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The Kalman Filter

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Historical notes on the Kalman filter

- Developed in 1960

  R. E. Kalman, "A New Approach to Linear Filtering and Prediction Problems," *Journal of Basic Engineering*, vol. 82, pp. 35-45, 1960

  www.cs.unc.edu/~welch/kalman/media/pdf/Kalman1960.pdf

- System + Measurement model: Effective estimation
- Recursive: low computational cost

Boom of applications in the past 50 years

Why is it called a “filter”? (1/2)

IN FREQUENCY ANALYSIS

To filter

=

to **remove frequencies** selectively

IN ESTIMATION THEORY

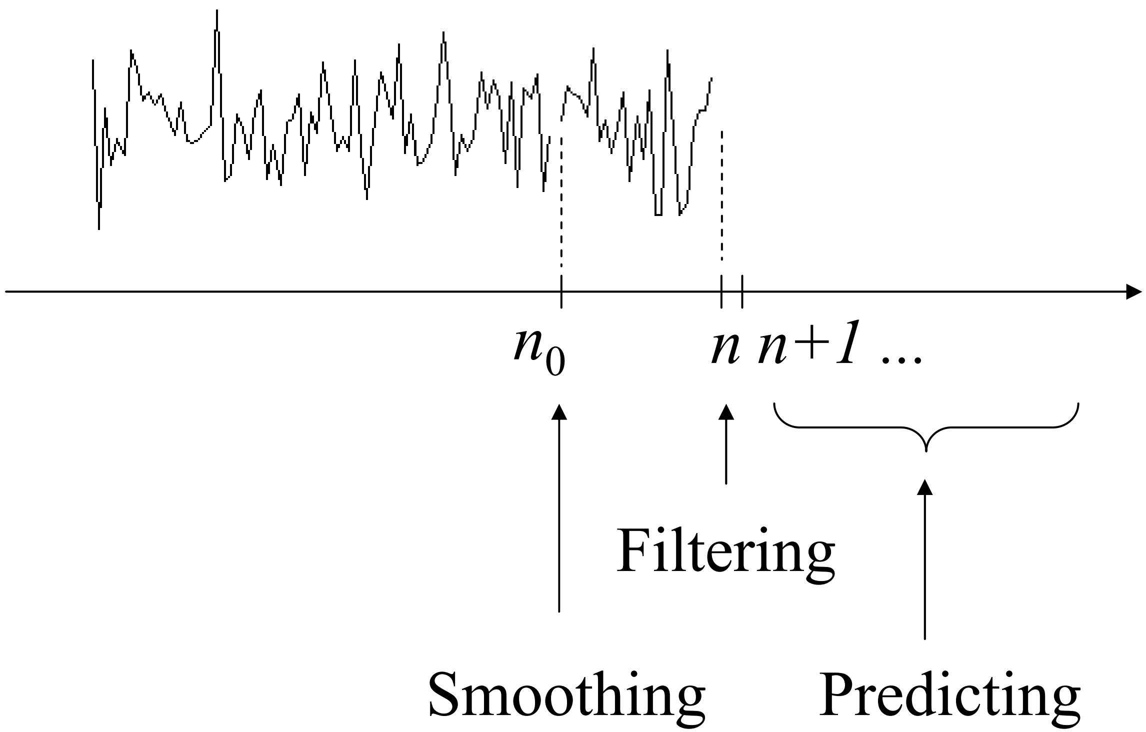
To filter

=

to **predict**

Why is it called a “filter”? (2/2)

Conventional terminology in estimation theory



The advantages of recursive estimation (1/6)

Suppose that we want to estimate the temperature in this room

Estimation problem

System: this room

State of the system: temperature

True value of the state: x

Measurements: n readings $z[1],\ldots,z[n]$ from n thermometers

The advantages of recursive estimation (2/6)

We assume the measurement model

$$z[n] = x + v[n]$$

Where $v[n]$ is a Gaussian random variable with

- zero mean [E: Expected value]

$$E[v[n]] = 0$$

- standard deviation σ

$$\sigma = \sqrt{E[(v[n] - E[v[n]])^2]}$$

Compact notation

$$v[n] \sim N(0, \sigma^2)$$

Normal (or Gaussian) distribution

The advantages of recursive estimation (3/6)

We obtain an estimate $\hat{x}[n]$ of the temperature by averaging the n measurements

$$\hat{x}[n] = \frac{1}{n} \sum_{k=1}^{n} z[k]$$

[Sample mean estimator]

Now we perform a new measurement $z[n+1]$

We obtain the new estimate

$$\hat{x}[n+1] = \frac{1}{n+1} \sum_{k=1}^{n+1} z[k]$$

The advantages of recursive estimation (4/6)

What is the computational cost of this estimate?

$$\hat{x}[n] = \frac{1}{n}\sum_{k=1}^{n} z[k]$$

- The number of operations increases with *n*
- And we have to store an increasing (diverging) number of measurements...

The advantages of recursive estimation (5/6)

We observe that we can rewrite the estimate as

$$
\begin{aligned}
\hat{x}[n+1] &= \frac{1}{n+1}\sum_{k=1}^{n+1} z[k] \\
&= \frac{1}{n+1}\left(\sum_{k=1}^{n} z[k] + z[n+1]\right) \\
&= \frac{1}{n+1}\sum_{k=1}^{n} z[k] + \frac{1}{n+1} z[n+1] \\
&= \hat{x}[n] + \frac{1}{n+1} z[n+1]
\end{aligned}
$$

The advantages of recursive estimation (6/6)

Recursive estimate

$$\hat{x}[n+1] = \hat{x}[n] + \frac{1}{n+1} z[n+1]$$

- The computational cost is fixed!
- And we have to store the previous estimate only

What are the performances of this estimator?

Performances of an estimator (1/2)

1. BIAS [Mean value of the estimation error]

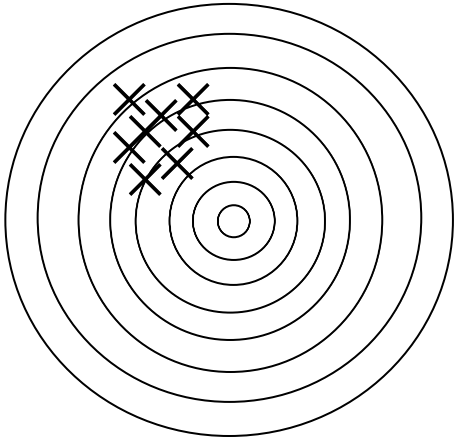
$$B = E[\hat{x}] - x$$

- $B = 0$: Unbiased estimator
- $B \neq 0$: Biased estimator

2. VARIANCE

$$\sigma^2 = E[(\hat{x} - E[x])^2]$$

Performances of an estimator (2/2)

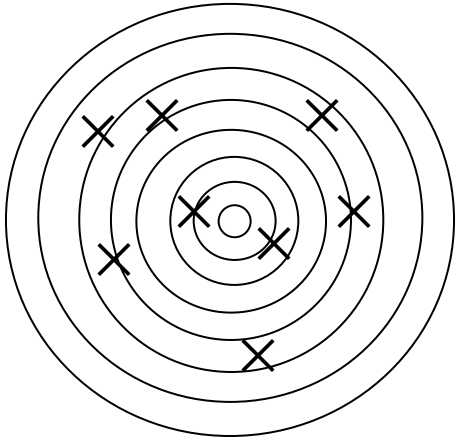


Biased

Low variance

$E[\hat{x}] \neq x$

$B \neq 0$

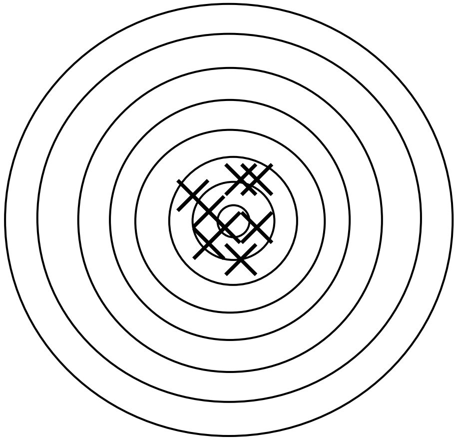


Unbiased

High variance

$E[\hat{x}] = x$

$B = 0$



Unbiased

Low variance

$E[\hat{x}] = x$

$B = 0$

Performances of the sample mean estimator (1/2)

1. BIAS

$$\begin{aligned} B &= E[\hat{x}] - x = \\ &= E\left[\frac{1}{n}\sum_{k=1}^{n} z[k] - x\right] \\ &= E\left[\frac{1}{n}\sum_{k=1}^{n} (x + v[k]) - x\right] \\ &= E\left[\frac{1}{n}\sum_{k=1}^{n} v[k]\right] \\ &= \frac{1}{n}\sum_{k=1}^{n} E\big[v[k]\big] \\ &= 0 \end{aligned}$$

$\Rightarrow$ Unbiased estimator

Performances of the sample mean estimator (2/2)

2. VARIANCE

$$\sigma_{\hat{x}}^{\;2} = E[(\hat{x} - E[x])^2]$$

$$= \frac{\sigma^2}{n}$$

$$\Downarrow$$

$$\sigma_{\hat{x}} = \frac{\sigma}{\sqrt{n}}$$

(Independent measurements)

Summarizing... (1/2)

Sample mean estimator

$$\hat{x}[n+1] = \hat{x}[n] + \frac{1}{n+1} z[n+1]$$

- Linear
- Recursive
- Optimal: minimum variance of the estimate

System model	Measurement model
x	$z[n] = x + v[n]$
(constant)	

Summarizing... (2/2)

Also the Kalman filter is

linear, recursive, and optimal

But the system model is not a constant!

System model

$$\underline{x}[n] = \Phi \underline{x}[n-1] + \underline{b}[n-1] + \underline{\eta}[n-1]$$

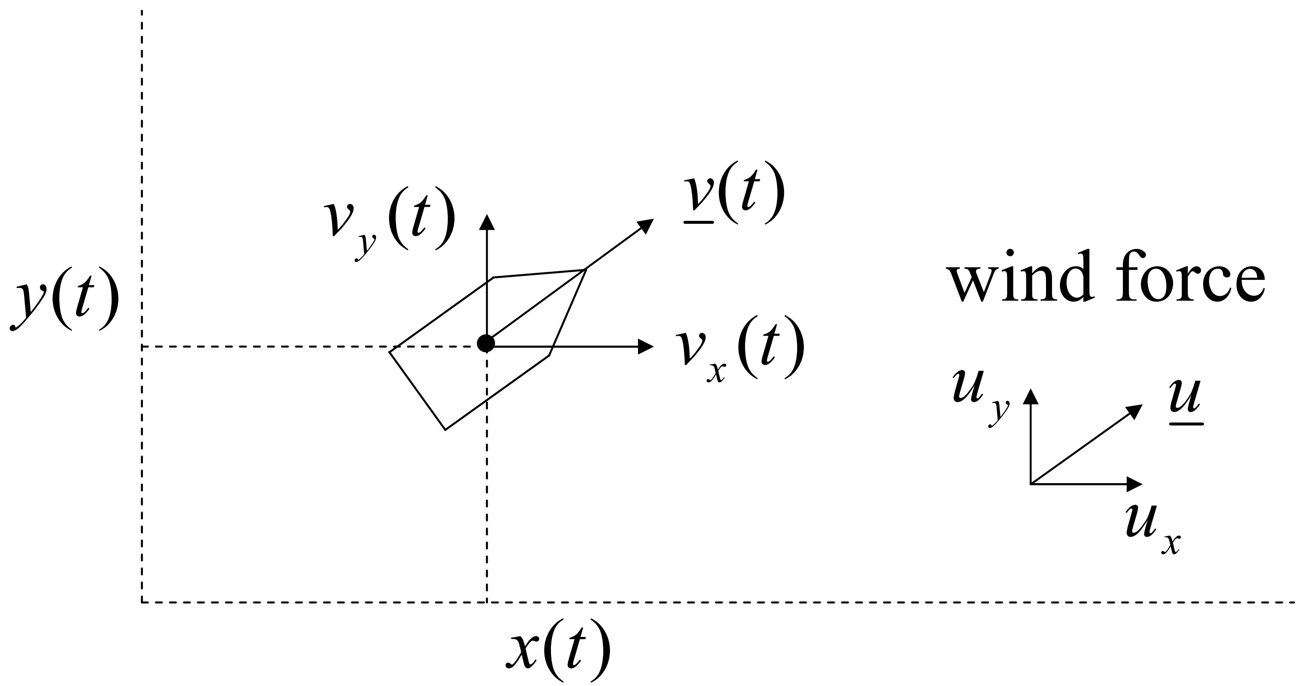
(dynamical system)

Measurement model

$$\underline{z}[n] = H \underline{x}[n] + \underline{v}[n]$$

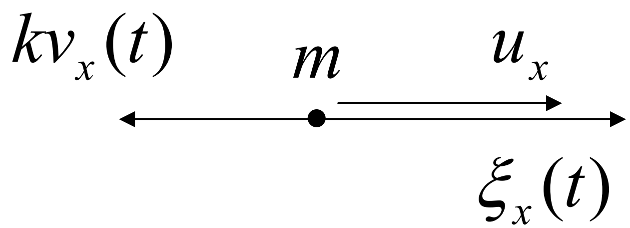
System model (1/10)

Example: a boat on the ocean



System model (2/10)

Balance of forces: x direction

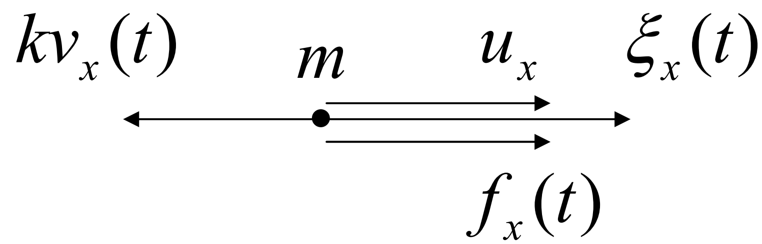


$u_x \longrightarrow$ Wind force

$kv_x(t) \longrightarrow$ Friction

$\xi_x(t) \longrightarrow$ Random force due to the waves

System model (3/10)



$$f_x(t) = ma_x(t)$$ [Newton's law]

$$\Downarrow$$

$$u_x + \xi_x(t) - kv_x(t) = ma_x(t)$$ [Balance of forces]

System model (4/10)

$$u_x + \xi_x(t) - kv_x(t) = ma_x(t)$$

We take $m = 1$

and we note that

$$a_x(t) = \dot{v}_x(t)$$

substituting, we have

$$\dot{v}_x + kv_x(t) = u_x + \xi_x(t)$$

System model (5/10)

$$\dot{v}_x + k v_x(t) = u_x + \xi_x(t)$$

We note that

$$v_x(t) = \dot{x}(t)$$

Substituting

$$\ddot{x}(t) + k\dot{x}(t) = u_x + \xi_x(t)$$

$$\ddot{y}(t) + k\dot{y}(t) = u_y + \xi_y(t)$$

Identical equation for the *y* axis

System model (6/10)

Summarizing

$$\ddot{x}(t) + k\dot{x}(t) = u_x + \xi_x(t)$$
$$\ddot{y}(t) + k\dot{y}(t) = u_y + \xi_y(t)$$

Initial conditions

$x(0)$	Initial x position
$\dot{x}(0)$	Initial x velocity
$y(0)$	Initial y position
$\dot{y}(0)$	Initial y velocity

System model (7/10)

Change of variables

$$x_1(t) = x(t)$$
$$x_2(t) = \dot{x}(t)$$
$$x_3(t) = y(t)$$
$$x_4(t) = \dot{y}(t)$$

$$\ddot{x}(t) + k\dot{x}(t) = u_x + \xi_x(t)$$
$$\ddot{y}(t) + k\dot{y}(t) = u_y + \xi_y(t)$$

$$\Downarrow$$

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -kx_2(t) + u_x + \xi_x(t)$$
$$\dot{x}_3(t) = x_4(t)$$
$$\dot{x}_4(t) = -kx_4(t) + u_y + \xi_y(t)$$

System model (8/10)

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -kx_2(t) + u_x + \xi_x(t)$$
$$\dot{x}_3(t) = x_4(t)$$
$$\dot{x}_4(t) = -kx_4(t) + u_y + \xi_y(t)$$

$$\boxed{\underline{\dot{x}}(t) = F\,\underline{x}(t) + B\underline{u}(t) + \underline{\xi}_x(t)}$$

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} \qquad F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -k & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -k \end{bmatrix} \qquad B\underline{u}(t) = \begin{bmatrix} 0 \\ u_x \\ 0 \\ u_y \end{bmatrix} \qquad \underline{\xi}(t) = \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \\ \xi_4(t) \end{bmatrix}$$

System model (9/10)

$$\underline{\dot{x}}(t) = F\,\underline{x}(t) + B\underline{u}(t) + \underline{\xi}_x(t)$$

$$\Downarrow$$

$$\boxed{\underline{x}[n] = \Phi\underline{x}[n-1] + \underline{b}[n-1] + \underline{\eta}[n-1]}$$

Where

$$\underline{x}[n] = \underline{x}(nT_s)$$

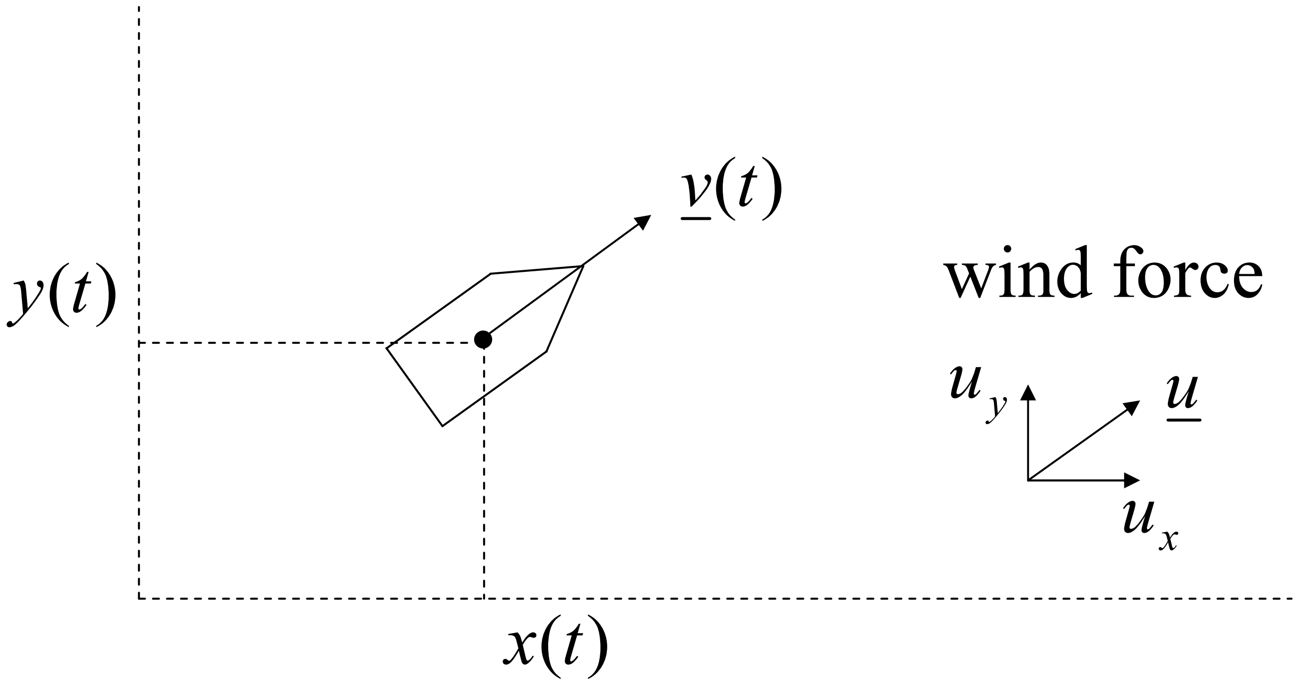
$$\Phi = e^{F\,T_s} \qquad \text{[Transition matrix]}$$

$$\underline{\eta}[n-1] \sim N(0,Q) \qquad \text{[Model noise]}$$

System model (10/10)

Key point: Physical systems can be modeled as

$$\boxed{\underline{x}[n] = \Phi\underline{x}[n-1] + \underline{b}[n-1] + \underline{\eta}[n-1]}$$

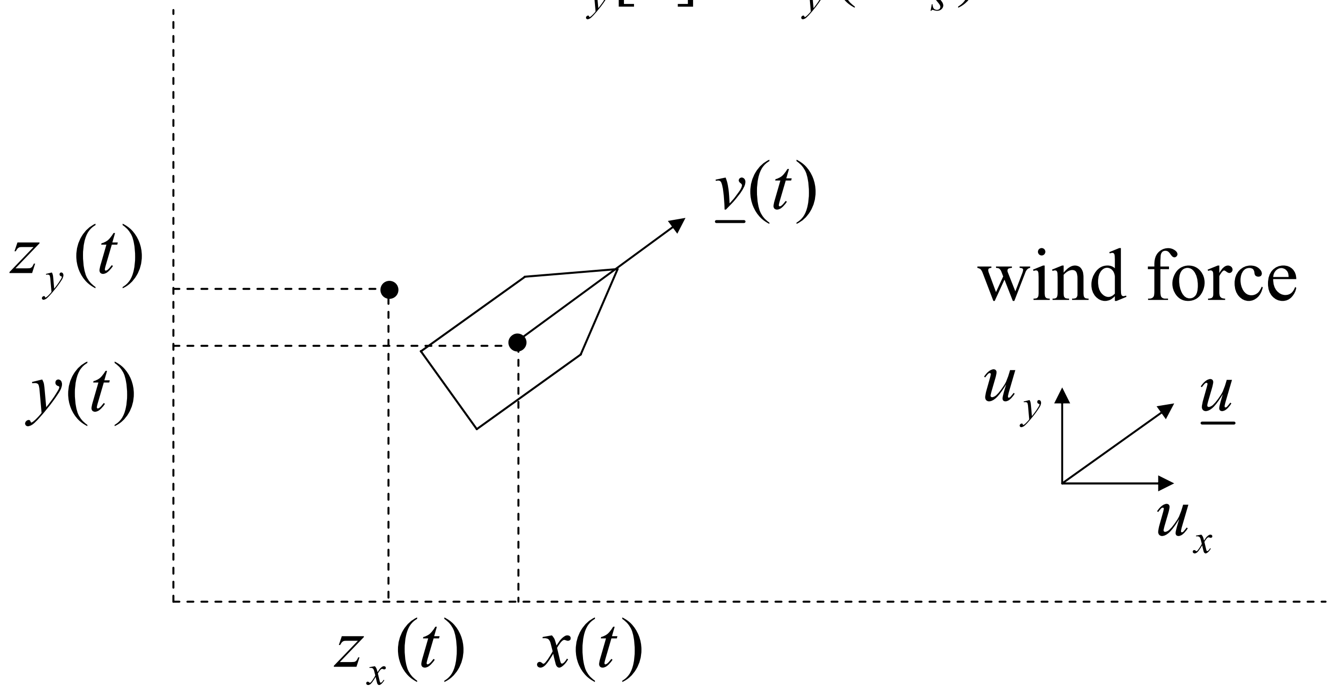


Measurement model (1/2)

Measurements performed by a GPS receiver

$$z_x[n] = z_x(nT_s)$$

$$z_y[n] = z_y(nT_s)$$



Measurement model (2/2)

Measurement model

$$\boxed{\underline{z}[n] = H\underline{x}[n] + \underline{v}[n]}$$

where

$$\underline{z}[n] = \begin{bmatrix} z_x[n] \\ z_y[n] \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

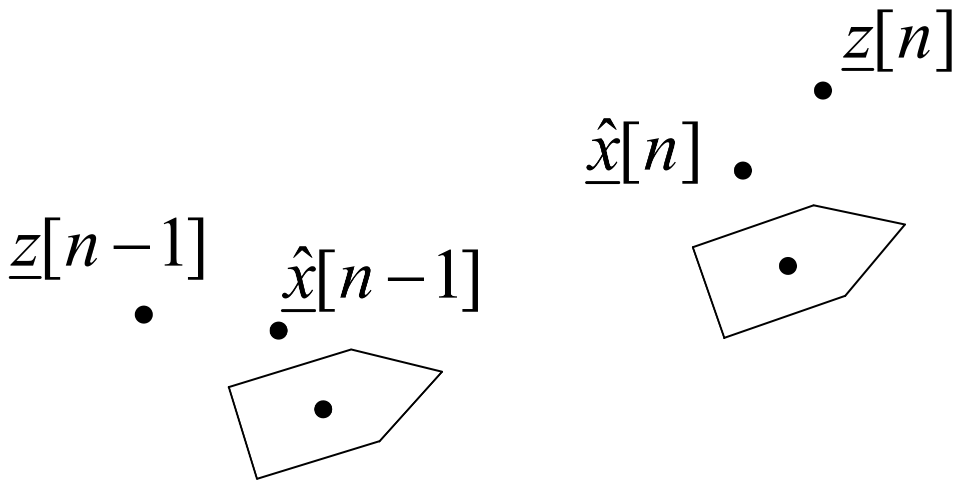
$\underline{v}[n] \sim N(0, R) \longrightarrow$ Measurement noise

Estimation problem (1/2)

How do we estimate the state of the boat?

What do we know:

1. Measurements $\longrightarrow$ $\underline{z}[n]$
2. Physics of the boat $\longrightarrow$ System model



Estimation problem (2/2)

We seek a linear, recursive, optimal estimate of the form

$$\underline{\hat{x}}[n] = L(\underline{\hat{x}}[n-1], \underline{z}[n])$$

where

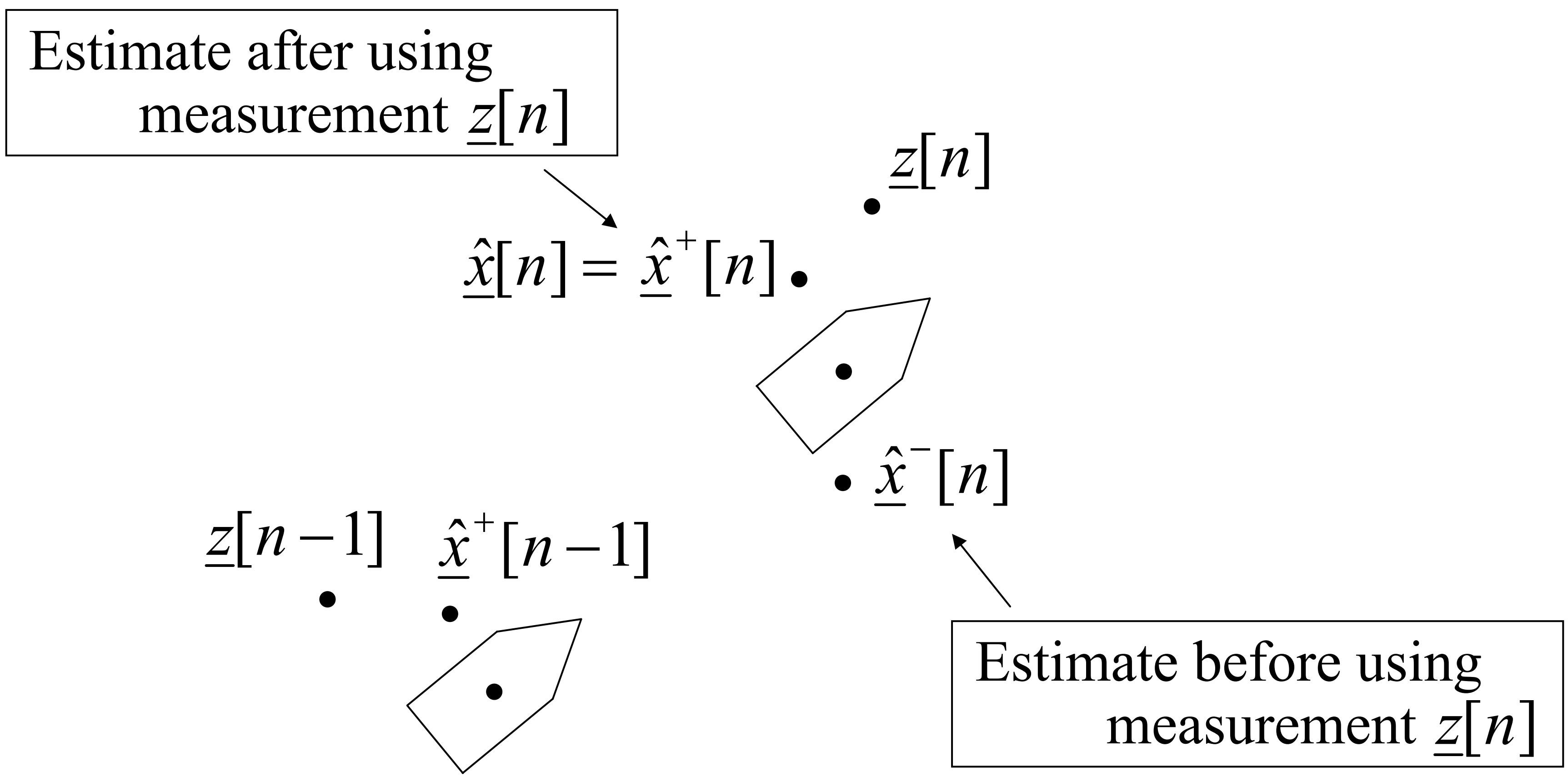
$$\underline{x}[n] = \Phi \underline{x}[n-1] + \underline{b}[n-1] + \underline{\eta}[n-1] \qquad \text{[System model]}$$

$$\underline{z}[n] = H \underline{x}[n] + \underline{v}[n] \qquad \text{[Measurement model]}$$

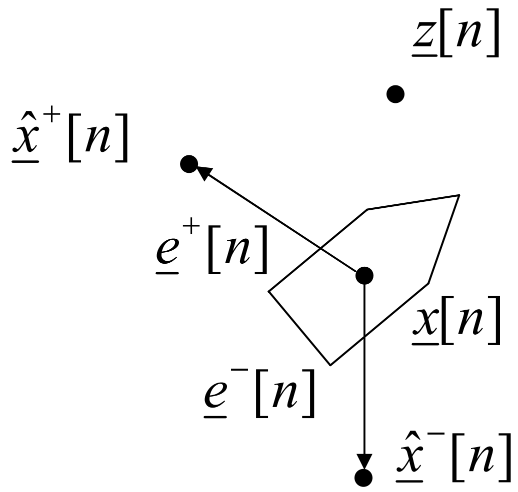
The estimate must be unbiased

The Kalman filter: Derivation (1/7)

We change notation slightly



The Kalman filter: Derivation (2/7)



Estimation errors

$$\underline{e}^+[n] = \underline{\hat{x}}^+[n] - \underline{x}[n]$$

$$\underline{e}^-[n] = \underline{\hat{x}}^-[n] - \underline{x}[n]$$

Linear, recursive estimate

$$\underline{\hat{x}}^+[n] = K'[n]\underline{\hat{x}}^-[n] + K[n]\underline{z}[n]$$

Unbiased

$$E[\underline{e}^+[n]] = 0$$

Optimal

minimize $\mathrm{E}[\|\underline{e}^+[n]\|]$ [Average length of $\underline{e}^+[n]$]

Find

$K[n]$

$K'[n]$

The Kalman filter: Derivation (3/7)

Constraint 1: Unbiased estimate

$$E[\underline{e}^{+}[n]] = 0$$

$$\Downarrow$$

$$\boxed{K'[n] = I - K[n]H}$$

Substituting, we obtain the **state estimate update**

$$\underline{\hat{x}}^{+}[n] = (I - K[n]H)\underline{\hat{x}}^{-}[n] + K[n]\underline{z}[n]$$

$$\underline{\hat{e}}^{+}[n] = (I - K[n]H)\underline{\hat{e}}^{-}[n] + K[n]\underline{v}[n]$$

$$P^{+}[n] = (I - K[n]H)P^{-}[n](I - K[n]H) + K[n]RK[n]^{T}$$

Error covariance matrix: $P^{+}[n] = E[\underline{e}^{+}[n]\underline{e}^{+}[n]^{T}]$

The Kalman filter: Derivation (4/7)

Constraint 2: Optimal estimate

$$\| \underline{e}^{+}[n] \| = \sqrt{e_1^{+}[n]^2 + \ldots + e_M^{+}[n]^2}$$

Equivalently, we consider

$$\| \underline{e}^{+}[n] \|^2 = e_1^{+}[n]^2 + \ldots + e_M^{+}[n]^2$$

We seek $K[n]$ which minimizes

$$\mathrm{E}[\| \underline{e}^{+}[n] \|^2] = E[e_1^{+}[n]^2 + \ldots + e_M^{+}[n]^2]$$

Since the expected value E is linear

$$\mathrm{E}[\| \underline{e}^{+}[n] \|^2] = E[e_1^{+}[n]^2] + \ldots + E[e_M^{+}[n]^2]$$

this is a known quantity...

The Kalman filter: Derivation (5/7)

Error covariance matrix

$$
\begin{aligned}
P^{+}[n] &= E[\underline{e}^{+}[n]\underline{e}^{+}[n]^{T}] \\
&= E\left[\begin{bmatrix} e_1^{+}[n] \\ \vdots \\ e_M^{+}[n] \end{bmatrix}\begin{bmatrix} e_1^{+}[n] & \cdots & e_M^{+}[n] \end{bmatrix}\right] \\
&= \begin{bmatrix} E[e_1^{+}[n]^2] & \cdots & E[e_1^{+}[n]e_M^{+}[n]] \\ \vdots & \ddots & \vdots \\ E[e_1^{+}[n]e_M^{+}[n]] & \cdots & E[e_M^{+}[n]^2] \end{bmatrix}
\end{aligned}
$$

The sum of the entries on the diagonal is

$$\text{trace}\, P^{+}[n] = E[e_1^{+}[n]^2] + \ldots E[e_M^{+}[n]^2]$$

Therefore

$$\mathrm{E}[\| \underline{e}^{+}[n] \|^2] = \text{trace}\, P^{+}[n]$$

The Kalman filter: Derivation (6/7)

We look for $K[n]$ which minimizes

$$\text{trace}\, P^{+}[n]$$

Minimum problem

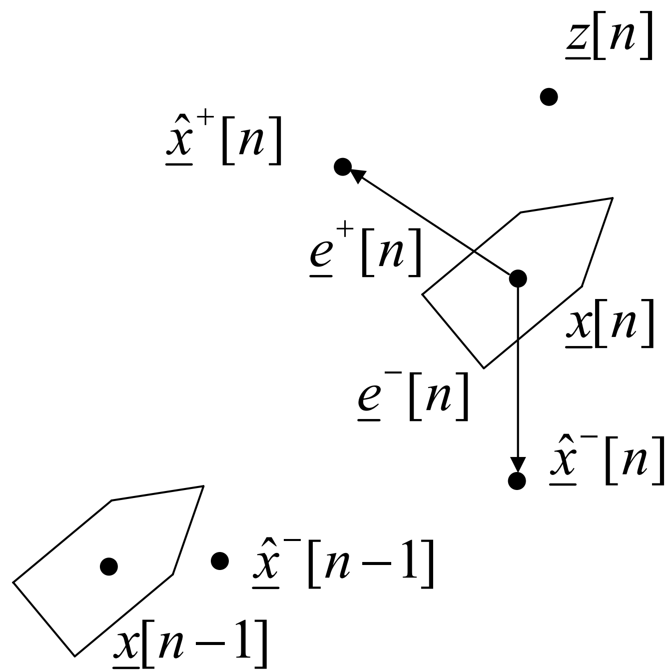
$$\frac{\partial\, \text{trace}\, P^{+}[n]}{\partial K[n]} = 0$$

The solution is

$$\boxed{K[n] = P^{-}[n] H^{T} (H P^{-}[n] H^{T} + R)^{-1}}$$

One more thing: how do we extrapolate the estimate at time n-1 to the estimate at time n?

The Kalman filter: Derivation (7/7)

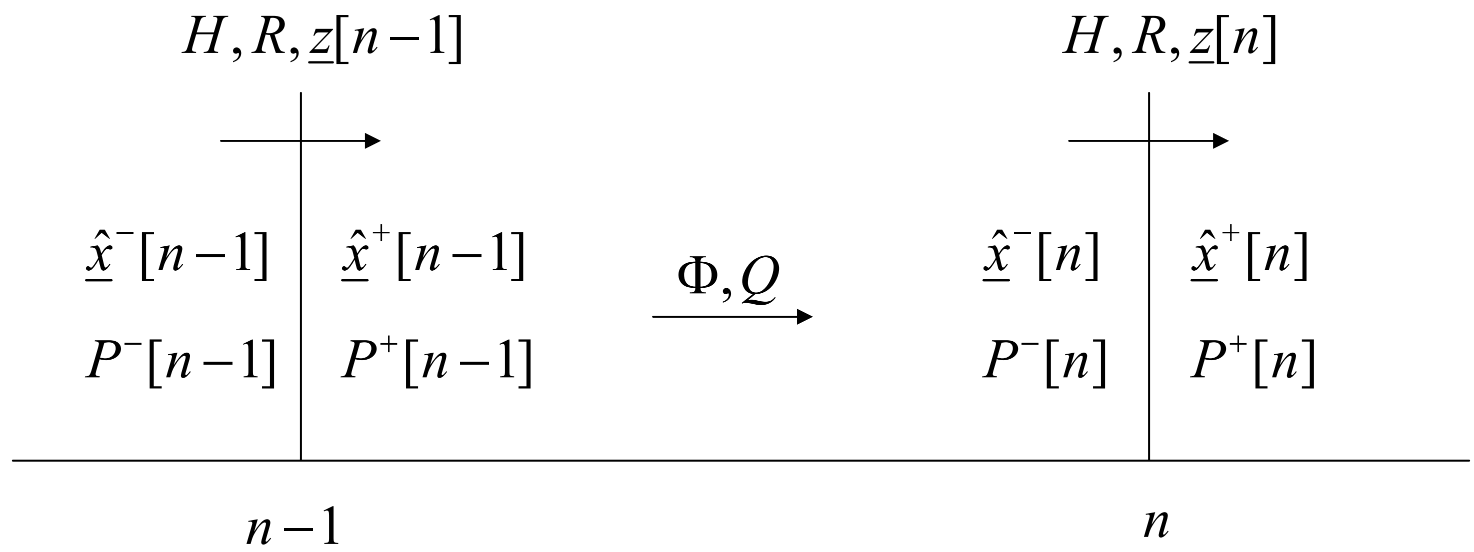


State estimate extrapolation

$$\underline{\hat{x}}^{-}[n] = \Phi \underline{\hat{x}}^{+}[n-1] + \underline{b}[n-1]$$ ← Unbiased estimate

$$P^{-}[n] = \Phi P^{+}[n-1]\Phi^{T} + Q$$

The Kalman filter: Timing diagram



The Kalman filter: Summary of equations

System model

$$\underline{x}[n] = \Phi \underline{x}[n-1] + \underline{b}[n-1] + \underline{\eta}[n-1] \longrightarrow \underline{\eta}[n-1] \sim N(0, Q)$$

Measurement model

$$\underline{z}[n] = H \underline{x}[n] + \underline{v}[n] \longrightarrow v[n] \sim N(0, R)$$

Initial conditions

$$\hat{\underline{x}}^{+}[0] = E[x[0]]$$

$$P^{+}[0] = E[(\hat{x}[0] - x[0])(\hat{x}[0] - x[0])^{T}]$$

State estimate extrapolation

$$\hat{\underline{x}}^{-}[n] = \Phi \hat{\underline{x}}^{+}[n-1]$$

$$P^{-}[n] = \Phi P^{+}[n-1] \Phi^{T} + Q$$

State estimate update

$$\hat{\underline{x}}^{+}[n] = (I - K[n]H) \hat{\underline{x}}^{-}[n] + K[n] \underline{z}[n]$$

$$P^{+}[n] = (I - K[n]H) P^{-}[n] (I - K[n]H) + K[n] R K[n]^{T}$$

The atomic clock signal (1/3)

Signal generated by an ideal clock (oscillator)

$$u(t) = U_0 \sin(2\pi\nu_0 t)$$

We define the ideal clock reading

$$h_0(t) = t$$

$$\Downarrow$$

$$u(t) = U_0 \sin(2\pi\nu_0 h_0(t))$$

Signal generated by a real clock (oscillator)

$$u(t) = (U_0 + \varepsilon(t))\sin(2\pi\nu_0 t + \varphi(t))$$

amplitude fluctuations [negligible]

phase fluctuations

The atomic clock signal (2/3)

We rewrite the clock signal as

$$u(t) = U_0 \sin(2\pi \nu_0 h(t))$$

where $h(t)$ is the clock reading

$$h(t) = t + \frac{1}{2\pi \nu_0} \varphi(t)$$

substituting

$$h(t) = h_0(t) + x(t)$$

where $x(t)$ is the time deviation

$$\boxed{x(t) = \frac{1}{2\pi \nu_0} \varphi(t)}$$

The atomic clock signal (3/3)

Instantaneous frequency of the oscillation $u(t)$

$$\nu(t) = \nu_0 + \frac{1}{2\pi}\frac{d\varphi(t)}{dt}$$

Normalized frequency deviation

$$\boxed{y(t) = \frac{\nu(t) - \nu_0}{\nu_0}}$$

It is

$$y(t) = \frac{dx(t)}{dt}$$

$x(t)$ and $y(t)$ have a noise-like nature

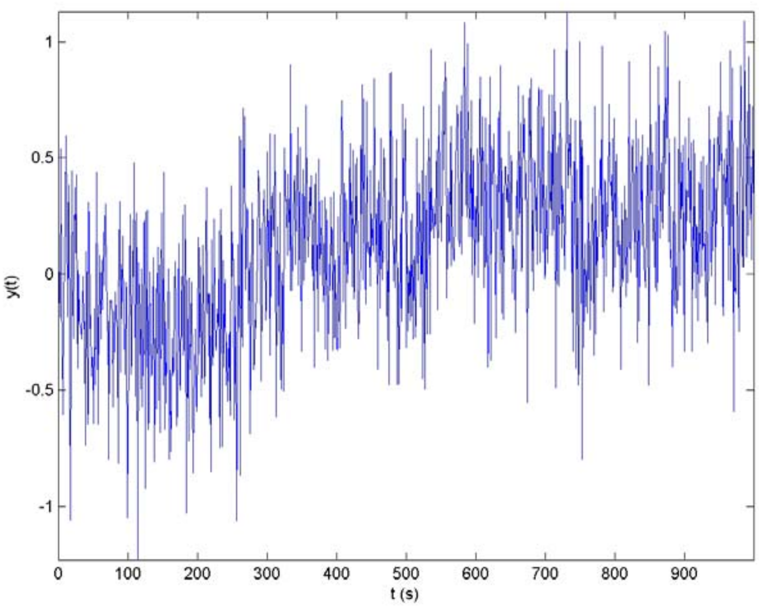
- they are referred to as clock noise
- they can be modeled as the sum of $1/f^{\alpha}$ noise components

The two-state model of clock noise (1/9)

Similarly to the boat, clock noise can be modeled with a random dynamical system

Model for clock noise (experimental evidence)

$$\boxed{y(t) = \text{white noise} + \text{random walk}}$$



The two-state model of clock noise (2/9)

$$y(t) = \xi_1(t) + W_2(t)$$

White noise

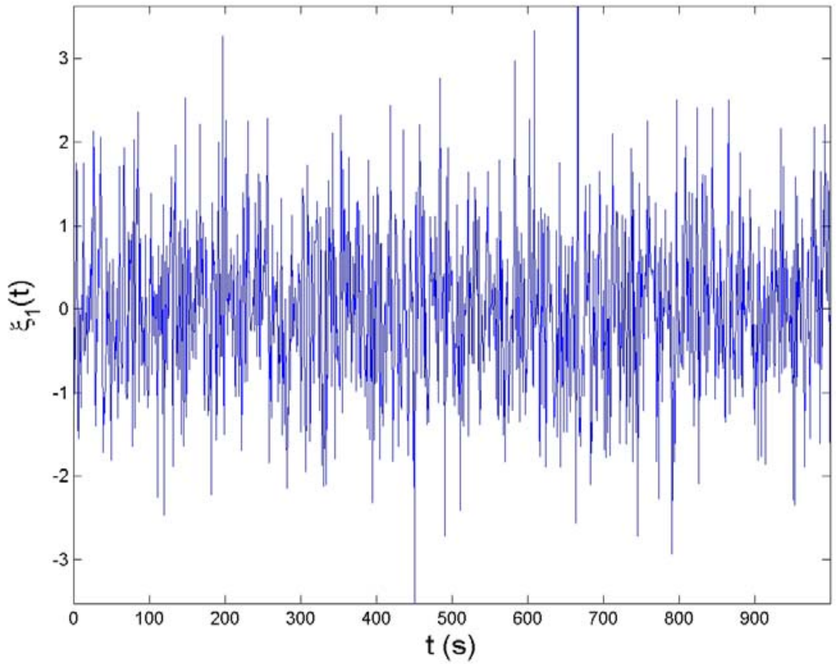
White Frequency Noise
(WFN)

Random Walk

Random Walk Frequency Noise
(RWFN)

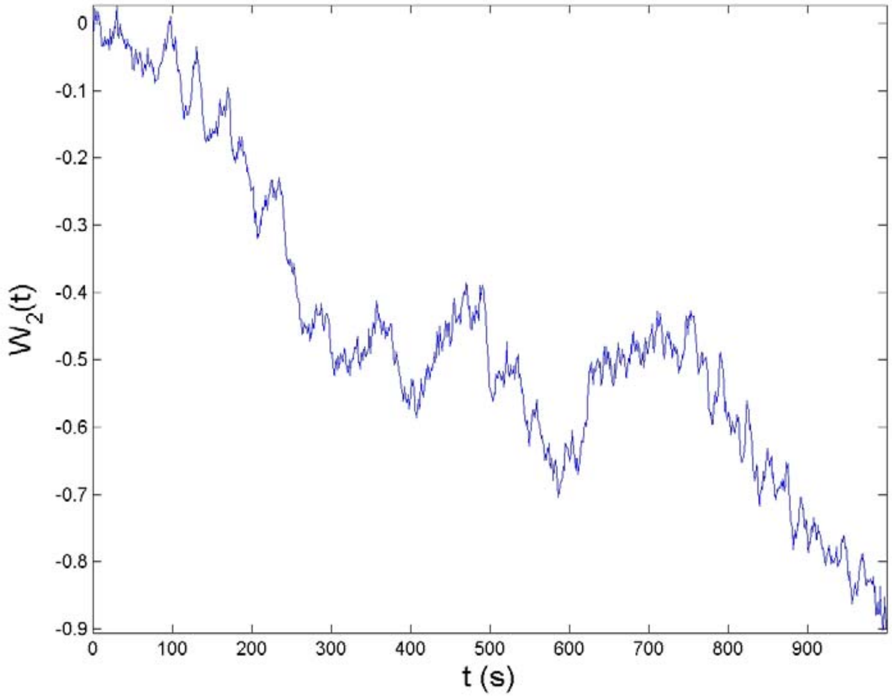
The two-state model of clock noise (3/9)

White Gaussian noise $\xi_1(t)$



The two-state model of clock noise (4/9)

Wiener process $W_2(t)$



Wiener process ≡ Random walk

The two-state model of clock noise (5/9)

The Wiener process is defined as

$$\frac{dW_2(t)}{dt} = \xi_2(t)$$
$$W_2(0) = 0$$

white Gaussian noise

Therefore

$$\boxed{W_2(t) = \int_0^t \xi_2(t')\,dt'}$$

We derive the two-state model in matrix form

The two-state model of clock noise (6/9)

$$y(t) = \xi_1(t) + W_2(t)$$

$$y(t) = \xi_1(t) + \int_0^t \xi_2(t')\,dt'$$

The phase deviation is given by

$$x(t) = \int_0^t y(t')\,dt'$$

Substituting

$$x(t) = \int_0^t \xi_1(t')\,dt' + \iint_0^t \xi_2(t')dt'$$

We set $x_1(t) = x(t)$

$$x_1(t) = \int_0^t \xi_1(t')\,dt' + \iint_0^t \xi_2(t')dt'$$

The two-state model of clock noise (7/9)

$$x_1(t) = \int_0^t \xi_1(t')\,dt' + \iint_0^t \xi_2(t')dt'$$

We evaluate the derivative of $x_1(t)$ $(= y(t))$

$$\dot{x}_1(t) = \xi_1(t) + \underbrace{\int_0^t \xi_2(t')\,dt'}_{x_2(t)}$$

$$\boxed{\dot{x}_1(t) = x_2(t) + \xi_1(t)}$$

We evaluate the derivative of $x_2(t)$

$$\boxed{\dot{x}_2(t) = \xi_2(t)}$$

The two-state model of clock noise (8/9)

Scalar equations

$$\dot{x}_1(t) = x_2(t) + \xi_1(t)$$
$$\dot{x}_2(t) = \xi_2(t)$$

Matrix equation

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix}$$

Model for clock noise

$$\boxed{\dot{\underline{x}}(t) = F\,\underline{x}(t) + \underline{\xi}(t)}$$

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \underline{\xi}(t) = \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix}$$

The two-state model of clock noise (9/9)

Continuous-time model

$$\dot{\underline{x}}(t) = F\,\underline{x}(t) + \underline{\xi}(t)$$

Discrete-time model

$$\underline{x}[n] = \Phi\,\underline{x}[n-1] + \underline{\eta}[n-1]$$

$$\Phi = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}$$

$$\underline{\eta}[n-1] \sim N(0, Q)$$

Exact transition matrix

Covariance matrix of clock noise

The Kalman filter for time scales (1/2)

The Kalman filter is an ideal tool for defining an atomic time scale

1. The model of the clock noise is a linear dynamical system
2. The measurement equation is linear
3. The time scale has to be generated in real time (the computational cost must be low)

The Kalman filter for time scales (2/2)

The GPS composite clock algorithm generates GPS time by

1. Processing the measurements from *N* clocks
2. Predicting the reading of each clock
3. Defining the implicit ensemble mean (IEM)

GPS time is continually steered versus UTC(USNO)

The GPS clock model (1/8)

- Cesium clocks are modeled by the two-state model
- Rubidium clocks need a third state

We review the GPS composite clock by following Brown's paper

K. R. Brown, "The theory of the GPS composite clock," *ION GPS-91*, September 1991, Albuquerque, USA, pp. 223-241

The GPS clock model (2/8)

The algorithm is given for N clocks:

We consider the case N=3

We write a random differential equation for the clocks in the ensemble

$$\dot{\underline{h}}(t) = F\,\underline{h}(t) + \underline{\xi}(t)$$

where

$$\underline{h}(t) = \begin{bmatrix} \underline{h}_1(t) \\ \underline{h}_2(t) \\ \underline{h}_3(t) \end{bmatrix}$$

and

$$\underline{h}_1(t) = \begin{bmatrix} h_{1,1}(t) \\ h_{1,2}(t) \end{bmatrix}$$

The GPS clock model (3/8)

The matrix F is given by

$$F = \begin{bmatrix} F_1 & 0 & 0 \\ 0 & F_2 & 0 \\ 0 & 0 & F_3 \end{bmatrix}$$

All the clocks are identical and

$$F_1 = F_2 = F_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The GPS clock model (4/8)

Every element of $\underline{\xi}(t)$ is a white Gaussian noise

$$\underline{\xi}(t) = \begin{bmatrix} \underline{\xi}_1(t) \\ \underline{\xi}_2(t) \\ \underline{\xi}_3(t) \end{bmatrix}$$

where

$$\underline{\xi_1}(t) = \begin{bmatrix} \xi_{1,1}(t) \\ \xi_{1,2}(t) \end{bmatrix}$$

with autocorrelation matrix

$$r_{\xi_1}(t_1, t_2) = E[\xi_1(t_1)\xi_1^T(t_2)] = Q_1\delta(t_1 - t_2)$$

$$Q_1 = \begin{bmatrix} q_{1,1} & 0 \\ 0 & q_{1,2} \end{bmatrix}$$

The GPS clock model (5/8)

We can write

$$\underline{h}(t) = \underline{h_0}(t) + \underline{x}(t)$$

where

$$\underline{h_0}(t) = H_R h_0(t)$$

H_R is a replication matrix

$$H_R = \begin{bmatrix} I_2 \\ I_2 \\ I_2 \end{bmatrix}$$

where

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The GPS clock model (6/8)

Some simple algebra shows that

$$\dot{\underline{x}}(t) = F\,\underline{x}(t) + \underline{\xi}(t)$$

We recall the equation for the clock readings

$$\dot{\underline{h}}(t) = F\,\underline{h}(t) + \underline{\xi}(t)$$

Therefore, the same random differential equation holds for the clock readings and the time deviations

The GPS clock model (7/8)

In discrete time

$$\underline{h}[n] = \Phi \underline{h}[n-1] + \underline{\eta}[n-1]$$

where, as usual

$$\underline{h}[n] = \underline{h}(nT_s)$$

It is

$$\Phi = \begin{bmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & \phi_3 \end{bmatrix}$$

where

$$\phi_1 = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}$$

The GPS clock model (8/8)

$\underline{\eta}[n-1]$ is a vector of Gaussian random variables

$$\underline{\eta}[n-1] = \begin{bmatrix} \underline{\eta}_1[n-1] \\ \underline{\eta}_2[n-1] \\ \underline{\eta}_3[n-1] \end{bmatrix} \quad \Rightarrow \quad C_{\underline{\eta}} = \begin{bmatrix} C_{\eta_1} & 0 & 0 \\ 0 & C_{\eta_2} & 0 \\ 0 & 0 & C_{\eta_3} \end{bmatrix}$$

It is

$$\underline{\eta_1}[n-1] \sim N(0, C_{\eta_1})$$

where

$$C_{\eta_1} = \begin{bmatrix} q_{1,1}T_s + \dfrac{q_{1,2}T_s^{\,3}}{3} & \dfrac{q_{1,2}T_s^{\,2}}{2} \\ \dfrac{q_{1,2}T_s^{\,2}}{2} & q_{1,2}T_s \end{bmatrix}$$

The GPS clock measurement model (1/2)

The input of the Kalman filter are the *N*-1 time differences between the *N* clocks, corrupted by measurement noise

$$z_1[n] = h_{1,1}[n] - h_{2,1}[n] + v_1[n]$$
$$z_2[n] = h_{2,1}[n] - h_{3,1}[n] + v_2[n]$$

We can write these measurements as

$$\underline{z}[n] = H\,\underline{h}[n] + \underline{v}[n]$$

where

$$\underline{z}[n] = \begin{bmatrix} z_1[n] \\ z_2[n] \end{bmatrix} \qquad \underline{v}[n] = \begin{bmatrix} v_1[n] \\ v_2[n] \end{bmatrix}$$

$$\underline{v}[n] \sim N(0, R)$$

Covariance of the measurement noise

The GPS clock measurement model (2/2)

The matrix H which performs the time differences is defined as

$$H = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}$$

With N=3 clocks we have N-1=2 measurements only...

One measurement is missing!

The GPS Implicit Ensemble Mean (1/15)

The Kalman filter uses the system model and the measurements to obtain an estimate $\underline{\hat{x}}[n]$ of the clock time deviations, along with its covariance $C_{\underline{\hat{x}}}[n]$

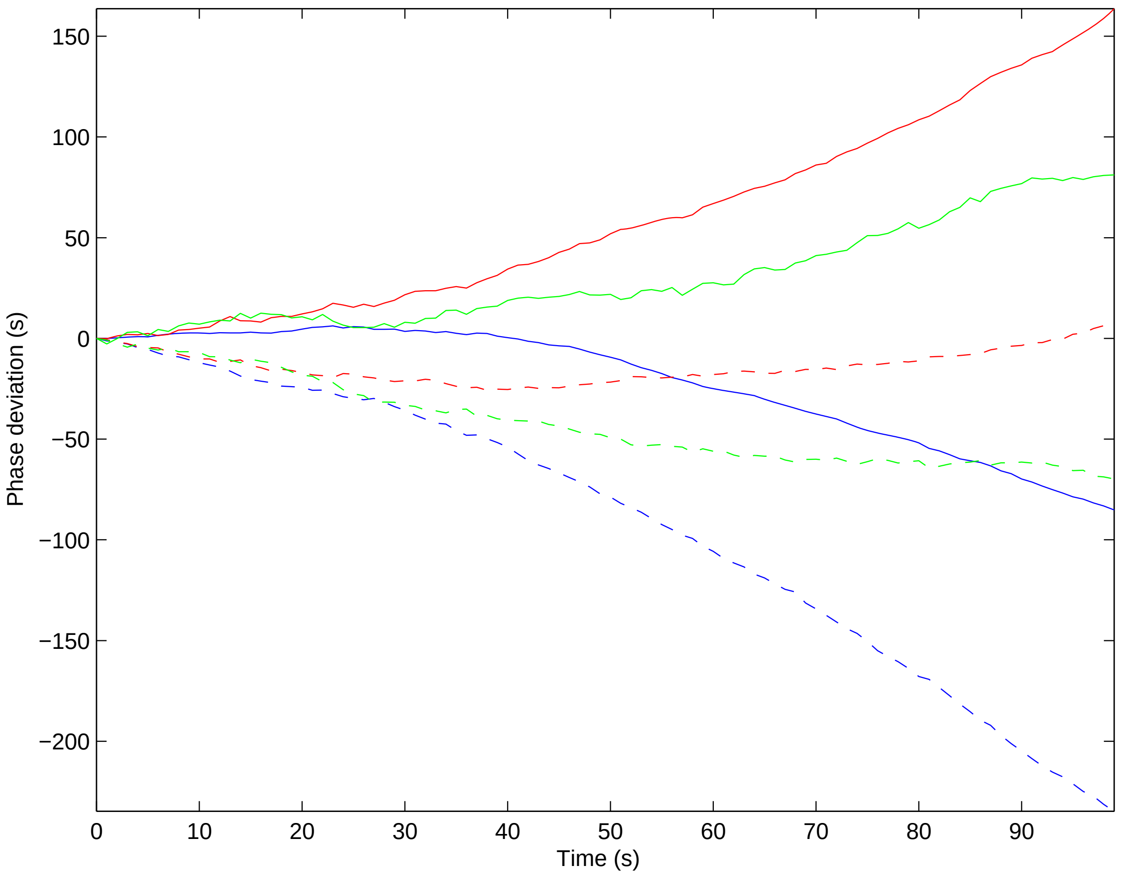
Then the IEM is built from the estimated time deviations

We simulate N=3 clocks by using

$$Q_1 = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 10^{-2} \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & 10^{-2} \end{bmatrix}, \qquad Q_3 = \begin{bmatrix} 4 & 0 \\ 0 & 10^{-2} \end{bmatrix}$$

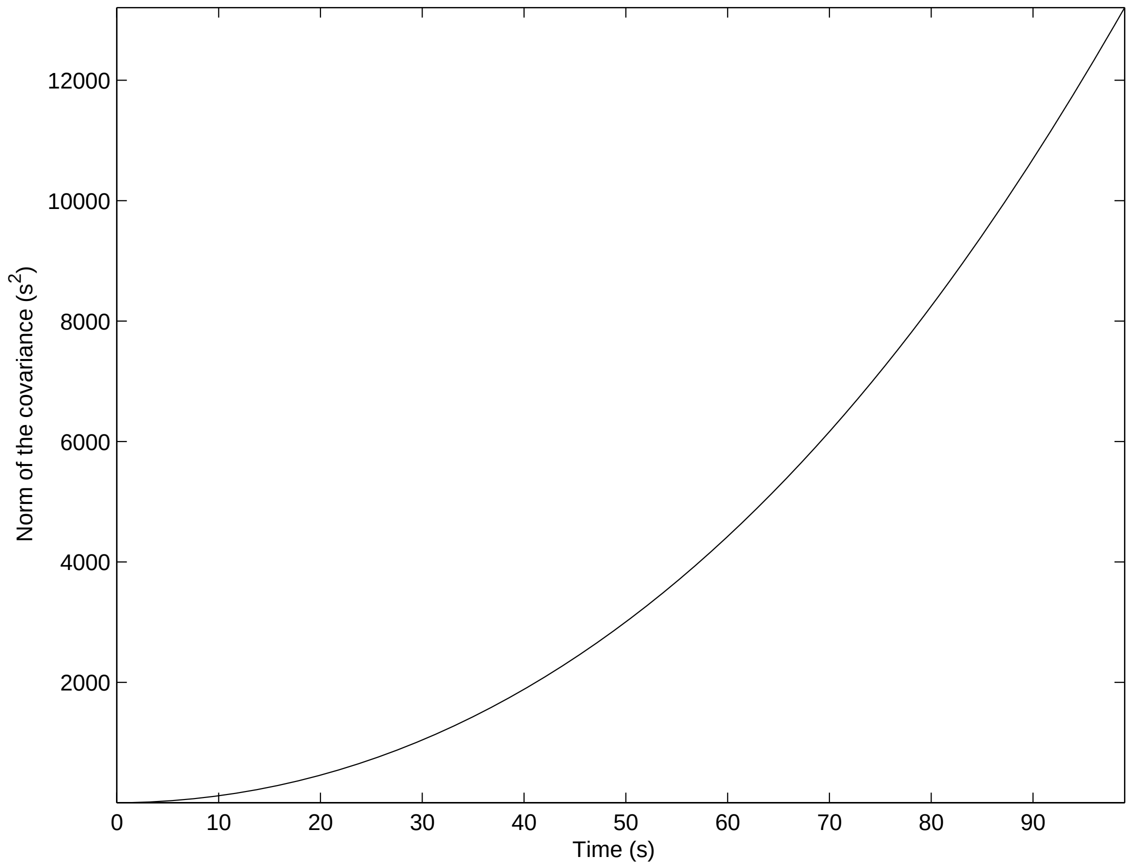
$$T_s = 1\text{s}$$

The GPS Implicit Ensemble Mean (2/15)



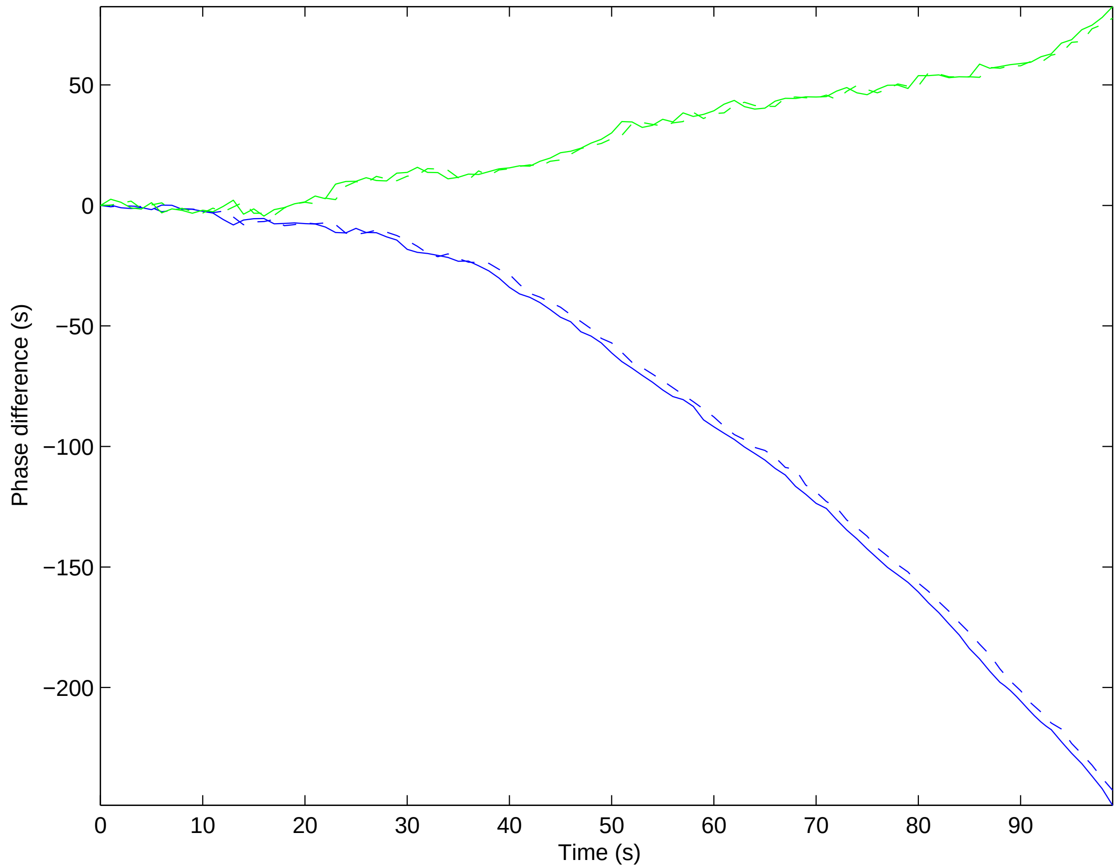
Simulated and estimated clock time deviations

The GPS Implicit Ensemble Mean (3/15)



Growth of the covariance matrix

The GPS Implicit Ensemble Mean (4/15)



Simulated and estimated clock differences

The GPS Implicit Ensemble Mean (5/15)

The IEM is defined as

$$\underline{\bar{h}}_0[n] = W_{\mathrm{IEM}}[n]\underline{\Omega}[n]$$

where

$$\underline{\bar{h}}_0[n] = \begin{bmatrix} \bar{h}_{0,1}[n] \\ \bar{h}_{0,2}[n] \end{bmatrix}$$

and $\underline{\Omega}[n]$ is the vector of corrected clocks

$$\underline{\Omega}[n] = \underline{h}[n] - \underline{\hat{x}}[n]$$

where

$$\underline{\Omega}[n] = \begin{bmatrix} \underline{\omega}_1[n] \\ \underline{\omega}_2[n] \\ \underline{\omega}_3[n] \end{bmatrix}, \qquad \underline{\omega}_1[n] = \begin{bmatrix} \omega_{1,1}[n] \\ \omega_{1,2}[n] \end{bmatrix}$$

The GPS Implicit Ensemble Mean (6/15)

The IEM is a weighted average of the corrected clocks
The weights are given by

$$W_{\mathrm{IEM}}[n] = [H_R{}^T C_{\underline{\hat{x}}}{}^{-1}[n] H_R]^{-1} H_R{}^T C_{\underline{\hat{x}}}{}^{-1}[n]$$

The deviation of the IEM from the ideal time is given by

$$\underline{TA}[n] = \overline{\underline{h}}_0[n] - \underline{h}_0[n]$$

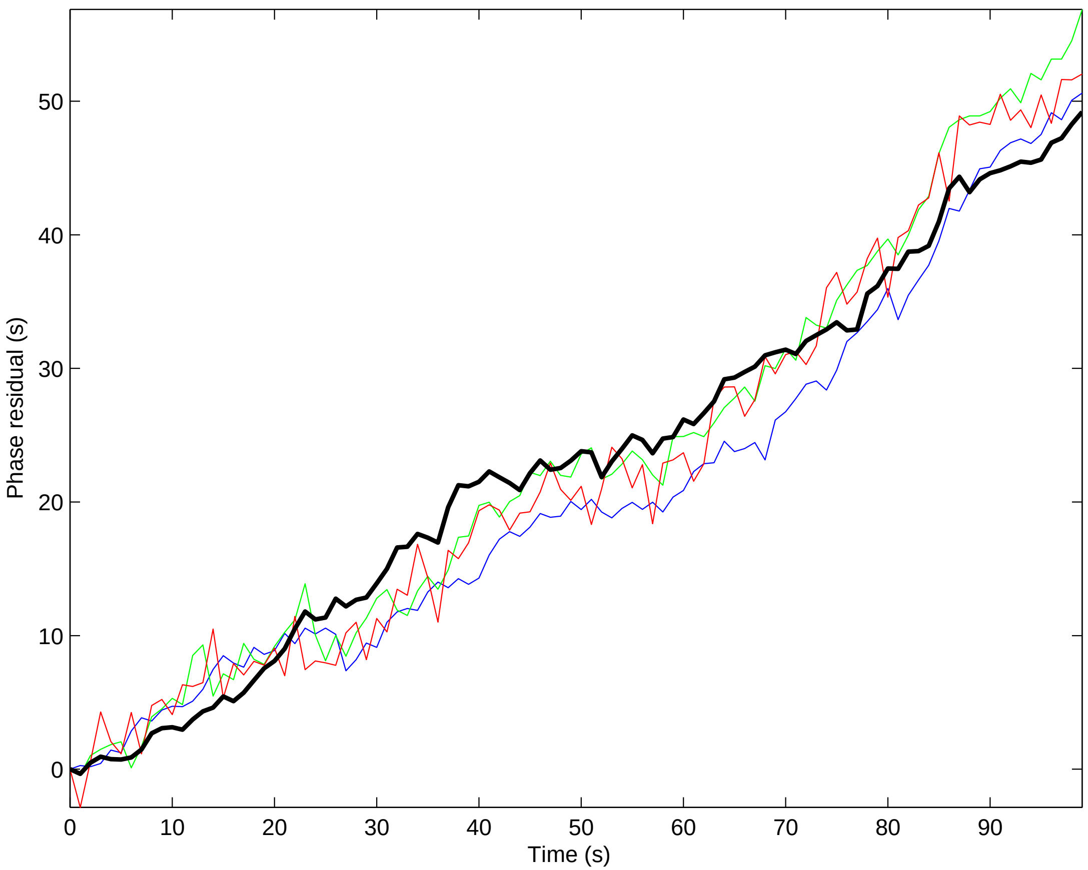
where

$$\underline{TA}[n] = \begin{bmatrix} TA_1[n] \\ TA_2[n] \end{bmatrix}$$

It can be shown that

$$\underline{TA}[n] = W_{\mathrm{IEM}}[n][\underbrace{\underline{x}[n] - \underline{\hat{x}}[n]}_{\text{Clock residuals } \underline{E}[n]}]$$

The GPS Implicit Ensemble Mean (7/15)

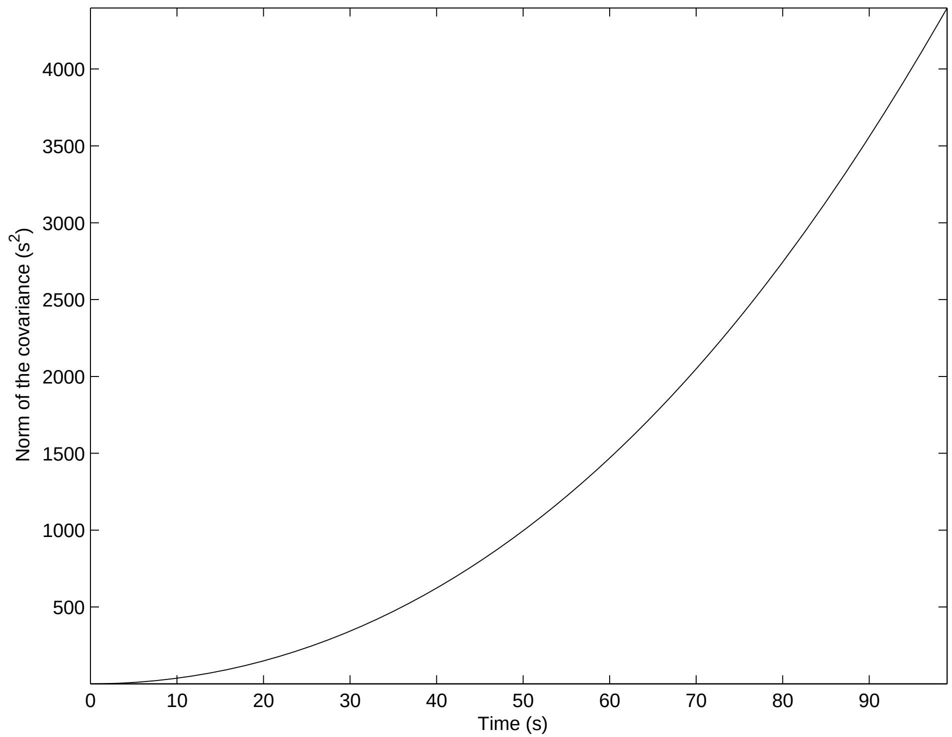


Clock residuals and implicit ensemble mean

The GPS Implicit Ensemble Mean (8/15)

Also the IEM diverges, as can be seen from its covariance matrix

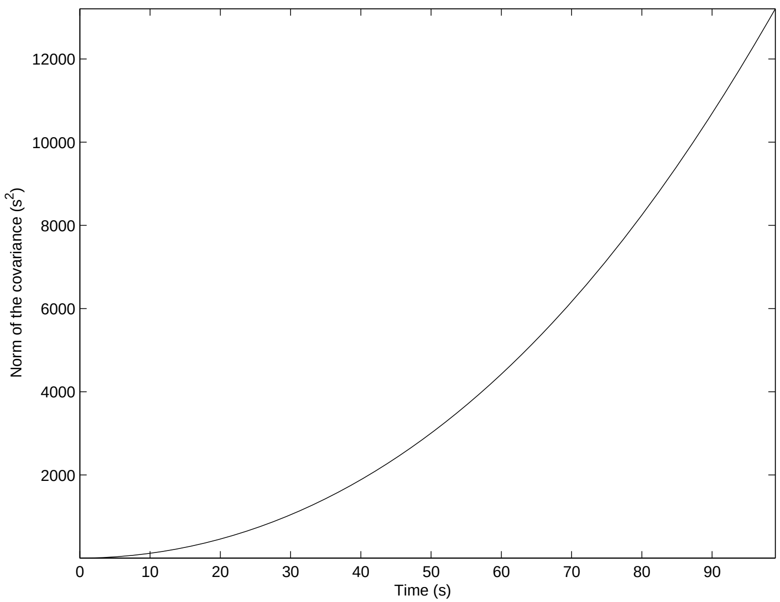
$$C_{\underline{TA}}[n] = [H_R^{\ T} C_{\underline{\hat{x}}}^{-1}[n] H_R]^{-1}$$



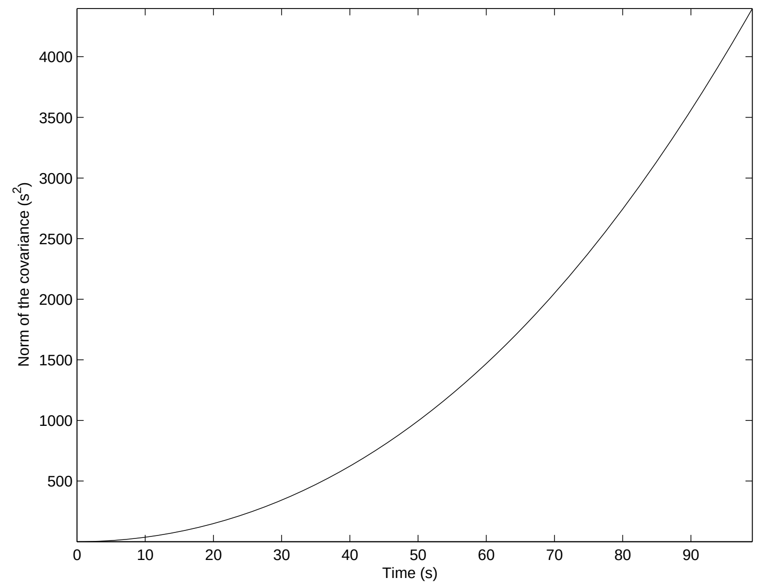
Growth of the covariance of the IEM

The GPS Implicit Ensemble Mean (9/15)

The IEM diverges slower than the clock estimates



Covariance of the clock estimates



Covariance of the IEM

The GPS Implicit Ensemble Mean (10/15)

The clock residuals remain close to the IEM

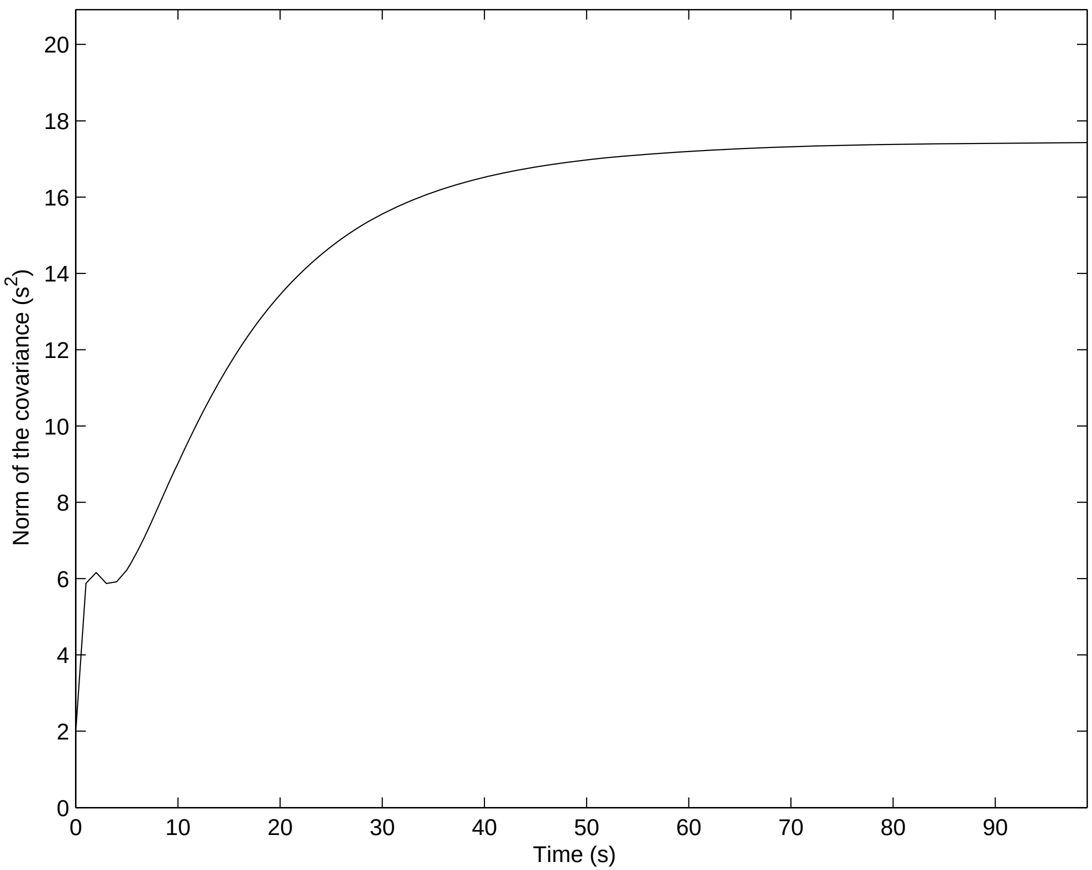
$$\underline{\Gamma}[n] = \underline{E}[n] - \underline{TA}[n]$$

Deviation of the clock residuals from the IEM

$$C_{\underline{\Gamma}}[n] = C_{\underline{\hat{x}}}[n] - H_R[H_R{}^T C_{\underline{\hat{x}}}{}^{-1}[n] H_R]^{-1} H_R{}^T$$

Covariance matrix of the deviation of the clock residuals from the IEM

The GPS Implicit Ensemble Mean (11/15)



Covariance of the difference between the clock residuals and the IEM

The GPS Implicit Ensemble Mean (12/15)

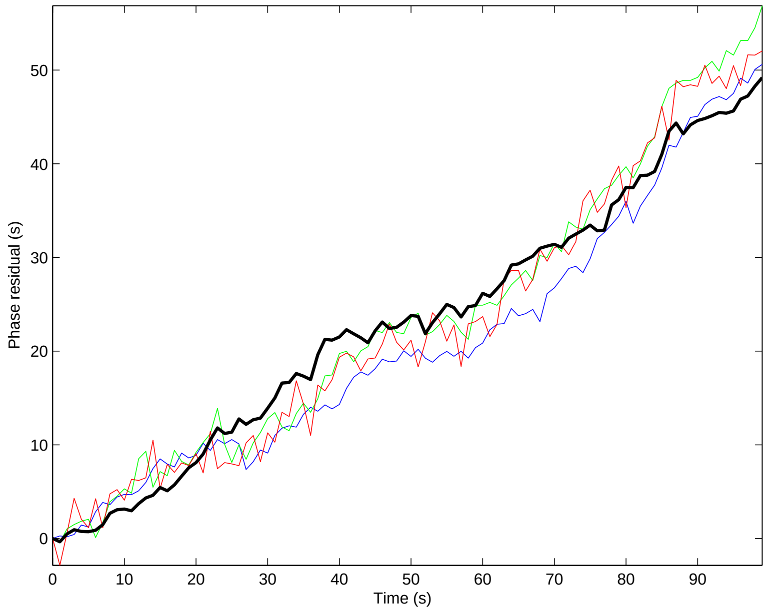
The clock residuals remain close to the IEM

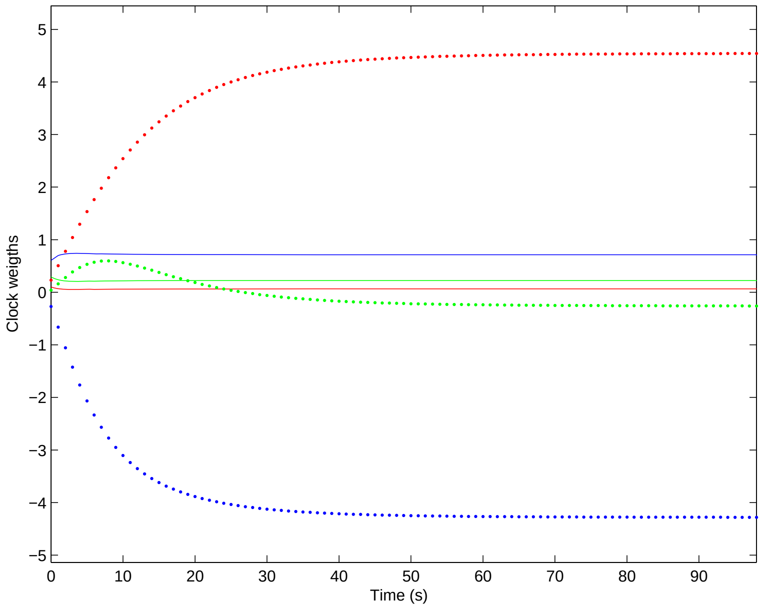
Every clock residual is a good representation of the IEM

GPS uses the clock residuals rather than the IEM

The GPS Implicit Ensemble Mean (13/15)

Note: The IEM is not bounded by the clock residuals





The reason is that the clock weights can be negative

The GPS Implicit Ensemble Mean (14/15)

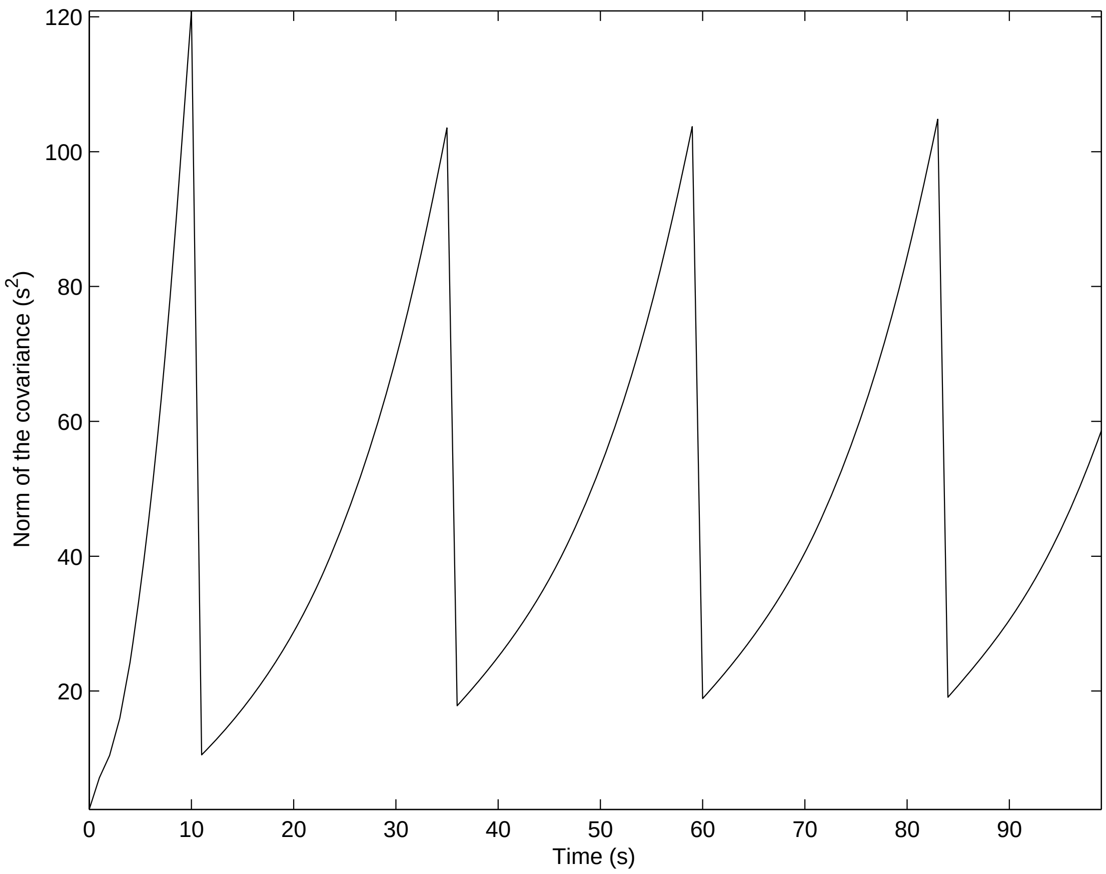
Eventually, the divergence of the covariance matrix results in a numeric overflow

We shrink the covariance matrix

$$C_{\underline{\hat{x}}}'[n] = C_{\underline{\hat{x}}}[n] - H_R[H_R{}^T C_{\underline{\hat{x}}}{}^{-1}[n] H_R]^{-1} H_R{}^T$$

This formula provides maximal reduction

The GPS Implicit Ensemble Mean (15/15)



Control of covariance

Conclusions

- The Kalman filter estimates the current state of a dynamical system from noisy measurements
- It is linear, unbiased, optimal, and recursive
- The atomic clock noise can be modeled by a dynamical system
- The Kalman filter is used in several applications in atomic timing
- GPS time is generated by a Kalman filter: it is the GPS composite clock algorithm

References

Two state-model

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