Fundamentals of Magnetism

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Part I:  $H,M,B,\chi,\mu$

Part II:  $M(H)$
Magnetic Field from Current in a Wire

Ampere’s Circuital Law:
\[ \oint_c \vec{H} \cdot d\vec{l} = I \]

Biot-Savart Law:
\[ d\vec{H} = \frac{I d\vec{l} \times \hat{R}}{4\pi |\vec{R}|^2} \]

\[ H = \frac{I}{2\pi r} \quad [\text{A/m}] \]
Use Biot-Savart Law:

\[ d\vec{H} = \frac{I d\vec{l} \times \hat{R}}{4\pi |\vec{R}|^2} \]

In the center:

\[ H = \frac{I}{2r} \quad \text{[A/m]} \]
Magnetic Field in a Long Solenoid

Inside, field is uniform with:

\[ H = \frac{NI}{L} \text{ [A/m]} \]

\[ H = J_S \text{ [A/m]} \]
Magnetic Field from Small Loop (Dipole)
Magnetic Field from Small Loop (Dipole)

For $R \gg r$

$$H = \frac{IA}{4\pi R^3} \left[ 2\hat{r} \cos(\theta) + \hat{\theta} \sin(\theta) \right] \quad [A/m]$$

Magnetic (dipole) moment:

$$\vec{m} = I\hat{A} \quad [Am^2]$$

Compare to Electric dipole:
Magnetic field from a “monopole”

\[ H = \frac{q_m}{4\pi R^2} \quad [\text{A/m}] \]
Magnetic Field of a Dipole

For $R \gg r$

$$H = \frac{q_m d}{4\pi R^3} \left[ 2\hat{r} \cos(\theta) + \hat{\theta} \sin(\theta) \right] \quad \text{[A/m]}$$

Magnetic (dipole) moment:

$$\vec{m} = q_m \vec{d} \quad \text{[Am^2]}$$
Orbital Magnetic Moment

• For an electron with charge $q_e$ orbiting at a radius $R$ with frequency $f$, the Orbital Magnetic Moment is
  \[ m = IA = -q_e f \pi R^2 \quad [\text{Am}^2] \]

• It also has an Orbital Angular Momentum
  \[ L = m_e \nu R = m_e 2\pi f R \]

• Note: \[ \frac{m}{L} = \frac{-q_e}{2m_e} \]
m and $L$ are in opposite directions!

• Electrons orbiting a nucleus are like a circulating current producing a magnetic field.

• Electrons orbiting a nucleus are like a circulating current producing a magnetic field.
Orbital Moment is Quantized

Bohr model of the atom

• DeBroglie wavelength is:
\[ \lambda_{dB} = \frac{h}{m_e \nu} \]

• Bohr Model: orbit must be integer number of wavelengths
\[ 2\pi R = N\lambda_{dB} = N \frac{h}{m_e \nu} = N \frac{h}{m_e 2\pi Rf} \]

• Thus the orbital magnetic moment is quantized:
\[ m = -q_e f \pi R^2 = N \frac{h}{2\pi} \frac{q_e}{2m_e} = N \frac{\hbar q_e}{2m_e} \]

• Magnetic moment restricted to multiples of

\[ \mu_B = \frac{\hbar q_e}{2m_e} = 9.2742 \times 10^{-24} [Am^2] \]
Spin

- Spin is a property of subatomic particles (just like charge or mass.)
- A particle with spin has a magnetic dipole moment and angular momentum.
- Spin may be thought of *conceptually* as arising from a spinning sphere of charge. (However, note that neutrons also have spin but no charge!)

Pauli and Bohr contemplate the “spin” of a tippy-top
Electron Spin

- Electrons, protons and neutrons (fermions) have spin, $s = \frac{1}{2}$
- Photons (boson) have spin 1

- When measured in a particular direction, the measured angular momentum of an electron is

$$L_z = s_z \hbar = \pm \frac{\hbar}{2} \quad s_z = -s, -s + 1 \ldots s$$

- When measured in a particular direction, the measured magnetic moment of an electron is

$$m_z = s_z \frac{\hbar q_e}{m_e} = \pm \mu_B$$

- We say “spin up” and “spin down”

- Note: $\frac{m}{L} = \frac{-q_e}{m_e}$

<table>
<thead>
<tr>
<th>Compare:</th>
<th>Spin</th>
<th>Orbital</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_z$</td>
<td>$\pm \mu_B$</td>
<td>$m_z = N\mu_B$</td>
</tr>
<tr>
<td>$\frac{m}{L}$</td>
<td>$\frac{-q_e}{m_e}$</td>
<td>$\frac{-q_e}{2m_e}$</td>
</tr>
</tbody>
</table>
Demonstrated that magnetic moment is quantized with $\pm \mu_B$

Gerlach's postcard, dated 8 February 1922, to Niels Bohr. It shows a photograph of the beam splitting, with the message, in translation: "Attached [is] the experimental proof of directional quantization. We congratulate [you] on the confirmation of your theory." (Physics Today December 2003)
Quantum Numbers of Electron Orbitals

Electrons bound to a nucleus move in orbits identified by quantum numbers (from solution of Schrödinger's equation):

- **n** = 1... principal quantum number, identifies the “shell”
- **l** = 0..n-1 angular momentum quantum number, type of orbital, e.g. s, p, d, f
- **m_l** = -l..l magnetic quantum number
- **m_s** = +½ or -½ spin quantum number

For example, the electron orbiting the hydrogen nucleus (in the ground state) has:

- **n** = 1, **l** = 0, **m_l** = 0, **m_s** = +½

In spectroscopic notation: 1s\(^1\)

Spectroscopic notation:
- l=0  s
- l=1  p
- l=2  d
- l=3  f

For the electron configuration 1s\(^2\)2s\(^2\):
- **n** = 1, **l** = 0, **m_l** = 0, **m_s** = +½
- **n** = 2, **l** = 0, **m_l** = 0, **m_s** = +½
Spin and Orbital Magnetic Moment

- Total orbital magnetic moment (sum over all electron orbitals)
  \[ m_{\text{tot orbital}} = \mu_B \sum m_\ell \]
- Total spin magnetic moment
  \[ m_{\text{tot spin}} = 2\mu_B \sum m_s \]

E.g. Iron:

1s\(^2\)2s\(^2\)2p\(^6\)3s\(^2\)3p\(^6\)3d\(^6\)4s\(^2\)

How to fill remaining d orbitals (\(\ell=2\)) 6 electrons for 10 spots:

<table>
<thead>
<tr>
<th>(m_s=+1/2)</th>
<th>(m_\ell=-2)</th>
<th>(m_\ell=-1)</th>
<th>(m_\ell=0)</th>
<th>(m_\ell=+1)</th>
<th>(m_\ell=+2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\uparrow)</td>
<td>(\uparrow)</td>
<td>(\uparrow)</td>
<td>(\uparrow)</td>
<td>(\uparrow)</td>
<td></td>
</tr>
</tbody>
</table>

\(m_s=-1/2\)

Hund’s rules:
Maximize \(\Sigma m_s\)
Then maximize \(\Sigma m_\ell\)

\[ m_{\text{tot orbital}} = 2\mu_B \quad m_{\text{tot spin}} = 4\mu_B \]
Periodic Table

Atomic Properties of the Elements

Frequently used fundamental physical constants

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>speed of light in vacuum c</td>
<td>299,792,458 m/s</td>
</tr>
<tr>
<td>Planck constant c</td>
<td>6.6261 x 10^-34 J s (exact)</td>
</tr>
<tr>
<td>elementary charge e</td>
<td>1.6022 x 10^-19 C</td>
</tr>
<tr>
<td>electron mass m_e</td>
<td>9.1094 x 10^-31 kg</td>
</tr>
<tr>
<td>proton mass m_p</td>
<td>1.6726 x 10^-27 kg</td>
</tr>
<tr>
<td>fine-structure constant (\alpha)</td>
<td>1/137.0356</td>
</tr>
<tr>
<td>Rydberg constant (R_\infty)</td>
<td>10.973 32 m (^1)</td>
</tr>
<tr>
<td>(c_0), the speed of light in vacuum in (m/s)</td>
<td>299,792,458</td>
</tr>
<tr>
<td>Boltzmann constant (k)</td>
<td>1.3807 x 10^-23 J K^-1</td>
</tr>
</tbody>
</table>

Physics Laboratory: physics.nist.gov

Standard Reference Data Group: www.nist.gov/srd

For a description of the data, visit physics.nist.gov/data

NIST SP 966 (September 2003)
Magnetization, $\vec{M}$

Atomic magnetic moment: $\vec{m}$ [Am$^2$]

Atomic density: N/vol [1/m$^3$]

Magnetic moment per unit volume:

$$\vec{M} \overset{\text{def}}{=} \frac{\vec{m}}{\text{vol}}$$ [A/m]
Equivalent Surface Current

Atomic magnetic moment:
\[ \vec{m} = I \vec{A} \] \hspace{1cm} \text{[Am}^2\text{]}

Atomic density: \( N/\text{vol} \)
\[ [1/\text{m}^3] \]

Magnetic moment per unit volume:
\[ \vec{M} \overset{\text{def}}{=} \frac{\vec{m}}{\text{vol}} \] \hspace{1cm} \text{[A/m]}
Equivalent Surface Current

Atomic magnetic moment:

$\vec{m} = I \vec{A}$ \([\text{Am}^2]\)

Atomic density: \(N/\text{vol}\)

[1/\text{m}^3]

Equivalent surface current:

$\vec{J}_S = \vec{M} \times \hat{n}$ \([\text{A/m}]\)

What is \(H\) here?
**Magnetic Pole Model**

Atomic magnetic moment:

\[ \vec{m} = q_m \vec{d} \quad [\text{Am}^2] \]

Atomic density: \( N/\text{vol} \) 

[1/m³]

Magnetic moment per unit volume:

\[ \vec{M} \overset{\text{def}}{=} \frac{\vec{m}}{\text{vol}} \quad [\text{A/m}] \]

What is H here?
Equivalent Surface Pole Density

Atomic magnetic moment:
\[ \vec{m} = q_m \hat{d} \quad [\text{Am}^2] \]

Atomic density: N/vol
[1/m^3]

Surface charge density:
\[ \rho_{ms} = \frac{q_m}{\text{area}} \quad [\text{Am}] \quad [\text{m}^2] \]

Magnetic pole density:
\[ \rho_{ms} = \vec{M} \cdot \hat{n} \quad [\text{A/m}] \]
Example: NdFeB Cylinder

\[ M = 10^6 \, [A/m] \]
\[ \text{vol} = 14 \cdot 10^{-6} \, [m^3] \]
\[ m = 14 \, [Am^2] \]

\[ J_S = 10^6 \, [A/m] \]
\[ I = 20000 \, [A] \]
\[ A = 7.07 \cdot 10^{-4} \, [m^2] \]
\[ m = IA = 14 \, [Am^2] \]
Example: NdFeB Cylinder

\[ M = 10^6 \text{ [A/m]} \]

\[ \text{vol} = 14 \cdot 10^{-6} \text{ [m}^3\text{]} \]

\[ m = 14 \text{ [Am}^2\text{]} \]

\[ \rho_{ms} = 10^6 \text{ [A/m]} \]

\[ A = 7 \cdot 10^{-4} \text{ [m}^2\text{]} \]

\[ q_m = 700 \text{ [Am]} \]

\[ d = 2 \cdot 10^{-2} \text{ [m]} \]

\[ m = q_m d = 14 \text{ [Am}^2\text{]} \]
Example: NdFeB Cylinder

Result is the same external to magnetic material.

Hint: we are far away from all the dipoles!!!
Example: NdFeB Cylinder

Result is different inside the magnetic material.
The Constitutive Relation

\[ \vec{B} = \mu_0(\vec{H} + \vec{M}) \]

\( B \) = Magnetic flux density \hspace{1em} [Tesla]

\( H \) = Magnetic field, “Magnetizing force” \hspace{1em} [A/m]

\( M \) = Magnetization \hspace{1em} [A/m]

\( \mu_0 \) = Magnetic constant, “Permeability of free space” \hspace{1em} [Tesla-m/A] \hspace{1em} [Henry/m] \hspace{1em} [N/A^2]

Note, in vacuum, \( M \) is zero:

\[ \vec{B} = \mu_0 \vec{H} \]
What is $\mu_0$?

$\mu_0$ comes from the SI definition of the Ampere:

The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross section, and placed 1 meter apart in vacuum, would produce between these conductors a force equal to $2 \times 10^{-7}$ newton per meter of length.

\[ B = \mu_0 H = \frac{\mu_0 I_1}{2\pi r} \]

\[ \vec{F} = I_2 \vec{L} \times \vec{B} \]
Maxwell’s Equations (Magnetostatics)

**Differential form**
\[ \nabla \cdot \vec{B} = 0 \]
\[ \nabla \times \vec{H} = \vec{J} \]

**Integral form**
\[ \oiint \vec{B} \cdot d\vec{s} = 0 \]
\[ \oint \vec{H} \cdot d\vec{l} = I \]

With:
\[ \vec{B} = \mu_0 (\vec{H} + \vec{M}) \]
\[ \nabla \cdot \vec{B} = \mu_0 (\nabla \cdot \vec{H} + \nabla \cdot \vec{M}) \]
\[ \nabla \cdot \vec{H} = -\nabla \cdot \vec{M} = \rho_m \quad [\text{A/m}^2] \]

Magnetic charge density
i.e. magnetic charge/volume
Boundary Conditions

The perpendicular component of $B$ is continuous across a boundary.

The transverse component of $H$ is continuous across a boundary (unless there is a true current on the boundary).
Example: NdFeB Cylinder

\[\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})\]
Demagnetizing Fields

\[ H_d \propto M \]

\[ H_d = -NM \]

Demagnetizing Factor
Demagnetizing Factors

Demag. fields in ellipsoids of revolution are uniform.

$$H_d = -NM$$

$$N_x + N_y + N_z = 1$$
Demagnetizing Factors: Special Cases

\[ H_d = -NM \]

\[ N_x + N_y + N_z = 1 \]

**Sphere**

\[ N_x = N_y = N_z = \frac{1}{3} \]

**Long Rod**

\[ N_x = N_y = \frac{1}{2} \]

\[ N_z = 0 \]

**Thin Sheet**

\[ N_x = N_y = 0 \]

\[ N_z = 1 \]
Example: Ba-Ferrite Sphere

Barrium Ferrite:

\( M \approx 150 \) [kA/m]

\[
N_x = N_y = N_z = \frac{1}{3}
\]

\[
H_d = -NM
\]

\[
H_d = -50 \text{ [kA/m]}
\]

\[
\vec{B} = \mu_0(\vec{H} + \vec{M})
\]

\[
\vec{B} = \mu_0(100 \text{ [kA/m]})
\]

\[
\vec{B} = 0.126 \text{ [Tesla]}
\]
Example: Ba-Ferrite Ellipsoid

Barium Ferrite:

\[ M \approx 150 \text{ [kA/m]} \]

\[
\begin{align*}
N_x &= N_y = 0.45 \\
H_d &= -NM \\
H_d &= -67.5 \text{ [kA/m]} \\
\vec{B} &= \mu_0 (\vec{H} + \vec{M}) \\
\vec{B} &= \mu_0 (82.5 \text{ [kA/m]}) \\
\vec{B} &= 0.104 \text{ [Tesla]}
\end{align*}
\]
Example: Ba-Ferrite Ellipsoid

Barrium Ferrite:

\( M \sim 150 \) [kA/m]

\[ N_z = 0.1 \]

\[ H_d = -NM \]

\[ H_d = -15 \] [kA/m]

\[ \vec{B} = \mu_0 (\vec{H} + \vec{M}) \]

\[ \vec{B} = \mu_0 (135 \text{ [kA/m]}) \]

\[ \vec{B} = 0.17 \text{ [Tesla]} \]
Susceptibility and Permeability

Ideal permanent magnet

Real, useful magnetic material

Ideal, linear soft magnetic material

\[ M = \chi H \]

Susceptibility [unitless]

\[ \vec{B} = \mu_0(\vec{H} + \chi \vec{H}) \]

\[ \vec{B} = \mu_0(1 + \chi)\vec{H} \]

\[ \vec{B} = \mu_0 \mu_r \vec{H} \]

Rel. Permeability [unitless]
Example, Long Rod in a Long Solenoid

\[ I = 1 \text{A} \]

\[ N/L = 1000/m \]

\[ \mu_r = 1000 \quad \chi = 999 \]

\[ H = \frac{NI}{L} = 1 \text{[kA/m]} \]

\[ M = \chi H = \chi \frac{NI}{L} = 999 \text{[kA/m]} \]

\[ B = \mu_0 (H + M) = \mu_0 (1000 \text{kA/m}) = 1.25 \text{[Tesla]} \]

or:

\[ B = \mu_0 \mu_r H = 1.25 \text{[Tesla]} \]
Example, Soft Magnetic Ball in a Field

\[ M = \chi H = \chi (H_{\text{ext}} + H_d) = \chi (H_{\text{ext}} - NM) \]

\[ M = \frac{\chi}{1 + N\chi} H_{\text{ext}} \]

\[ \chi_{\text{eff}} = \frac{\chi}{1 + N\chi} < 3 \]
Example, Thin Film in a Field

\[ M = \chi H_{\text{inside}} = \chi (H_{\text{ext}} + H_d) = \chi (H_{\text{ext}} - M) \]

\[ M = \frac{\chi}{1 + \chi} H_{\text{ext}} \]

\[ H_{\text{inside}} = \frac{1}{1 + \chi} H_{\text{ext}} \]

\[ B_{\text{inside}} = B_{\text{ext}} \]

Why?
Magnetic Energy and Shape Anisotropy

\[ E = \iiint_{vol} \mu_0 \vec{M} \cdot \vec{H} \, dv \]

**Higher energy**  
**Lower energy**

“Shape Anisotropy”

\[ E = \frac{1}{2} \mu_0 M_s^2 (N_x - N_y) \sin^2(\theta) \]
Coffee Break!