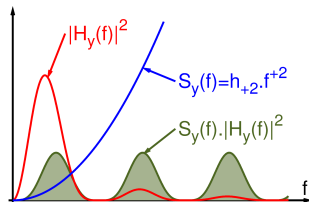
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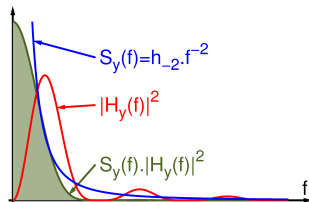
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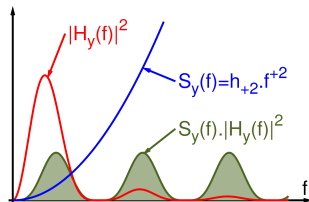
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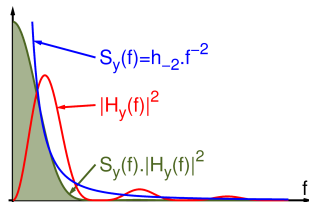
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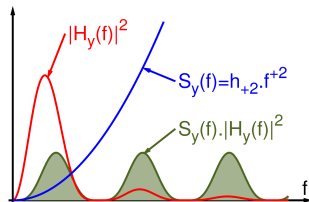
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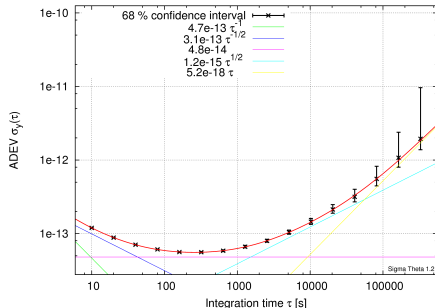
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Link between noise levels and variance responses

$$\sigma_y^2(\tau) = 2 \int_0^{+\infty} h_\alpha f^\alpha \frac{\sin^4(\pi \tau f)}{(\pi \tau f)^2} df$$

f_h is the high cut-off frequency



$S_y(f)$	$h_{-2}f^{-2}$	$h_{-1}f^{-1}$	h_0f^0	$h_{+1}f^{+1}$	$h_{+2}f^{+2}$
$\sigma_y^2(\tau)$	$\frac{2\pi^2 h_{-2} \tau}{3}$	$2 \ln(2) h_{-1}$	$\frac{h_0}{2\tau}$	$\frac{[1.04 + 3 \ln(2\pi f_h \tau)] h_{+1}}{4\pi^2 \tau^2}$	$\frac{3h_{+2} f_h}{4\pi^2 \tau^2}$

Practical calculation of the Allan variance

Calculation from time error samples

Calculation from frequency deviation

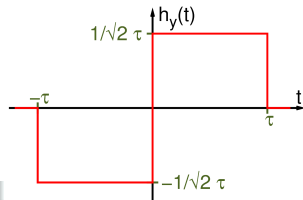
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$$\sigma_y^2(\tau) = \int_0^\infty S_y(f) |H_y(f)|^2 df$$

Calculation from time error samples

$$\begin{aligned} \bullet \sigma_y^2(\tau) &= \int_0^\infty S_x(f) |j2\pi f H_y(f)|^2 df \\ &= \left\langle [x(t) * h_x(t)]^2 \right\rangle \quad \text{with } h_x(t) = \frac{dh_y(t)}{dt} \end{aligned}$$

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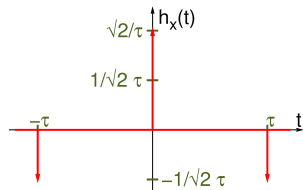
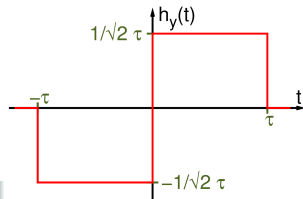
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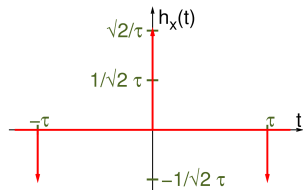
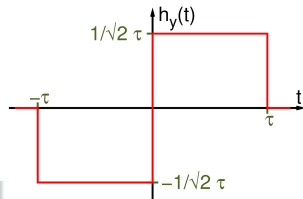
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Practical calculation of the Allan variance

Calculation from spectral density

Calculation from frequency deviation

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Calculation from spectral density

From a Phase Noise Measurement System:

$S_y(f_k)$ with $f_k \in \{f_1, 2f_1, \dots, kf_1, \dots, Nf_1\}$

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f_h is the bandwidth of the system

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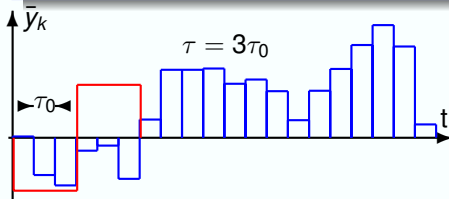
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Allan variance with or without overlapping

Allan variance with overlapping

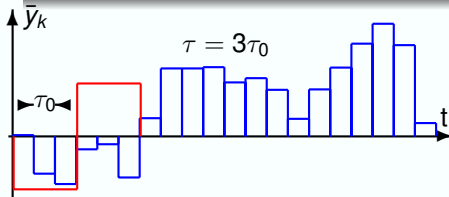


τ_0 -steps moving average

Benefits and drawbacks :

- lower dispersion
- more correlated estimates

Allan variance without overlapping



Shifted by τ -steps :

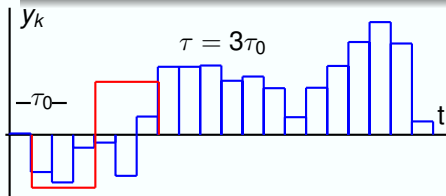
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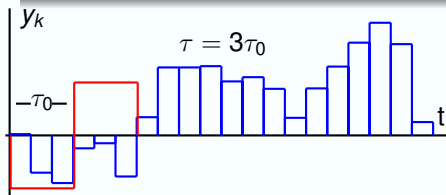


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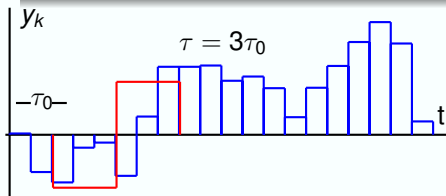
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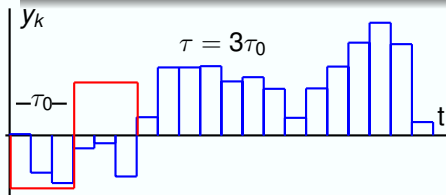


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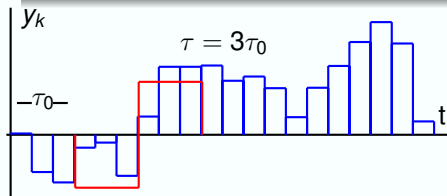
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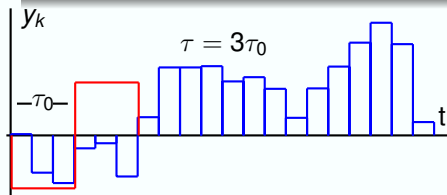


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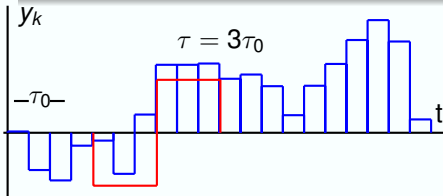
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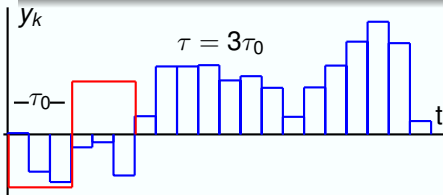


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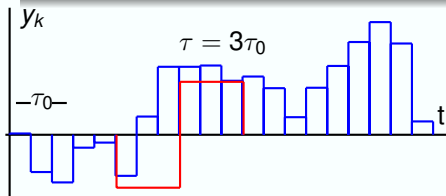
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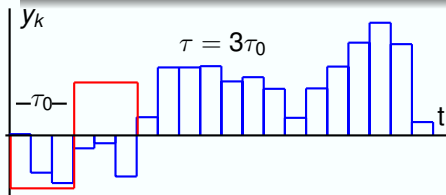


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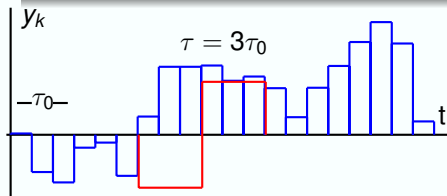
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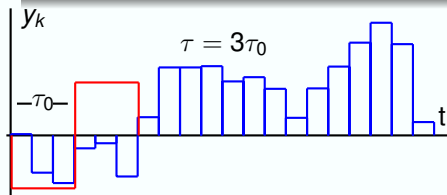


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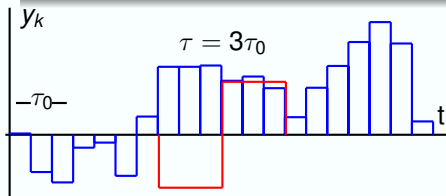
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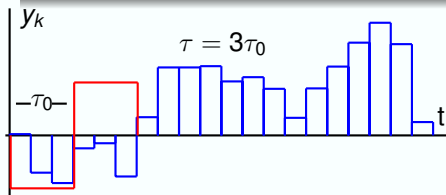


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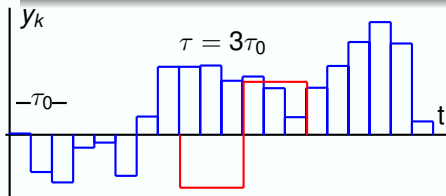
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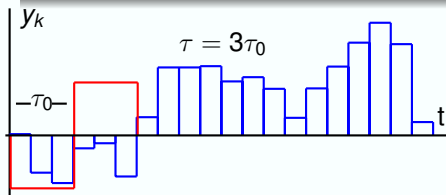


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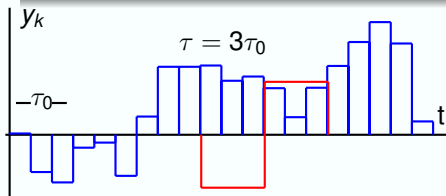
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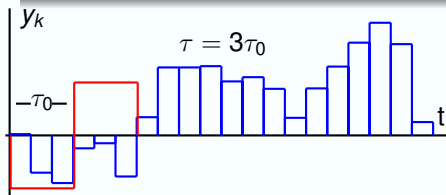


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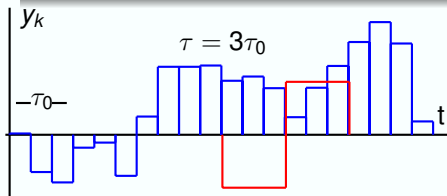
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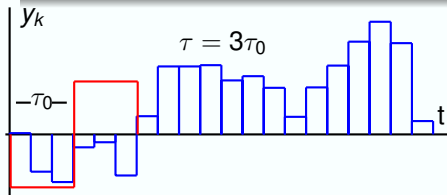


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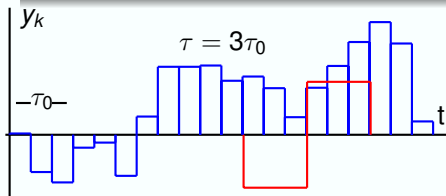
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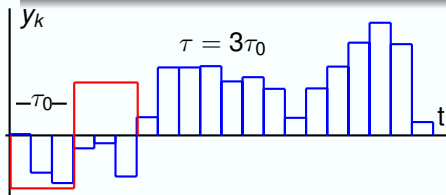


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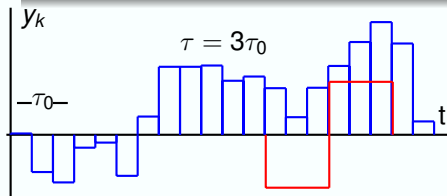
$$\tau = 3\tau_0 \Leftrightarrow \bar{Y}_1 = (\bar{y}_1 + \bar{y}_2 + \bar{y}_3)/3$$

Benefits and drawbacks :

- less correlated estimates
- higher dispersion

Allan variance with or without overlapping

Allan variance with overlapping

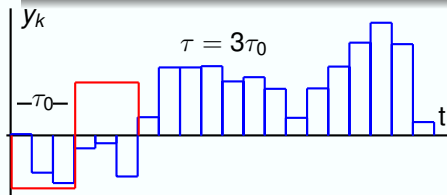


τ_0 -steps moving average

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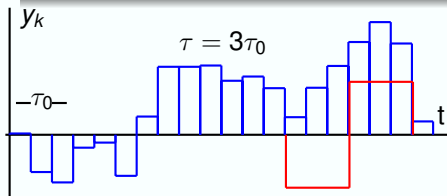
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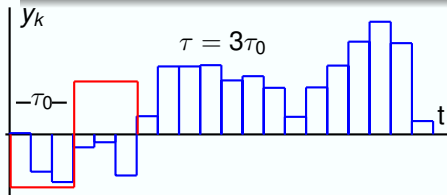


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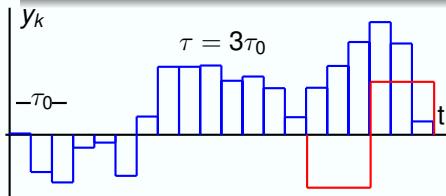
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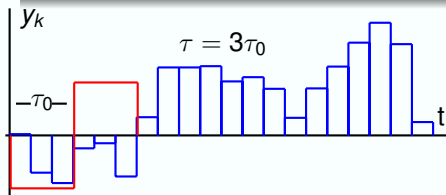


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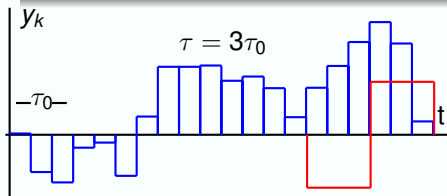
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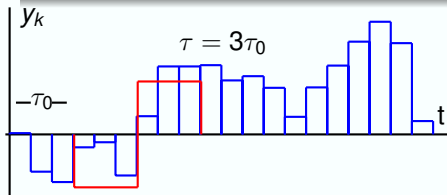


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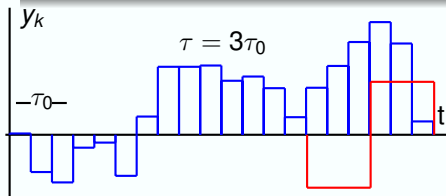
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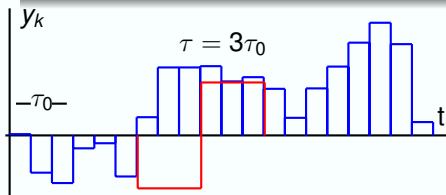


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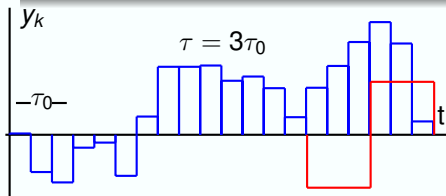
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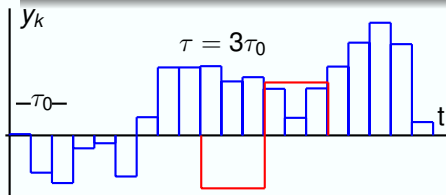


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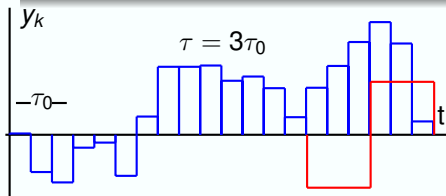
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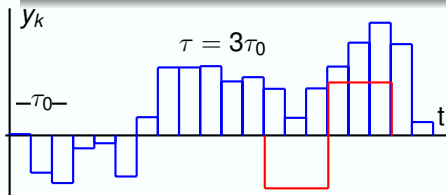


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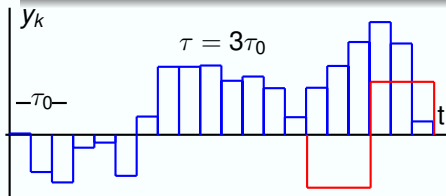
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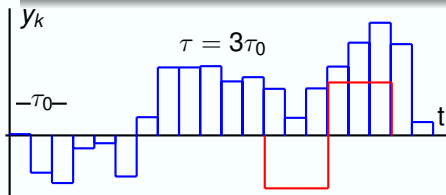


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Allan variance versus Allan deviation

$$\text{ADEV}(\tau) = \sigma_y(\tau) = \sqrt{\sigma_y^2(\tau)}$$

Physical meaning

- $\sigma_y(\tau) \equiv \frac{\Delta t}{\tau}$

Ex.: Cs clock $\sigma_y(\tau = 1\text{day}) = 10^{-14}$

$\Rightarrow \Delta t \approx 10^{-14} \cdot 10^5 = 10^{-9} = 1 \text{ ns over } 1 \text{ day}$

- $\sigma_y(\tau) \equiv \frac{\Delta f}{\nu_0}$ (during τ)

Ex.: H-Maser @ 100 MHz $\sigma_y(\tau = 1\text{hour}) = 10^{-14}$

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Benefits and drawbacks

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- 1 Introduction
- 2 Practical use of the Allan variance
- 3 Statistics of the Allan variance and the Allan deviation
 - Chi-square and Equivalent Degrees of Freedom
 - Confidence interval over the Allan variance/deviation measures
 - Parameter estimation
 - Increasing the number of edf: the Total variance
- 4 Prediction of very long term time stability

Chi-squared and Rayleigh distribution

Allan variance: $\sigma_y^2(\tau) = \frac{1}{2} \langle (\bar{y}_2 - \bar{y}_1)^2 \rangle$

Estimate: $\hat{\sigma}_y^2(\tau) = \frac{1}{2N} \sum_{i=1}^N (\bar{y}_2 - \bar{y}_1)^2$

- $\bar{y}_2 - \bar{y}_1$: Gaussian centered values
- $(\bar{y}_2 - \bar{y}_1)^2$: χ_1^2 distribution
- $\frac{1}{2N} \sum_{i=1}^N (\bar{y}_2 - \bar{y}_1)^2$: χ_N^2 distribution

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N is the number of Equivalent Degrees of Freedom (EDF)

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Reminder of the Equivalent Degrees of Freedom

Meaning of the EDF

$$\text{Mean}(\chi_\nu^2) = \nu \quad \text{and} \quad \text{Variance}(\chi_\nu^2) = 2\nu$$

The EDF ν contains the information about the dispersion of the random variable χ_ν^2

Estimation of the EDF

$$\hat{\sigma}_y^2(\tau) = \frac{1}{2N} \sum_{i=1}^N (\bar{y}_2 - \bar{y}_1)^2 \quad \Rightarrow \quad \chi_N^2 \text{ if } \{\bar{y}_1, \bar{y}_2, \dots\} \text{ uncorrelated!}$$

False:

- for low frequency noises (flicker and random walk FM)
- with overlapping variances

Algorithm for estimating the EDF:

- **C. Greenhall and W. Riley, 2003**, “*Uncertainty of Stability Variances Based on Finite Differences*” (35th PTTI).
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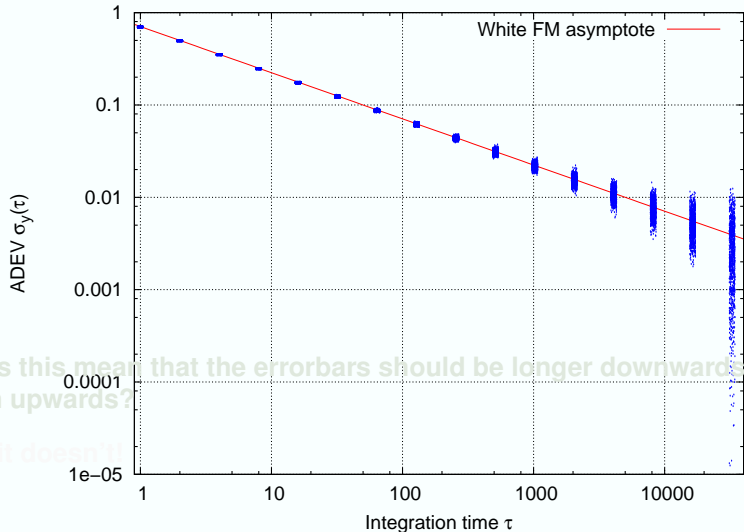
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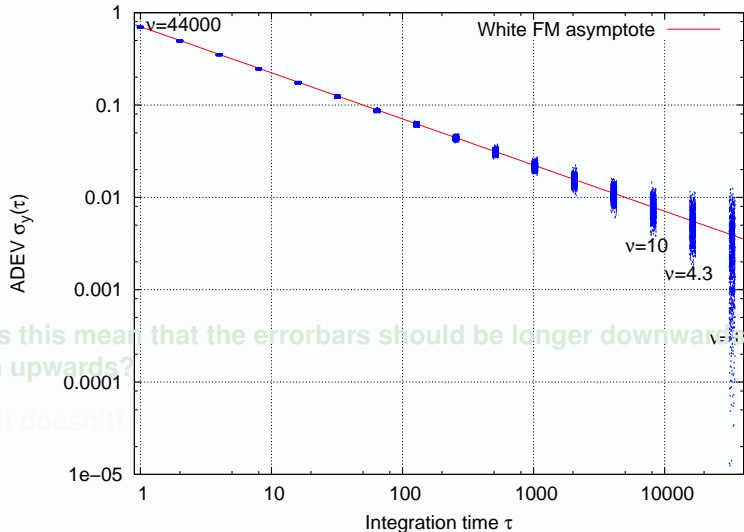
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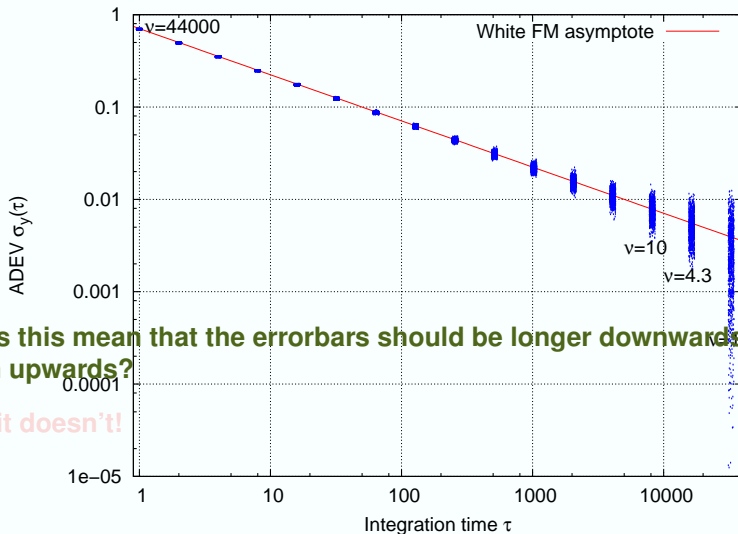
Dispersion of Allan deviation estimates



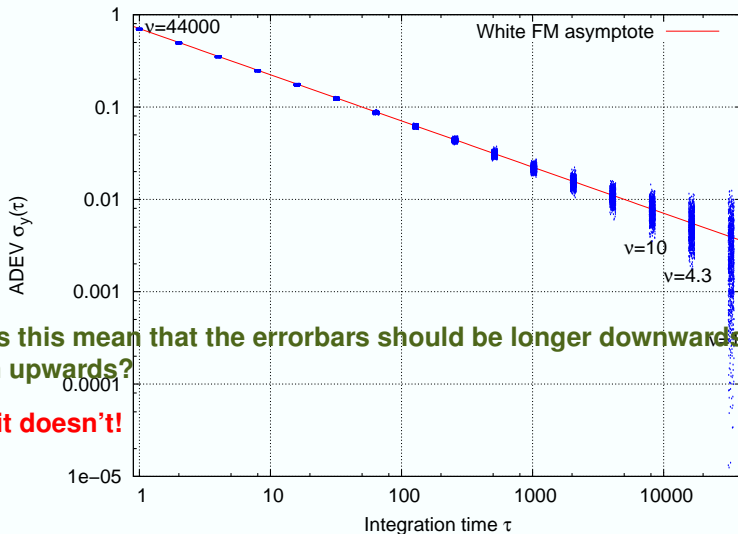
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Dispersion of Allan deviation estimates



Does this mean that the errorbars should be longer downwards than upwards?

No, it doesn't!

World of the model versus world of measures

- θ is the model parameter
- ξ is a measure of the parameter

Example:

Parameter $\sigma_y(\tau = 10 \text{ s}) = \sqrt{h_0/20}$ where h_0 is the white FM level

Measure $\hat{\sigma}_y(\tau)$ is a measure of $\sigma_y(\tau = 10 \text{ s})$

World of the model (direct problem):

Knowing the parameter θ_0 , how is the measure ξ distributed?

Only valid for simulations!

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It's the right question of the metrologist!

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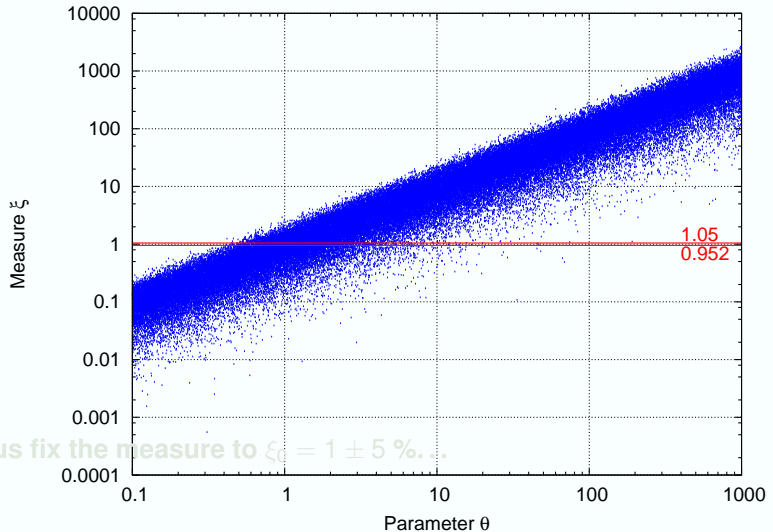
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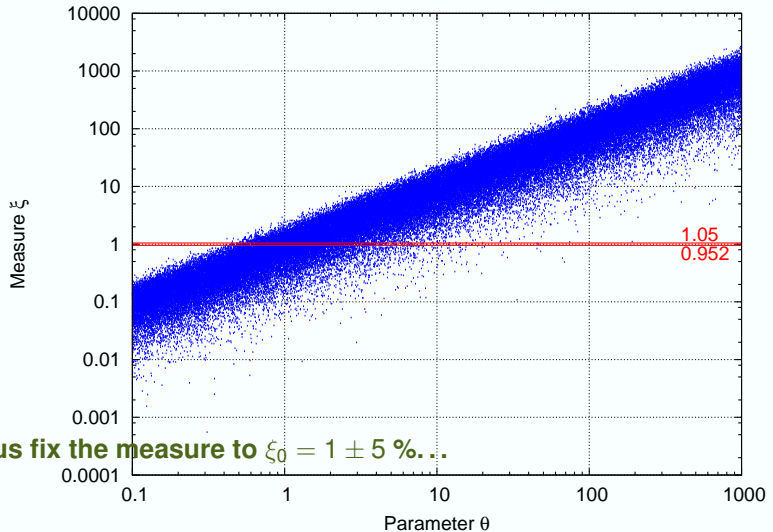
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It's the right question of the metrologist!

Model parameter and measure for a χ^2 distribution

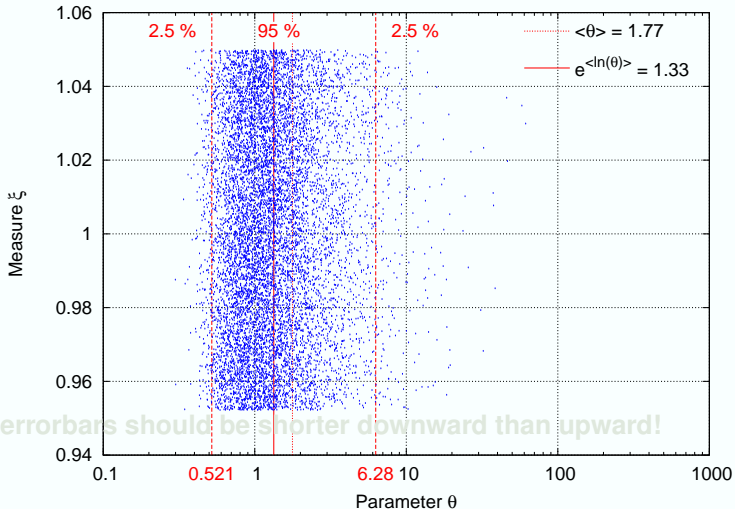


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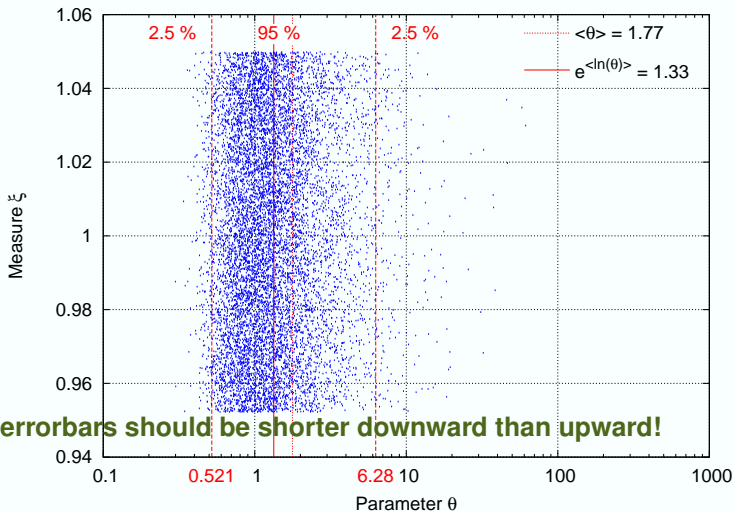
Model parameter values for a measure $\xi_0 \approx 1$

Theoretical results versus 20,000 simulations



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The errorbars should be shorter downward than upward!

Study of a χ distribution with 2 degrees of freedom

Direct problem

- Probability density function: $p(\chi) = \chi e^{-\chi^2/2}$
- The pdf is normalized: $\int_0^{\infty} p(\chi) d\chi = 1$
- Mathematical expectation: $\mu = \int_0^{\infty} \chi \cdot p(\chi) d\chi = \frac{\pi}{2}$
- Cumulative distribution function: $P(\chi) = \int_0^{\chi} p(y) dy = 1 - e^{-\chi^2/2}$
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Confidence Interval of a χ^2 random variable

- $\{\dots \chi_i \dots\}$ is a set of realizations of the random variable χ
 - $P^{-1}(0.025) \approx 0.22502$
 - $\Rightarrow \chi_i < 0.22502$ with 2.5% confidence
 - $P^{-1}(0.975) \approx 2.7162$
 - $\Rightarrow \chi_i < 2.7162$ with 97.5% confidence
- **Confidence Interval:**
 - $E(\chi) \approx 1.2533$
 - $0.22502 < \chi_i < 2.7162$ with 95% confidence
- **General case** of a random variable $x = k \cdot \chi$
 - Estimation of the scale factor: $k = \frac{E(x)}{E(\chi)} \approx \frac{\langle x \rangle}{\mu}$
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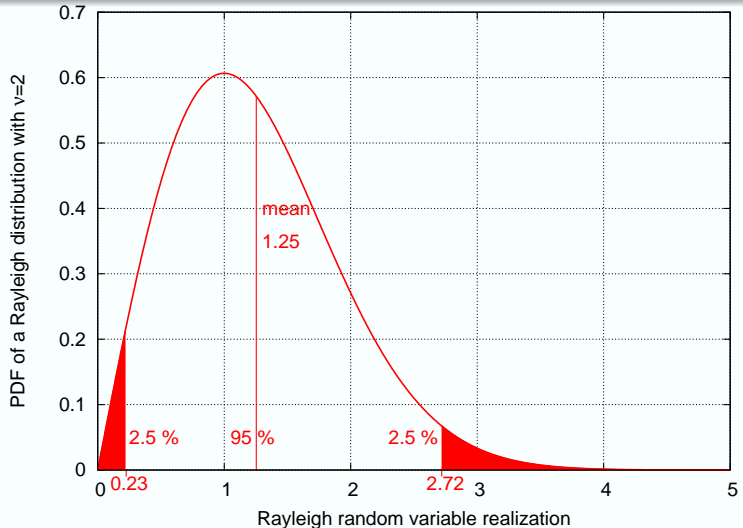
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Probability density function of a χ_2 distribution



Conditionnal probabilities

Reduced variable (I)

- Let us consider the standard χ_2^2 variable: $\chi_2^2 = X_1^2 + X_2^2$ where X_1 and X_2 are 2 Gaussian centered standard random variables

$$\Rightarrow E(\chi_2^2) = 2 \quad \Rightarrow \quad E\left(\frac{1}{2}\chi_2^2\right) = 1.$$

- We assume that $\hat{\sigma}_y^2(\tau) = \xi^2$ is χ_2^2 distributed and is an unbiased estimator of the parameter $\sigma_y^2(\tau) = \theta^2$:

$$E\left(\frac{\xi^2}{\theta^2}\right) = 1.$$

- We can then define the reduced variable χ_2^2 as:

$$\chi_2^2 = 2\frac{\xi}{\theta}.$$

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- By extension, we assume that $\hat{\sigma}_y(\tau) = \xi$ is χ_2 distributed and ξ is an estimator of the parameter $\sigma_y(\tau) = \theta$.
- We can then define the reduced variable χ as:

$$\chi = \sqrt{2} \frac{\xi}{\theta}.$$

- The differential $d\chi$ is then:

$$d\chi = \frac{\partial \chi}{\partial \xi} d\xi + \frac{\partial \chi}{\partial \theta} d\theta.$$

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Parameter estimation from a single measure

Usual frequentist reasoning

We assume that the measure ξ represents the estimate $\hat{\sigma}_y(\tau)$ and the parameter θ stands for the real unknown value $\sigma_y(\tau)$.

- **Reduced variable:** $\chi = \sqrt{2}\xi/\theta$
- **Low bound:** $B_{2.5\%} \approx 0.22502$
- **High bound:** $B_{97.5\%} \approx 2.7162$
- **95 % confidence interval:** $0.22502 < \sqrt{2}\xi/\theta < 2.7162$
- **Frequentist reversal:** $\frac{\sqrt{2}\xi_0}{2.7162} < \theta < \frac{\sqrt{2}\xi_0}{0.22502} @ 95 \%$

$\Rightarrow 0.52066 \cdot \xi_0 < \theta < 6.2847 \cdot \xi_0$ with 95 % confidence.

We obtain directly the same result from $P(\theta|\xi_0)$ (as well as from the Bayesian method with a total lack of knowledge prior).

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Generalization to a χ_ν distribution

- **Reduced variable:** $\chi = \sqrt{\nu} \frac{\xi}{\theta}$
- **pdf:** $p(\chi) = \frac{2^{1-\nu/2} \chi^{\nu-1} e^{-\chi^2/2}}{\Gamma(\nu/2)}$
- **Mathematical expectation:** $\mu_\nu = \sqrt{2} \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})}$
- **cdf:** $P(\chi) = \Gamma\left(\frac{\nu}{2}, \frac{\chi^2}{2}\right)$

Parameter estimation

Allan variance

- We assume that $\hat{\sigma}_y^2(\tau) = \xi^2$ is χ_2^2 distributed and is an estimator of the parameter $\sigma_y^2(\tau) = \theta^2$.
- **Reduced variable:** $\chi_2^2 = 2 \frac{\xi^2}{\theta^2}$
- **Mathematical expectation:** $\chi_2^2 = 2$
 $\Rightarrow \left\langle 2 \frac{\xi^2}{\theta^2} \right\rangle = 2 \Leftrightarrow \left\langle \frac{\xi^2}{\theta^2} \right\rangle = 1$
- **For a given parameter θ_0^2 :** $\xi^2 = \theta_0^2$
 The average of the measures given by the parameter θ_0^2 is equal to θ_0^2 : **ξ^2 is an unbiased estimator of θ_0^2 .**
- **For a given measure ξ_0^2 :** $\theta^2 = \xi_0^2$
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Parameter estimation

Allan deviation

- We assume that $\hat{\sigma}_y(\tau) = \xi$ is χ^2 distributed and is an estimator of the parameter $\sigma_y(\tau) = \theta$.
- **Reduced variable:** $\chi^2_2 = \sqrt{2} \frac{\xi}{\theta}$
- **Mathematical expectation:** $\chi^2_2 = \mu = \overline{\pi/2}$

$$\Rightarrow \sqrt{2} \frac{\xi}{\theta} = \frac{\overline{\pi}}{2} \Leftrightarrow \frac{\xi}{\theta} = \frac{\overline{\pi}}{4}$$
- **For a given parameter θ_0 :** $\langle \xi \rangle = \overline{\pi/4} \theta_0 \approx 1.128 \theta_0$
 ξ is a biased estimator of θ_0 (overestimated by 13%).
- **For a given measure ξ_0 :** $\langle \theta \rangle = \overline{4/\pi} \xi_0 \approx 0.886 \theta_0$
the measure ξ_0 should NOT be used for representing the parameter θ (underestimated by 13 %).

Never fit the curve of Allan deviation, always use the Allan variance!

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- **Mathematical expectation:** $\chi_2^2 = \mu = \overline{\chi_2^2}$

$$\Rightarrow \sqrt{2} \frac{\xi}{\theta} = \frac{\overline{\chi_2^2}}{2} \Leftrightarrow \frac{\xi}{\theta} = \frac{\overline{\chi_2^2}}{4}$$
- **For a given parameter θ_0 :** $\langle \xi \rangle = \overline{\chi_2^2} \theta_0 \approx 1.128 \theta_0$
 ξ is a biased estimator of θ_0 (overestimated by 13%).
- **For a given measure ξ_0 :** $\langle \theta \rangle = \frac{\xi_0}{\overline{\chi_2^2}} \approx 0.886 \theta_0$
the measure ξ_0 should NOT be used for representing the parameter θ (underestimated by 13 %).

Never fit the curve of Allan deviation, always use the Allan variance!

Parameter estimation

Allan deviation

- We assume that $\hat{\sigma}_y(\tau) = \xi$ is χ_2^2 distributed and is an estimator of the parameter $\sigma_y(\tau) = \theta$.
- **Reduced variable:** $\chi_2^2 = \sqrt{2} \frac{\xi}{\theta}$
- **Mathematical expectation:** $\chi_2^2 = \mu = \overline{\pi/2}$

$$\Rightarrow \sqrt{2} \frac{\xi}{\theta} = \frac{\overline{\pi}}{2} \Leftrightarrow \frac{\xi}{\theta} = \frac{\overline{\pi}}{4}$$
- **For a given parameter θ_0 :** $\langle \xi \rangle = \overline{\pi/4} \theta_0 \approx 1.128 \theta_0$
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- **For a given measure ξ_0 :** $\langle \theta \rangle = \overline{4/\pi} \xi_0 \approx 0.886 \theta_0$
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Parameter estimation

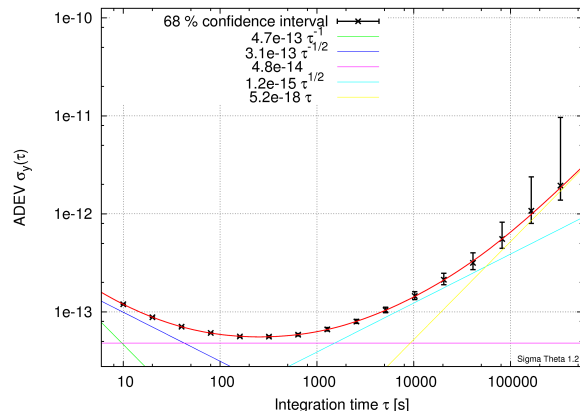
Allan deviation

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Increasing the number of edf: the Total variance



The longer the time duration, the larger the uncertainty.

What about very long term stability ?

In order to improve estimates for very long term, D. Howe developed:

- Total variance: *UFFC-47(5)*, 1102-1110 (2000)
- $\widehat{\text{Theo}}$: *Metrologia* 43, S322-S331 (2006)

Outline

- 1 Introduction
- 2 Practical use of the Allan variance
- 3 Statistics of the Allan variance and the Allan deviation
- 4 Prediction of very long term time stability
 - Fitting curve over variance measurement
 - Estimation of the noise levels from the fitting curve
 - Extrapolation to very long term time stability

Fitting curve over variance measurement (I)

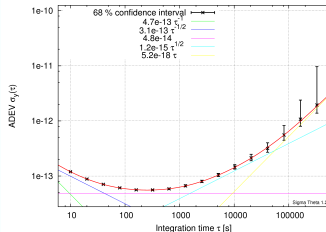
$$\sigma_y^2(\tau) = \sum_{i=0}^4 C_i \Phi_i(\tau) \quad \text{with} \quad \Phi_i(\tau) = \tau^{i-2}$$

How to estimate the C_i coefficients?

Classical least squares:

$$\sum_{j=1}^N \left(\hat{\sigma}_y^2(\tau_j) - \sum_{i=0}^4 C_i \Phi_i(\tau_j) \right)^2 \quad \text{is minimum}$$

- not suitable for high dynamic
- not suitable for positive or null values
- **not suitable for variance curves**



Fitting curve over variance measurement (II)

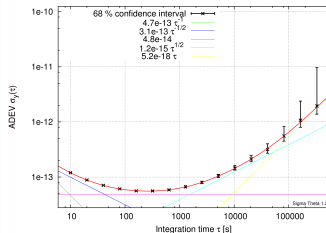
$$\sigma_y^2(\tau) = \sum_{i=0}^4 C_i \Phi_i(\tau) \quad \text{with} \quad \Phi_i(\tau) = \tau^{i-2}$$

How to estimate the C_i coefficients?

Relative least squares:

$$\sum_{j=1}^N \left[\frac{1}{\hat{\sigma}_y^2(\tau_j)} \left(\hat{\sigma}_y^2(\tau_j) - \sum_{i=0}^4 C_i \Phi_i(\tau_j) \right) \right]^2 \quad \text{is minimum}$$

- equivalent to a least square fit on log-log plot
- doesn't take into account the uncertainties over the Allan variance measures
- not suitable for variance curves



Fitting curve over variance measurement (III)

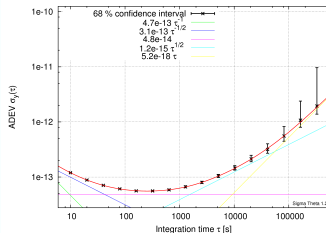
$$\sigma_y^2(\tau) = \sum_{i=0}^4 C_i \Phi_i(\tau) \quad \text{with} \quad \Phi_i(\tau) = \tau^{i-2}$$

How to estimate the C_i coefficients?

Weighted relative least squares:

$$\sum_{j=1}^N \frac{1}{\text{EDF} [\hat{\sigma}_y^2(\tau_j)]} \frac{1}{\hat{\sigma}_y^2(\tau_j)} \hat{\sigma}_y^2(\tau_j) - \sum_{i=0}^4 C_i \Phi_i(\tau_j) \quad \text{is minimum}$$

- equivalent to a least square fit on log-log plot
- takes into account the uncertainties over the Allan variance measures
- suitable for variance curves



Estimation of the noise levels from the fitting curve

$$\sigma_y^2(\tau) = \sum_{i=0}^4 C_i \Phi_i(\tau) \quad \text{with} \quad \Phi_i(\tau) = \tau^{i-2}$$

- $C_0 \tau^{-2}$ **White or Flicker PM:** $h_{+2} = \frac{4\pi^2 C_0}{3f_h}$ or $h_{+1} \approx 4\pi^2 C_0$
- $C_1 \tau^{-1}$ **White FM:** $h_0 = 2C_1$
- $C_2 \tau^0$ **Flicker FM:** $h_{-1} = \frac{C_2}{2 \ln(2)}$
- $C_3 \tau$ **Random Walk FM:** $h_{-2} = \frac{3C_3}{2\pi^2}$
- $C_4 \tau^2$ **Linear frequency drift:** $D_1 = \overline{2C_4}$

Uncertainties Δh_α ? See *Vernotte et al., IM-42(2), 342-350 (1993)*

Extrapolation to very long term time stability

Some recommendations

Is it possible to extrapolate the fit beyond the last Allan variance measure?

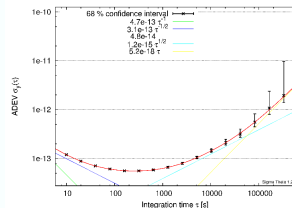
Sometimes yes, but very carefully !

We ought already to answer to the following questions. . .

- 1 Is the longest term noise or drift asymptote visible on the curve?
Flicker FM for Cesium, random walk FM and/or linear frequency drift otherwise
- 2 Is this asymptote well determined ?
This asymptote must be dominant for at least 2-3 octaves
- 3 Is the curve compatible with a null coefficient for the longest term noise or drift ?

The bottom uncertainty domains can fit correctly the other asymptotes

If you answered YES to the questions 1 and 2, and NO to the last question, you may try. . .



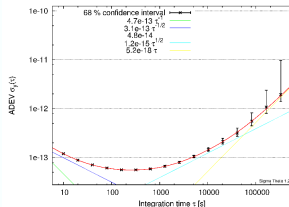
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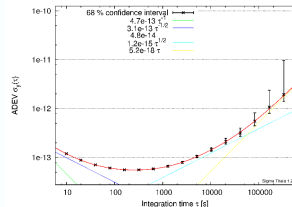
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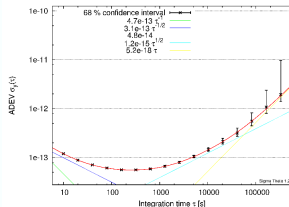
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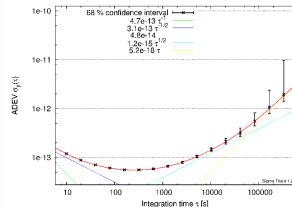
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