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Magnetization Dynamics

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Part I - Concepts and methods

- Competing energy contributions: micromagnetics
- Characteristic parameters: small <=> large, soft <=> hard
- Magnetization precession at constant energy
- Landau-Lifshitz-Gilbert equation

Part II - Mechanisms and examples

- Magnetization switching
- Ferromagnetic resonance
- Spin-transfer-driven magnetization dynamics
- Non-uniform magnetization configurations
- Thermal fluctuations

Bibliography (textbooks only)

- D.D. Stancil and A. Prabakhar, *Spin Waves*, Springer, New York, 2009.
- H. Suhl, *Relaxation Processes in Micromagnetics*, Oxford University Press, Oxford, 2007.
- A. G. Gurevich and G.A. Melkov, *Magnetization Oscillations and Waves*, CRC Press, Boca Raton, 1996. Extensive discussion of microwave phenomena and devices, yet through a somewhat clumsy presentation of the material.
- G. Bertotti, I.D. Mayergoyz, and C. Serpico, *Nonlinear Magnetization Dynamics in Nanosystems*, Springer, Oxford, 2009. Advanced treatment of nonlinear magnetization dynamics, mainly for the case of uniformly magnetized nanomagnets, and of spin-wave instabilities.
- A.H. Morrish, *The Physical Principles of Magnetism*, IEEE Press, Piscataway, 2001. Classical text with a lot of information about resonance phenomena (nuclear, paramagnetic, ferromagnetic).
- R.C. O'Handley, *Modern Magnetic Materials*, Wiley, New York, 2000. Comprehensive text with extensive information on behavior of various magnetic materials.
- A. Hubert and R. Schaefer, *Magnetic Domains*, Springer, Berlin, 1998. Text with useful chapters on micromagnetics, with discussion of the origin of magnetic domains. In addition, exceptional magnetic domain images.
- G. Bertotti, *Hysteresis in Magnetism*, Academic Press, San Diego, 1998. Pedagogical presentation of micromagnetics.

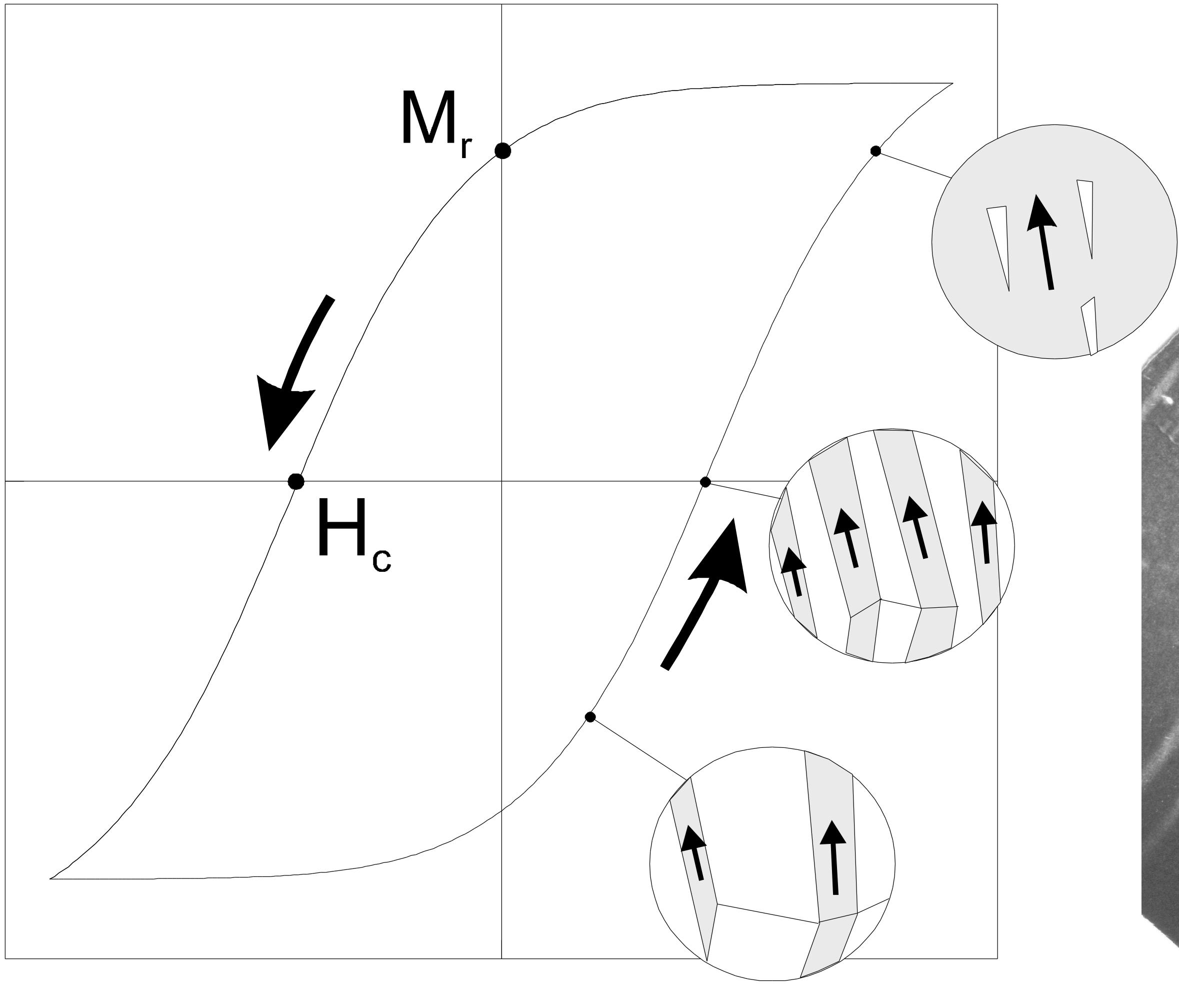
Part I - Concepts and methods

- Competing energy contributions: micromagnetics
- Characteristic parameters: small <=> large, soft <=> hard
- Magnetization precession at constant energy
- Landau-Lifshitz-Gilbert (LLG) equation

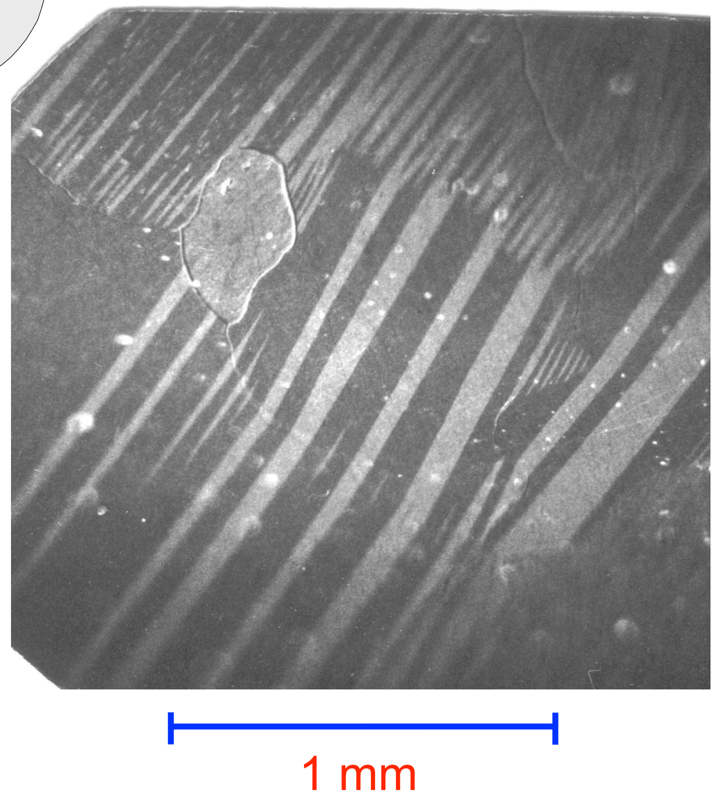
The micromagnetic approach

- Main contributions to the free energy of a ferromagnet:
    - exchange energy
    - magnetostatic energy
    - magnetocrystalline anisotropy energy
    - interaction with external field
- It is the competition between these energies that gives rise to magnetic domains and is eventually responsible for the hysteresis and switching phenomena observed in particles, films, etc.
- In micromagnetics, one is given the energy G_L of the ferromagnet, defined with respect to certain configurational coordinates X (both G_L and X will have to be defined in precise terms); then one looks for the set of local minima, characterized by $\partial G_L / \partial X = 0$ and $\partial^2 G_L / \partial X^2 > 0$, that represent possible metastable states for the system.
- The key complication is that X is not just a number, but represents the full magnetization vector field $\mathbf{M}(\mathbf{r})$ defined over the entire body volume.
- Thus, energy minimization has to be carried out in the infinite-dimensional functional space of all possible magnetization configurations (variational problem).
- The equations that express the condition of energy minimum for a given magnetization configuration are known as Brown's equations.
- This energy minimization program does not say anything about how the system will evolve if initially it is not in equilibrium; the Landau-Lifshitz-Gilbert (LLG) equation provides a suitable dynamic extension of micromagnetics for the description of out-of-equilibrium stuations.

Magnetization processes

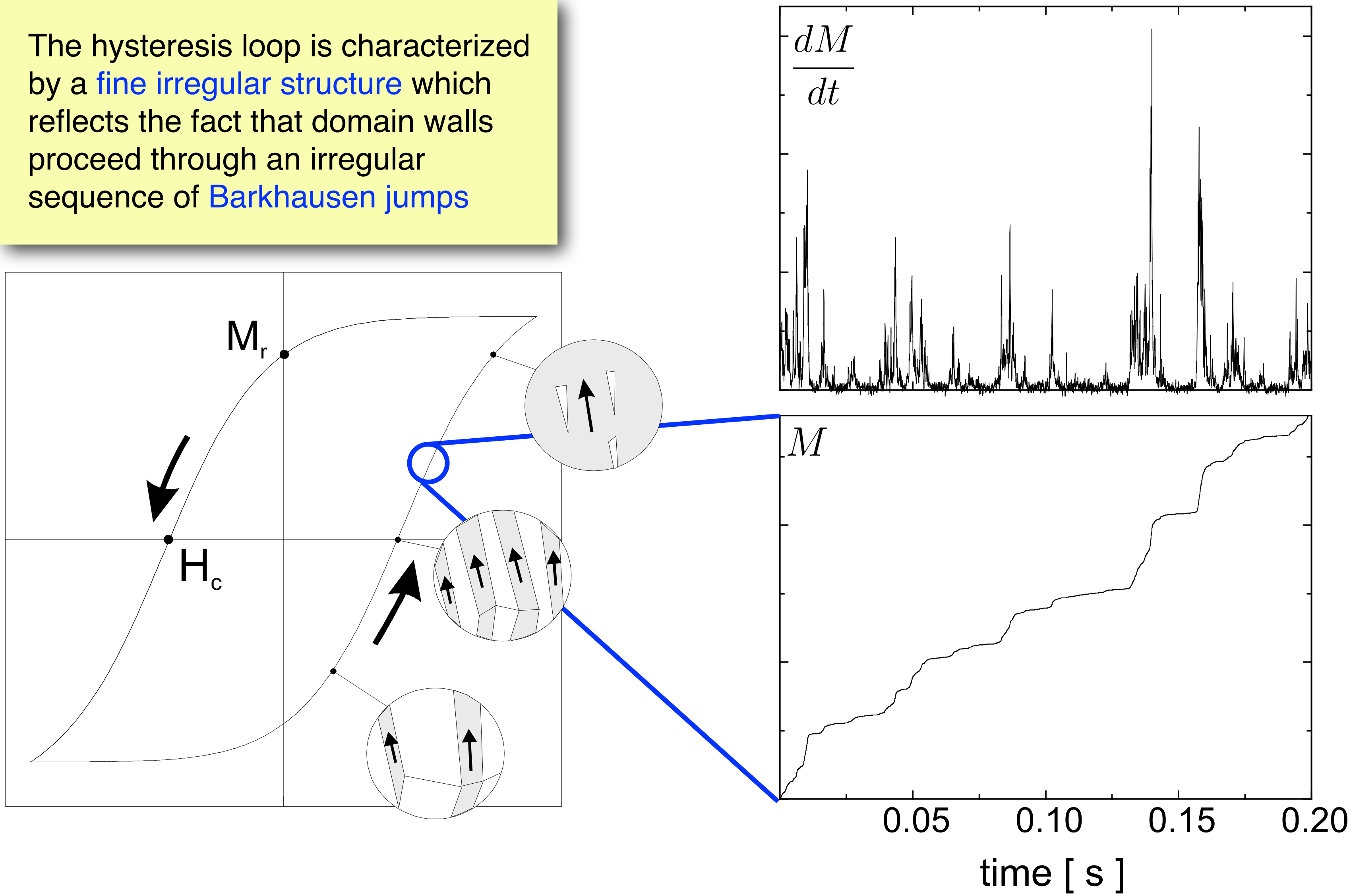


Hysteresis loop corresponds to evolution of magnetic domain structure

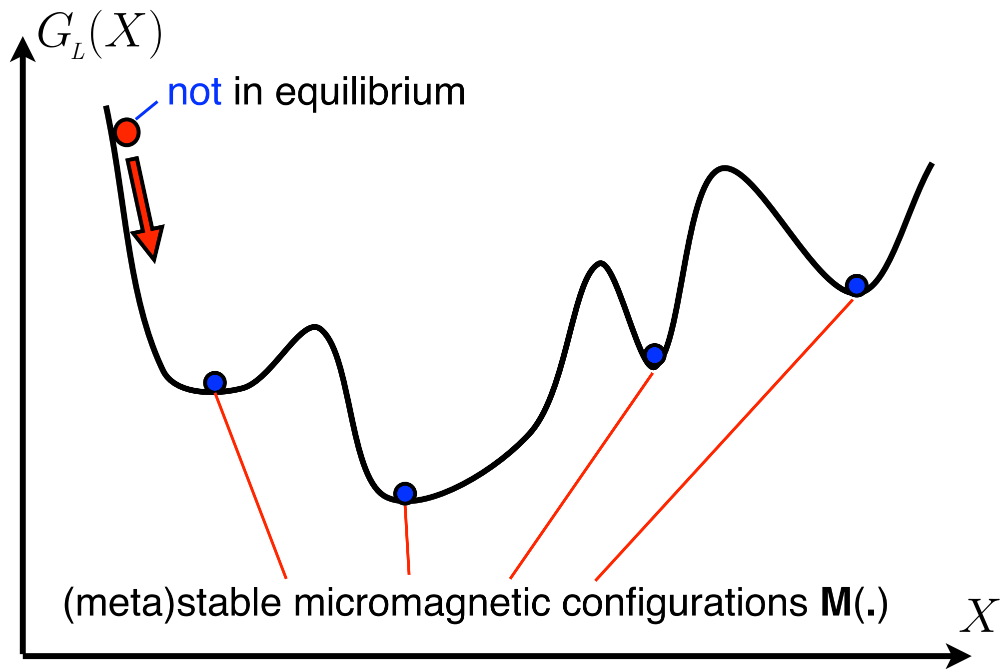


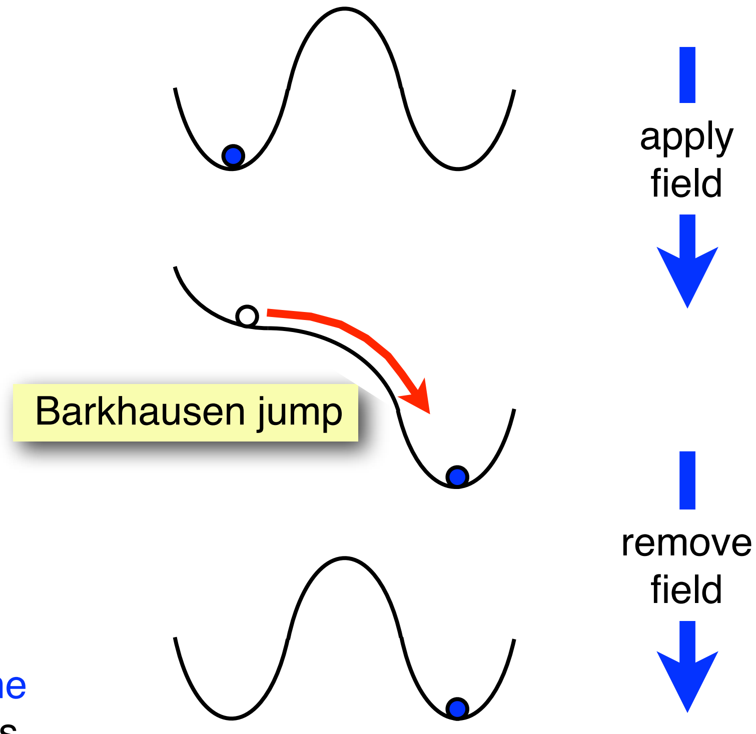
Magnetization processes

The hysteresis loop is characterized by a fine irregular structure which reflects the fact that domain walls proceed through an irregular sequence of Barkhausen jumps



Energy landscapes



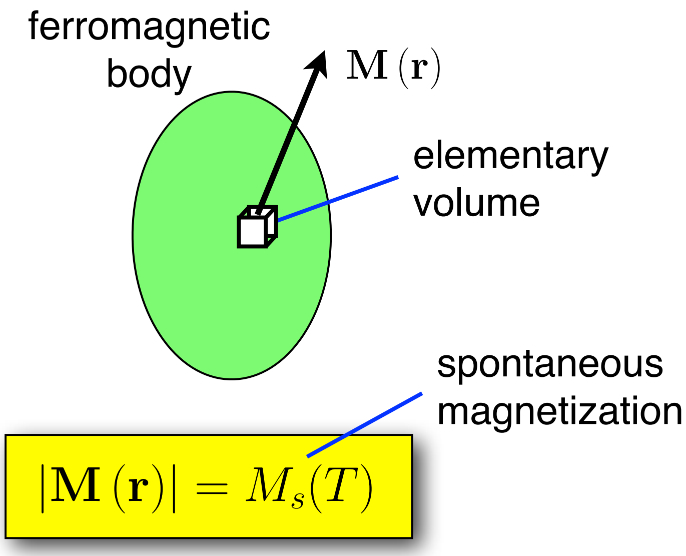


History dependence: the initial and final energy profiles are the same but the state occupied by the system is different depending on past history.

- The use of energy landscapes implies a separation of time scales: the relaxation time after which the system reaches equilibrium with respect to a particular value of X is much shorter than the time over which the system evolves from one value of X to another
- The number of stable magnetization configurations (local energy minima) can be very large due to structural disorder
- The state occupied by the system is history-dependent if the temperature is low enough

Elementary volumes and spontaneous magnetization

In micromagnetics, the ferromagnetic body is treated as a continuous medium with smooth magnetic properties. The smoothness comes from averaging over elementary volumes small enough with respect to the scale over which the magnetization varies significantly, but large enough with respect to atomic distances.



The local magnetization vector **M**(**r**) describes the magnetic state of the given elementary volume. We assume that its magnitude $|\mathbf{M}(\mathbf{r})|^2$ is not affected by external fields (exchange dominates with respect to thermal fluctuations).

The magnetic state of each elementary volume is thus defined by the vector:

$$\mathbf{m}(\mathbf{r}) = \frac{\mathbf{M}(\mathbf{r})}{M_s}$$

constant unit modulus but variable orientation

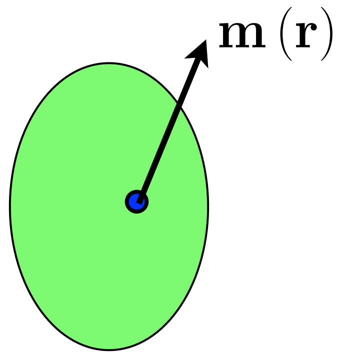
The state of the body is described by the magnetization vector field **m**(.) defined for each point inside the magnet. Although the magnetization magnitude is constant, its orientation can vary from point to point. It is the spatial variation of this orientation that defines the magnetic state of the magnet.

$$X \rightarrow \mathbf{m}(.) \qquad\qquad G_L(X) \rightarrow \int_V \left(f_{EX} + f_{AN} + f_M + \ldots\right) dV$$

Exchange energy

Exchange energy is caused by the fact that whenever there is some misalignment of neighboring magnetic moments, there is an energy cost involved.
The conclusion is evident if we consider for instance the Heisenberg Hamiltonian:

$$\mathcal{H} = -\sum_{<i,j>} J_{ij}\, \mathbf{S}_i \cdot \mathbf{S}_j$$



Body volume is *V*

If $\mathbf{S}_i$ is not parallel to its neighbor $\mathbf{S}_j$, the scalar product decreases and the energy (under positive J_{ij}) increases.

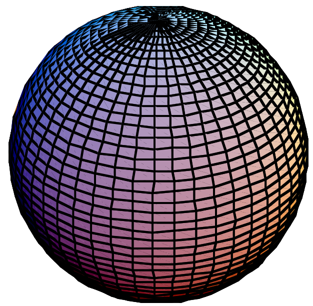
It is this non-uniformity energy that is usually meant when one speaks of exchange energy. in this sense, we have exchange energy only when the gradient of **m** takes non-zero values. If the variation from point to point is not too rapid, we can make a Taylor expansion of the exchange energy as a function of the magnetization gradients, and keep the lowest order terms. The exact form of the expansion will depend on the symmetry of the lattice hosting the magnetic moments. The leading term in the energy density consistent with cubic symmetry is the following:

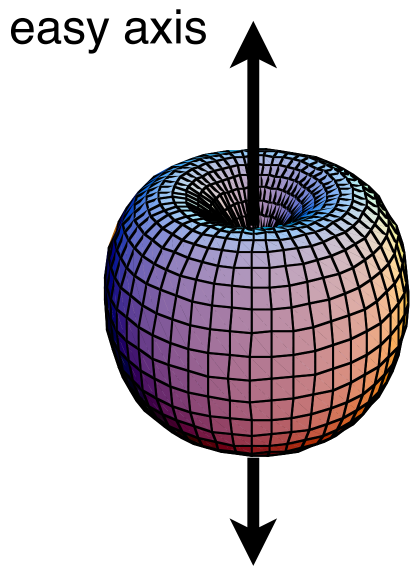
$$F_{EX} = \int_V A\left(|\nabla m_x|^2 + |\nabla m_y|^2 + |\nabla m_x|^2\right)\, dV$$

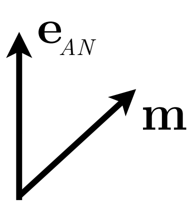
The parameter *A* is the exchange stiffness constant. Its typical value is of the order of 10^{-11} J/m.

Magnetocrystalline anisotropy energy

exchange is isotropic







symmetry is broken:

- uniaxial anisotropy
- cubic anisotropy
- ...

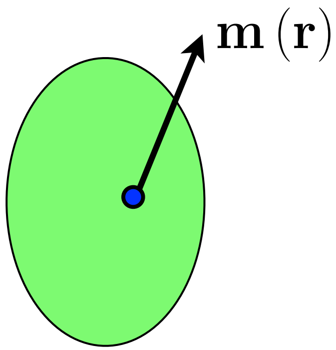
Magnetocrystalline anisotropy energy depends on the relative orientation of the local magnetization with respect to certain preferred directions. In a perfect single crystal these directions will be the same everywhere inside the body. However, in a polycrystal they will vary from point to point. In all cases, anisotropy energy has a purely local character.

this direction can be space-dependent

$$F_{AN} = -\int_V K_1 \left(\mathbf{m} \cdot \mathbf{e}_{AN}\right)^2 \, dV$$

This is the energy expression for the particular case of uniaxial anisotropy. The parameter K_1 is the anisotropy constant.

Magnetostatic energy



Magnetostatic energy is potential energy of magnetic moments in the magnetic field they themselves have created. Magnetostatics permits one to compute this energy if the vector field **m**(.) is known:

$$F_M = -\frac{\mu_0 M_s}{2} \int_V \mathbf{H}_M \cdot \mathbf{m} \, dV$$

Equivalent expression is:

$$F_M = \frac{\mu_0}{2} \int_{\text{all space}} |\mathbf{H}_M|^2 \, dV$$

The magnetostatic field is solution of magnetostatic Maxwell's equations with the usual interface conditions at the surface of the body:

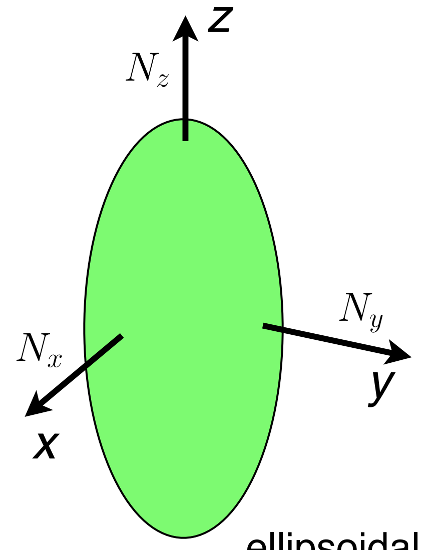
$$\nabla \cdot \mathbf{H}_M = -M_s \nabla \cdot \mathbf{m} \ , \ \nabla \times \mathbf{H}_M = 0$$ inside the magnet

$$\nabla \cdot \mathbf{H}_M = 0 \ , \ \nabla \times \mathbf{H}_M = 0$$ outside

The relation between $\mathbf{H}_M$ and **m** is not local because magnetic charges even far away from a certain point may affect the value of the magnetostatic field at that point. Although the magnetostatic energy is expressed as a volume integral, it is not true that it comes from local contributions as it is the case, for example, for crystal anisotropy energy. It is only after defining the geometry of the problem and after selecting a particular magnetization configuration **m**(.) for the entire magnet that it is possible to calculate the total magnetostatic energy for the magnet.

Shape anisotropy

Magnetostatic energy takes a particularly simple form for uniformly magnetized ellipsoidal bodies. This leads to the notion of shape anisotropy.



Consider an ellipsoidal body with principal axes along *x, y, z,* and corresponding demagnetizing coefficients N_x, N_y, N_z.

Assume that the body is uniformly magnetized, with normalized magnetization **m**. Then the magnetostatic energy is:

$$F_M = \frac{\mu_0 M_s^2 V}{2} \left(N_x m_x^2 + N_y m_y^2 + N_z m_z^2 \right)$$

When the body has spheroidal shape with symmetry axis along z (i.e., $N_x = N_y = N_\perp$), apart from inessential constant terms, the energy can be rewritten in a form identical to that for uniaxial anisotropy:

$$F_M = const - \frac{\mu_0 M_s^2 V}{2} (N_\perp - N_z) \, (\mathbf{m} \cdot \mathbf{e}_z)^2$$

$$\frac{\mu_0 M_s^2}{2} (N_\perp - N_z) \leftrightarrow K_1$$

Shape-anisotropy energy is just magnetostatic energy in a particular case.

Micromagnetic free energy

exchange energy — magnetocrystalline anisotropy — magnetostatic energy — external field interaction

$$G_L\left(\mathbf{m}(.);\mathbf{H}_a,T\right)=\int_V\left(A\left(\nabla\mathbf{m}\right)^2-K_1\left(\mathbf{m}\cdot\mathbf{e}_{AN}\right)^2-\frac{\mu_0 M_s}{2}\mathbf{H}_M\cdot\mathbf{m}-\mu_0 M_s\mathbf{H}_a\cdot\mathbf{m}\right)dV$$

exchange stiffness constant ~ 10^{-11} J/m

$$(\nabla\mathbf{m})^2=|\nabla m_x|^2+|\nabla m_y|^2+|\nabla m_z|^2$$

- The vector **m**(**r**) represents the normalized magnetization, measured in units of the spontaneous magnetization; it is characterized by unit magnitude everywhere:

$$|\mathbf{m}|^2=1 \qquad \mathbf{m}=\frac{\mathbf{M}}{M_s}$$

M_s: spontaneous magnetization

$$|\mathbf{M}(\mathbf{r})|=M_s(T)$$

- The magnetostatic field $\mathbf{H}_M$ is the solution of magnetostatic Maxwell equations under given **m**(.):

$$\nabla\cdot\mathbf{H}_M=-M_s\nabla\cdot\mathbf{m}\ ,\quad \nabla\times\mathbf{H}_M=0 \qquad \text{inside the magnet}$$

$$\nabla\cdot\mathbf{H}_M=0\ ,\quad \nabla\times\mathbf{H}_M=0 \qquad \text{outside}$$

- The energy G_L is not expressible in terms of a purely local energy density, because the magnetostatic field is known only after the magnetization configuration is specified for the entire body and magnetostatic Maxwell equations are solved.

Energy minimization

If the system energy is at a minimum when the magnetization is **m**(.), then, when **m**(.) is varied by the small amount $\mathbf{m}(\mathbf{r}) \to \mathbf{m}(\mathbf{r}) + \delta\mathbf{m}(\mathbf{r})$, the corresponding energy variation δG_L is such that $\delta G_L = 0$ to the first order in $\delta\mathbf{m}$ and $\delta G_L > 0$ to the second order in $\delta\mathbf{m}$ for every arbitrary variation $\delta\mathbf{m}$ that preserves the magnetization modulus, i.e., of the form $\delta\mathbf{m}(\mathbf{r}) = \mathbf{m}(\mathbf{r}) \times \delta\mathbf{v}(\mathbf{r})$, where $\delta\mathbf{v}(\mathbf{r})$ is a small arbitrary vector.

$$G_L(\mathbf{m}(.); \mathbf{H}_a, T) = \int_V \left(A(\nabla\mathbf{m})^2 + f_{AN}(\mathbf{m}) - \frac{\mu_0 M_s}{2} \mathbf{H}_M \cdot \mathbf{m} - \mu_0 M_s \mathbf{H}_a \cdot \mathbf{m} \right) dV$$

effective field

$$\delta G_L = -\mu_0 M_s \int_V (\mathbf{H}_{\text{eff}} \cdot \delta\mathbf{m})\, dV + 2A \oint_S \left(\frac{\partial \mathbf{m}}{\partial n} \cdot \delta\mathbf{m} \right) dS$$

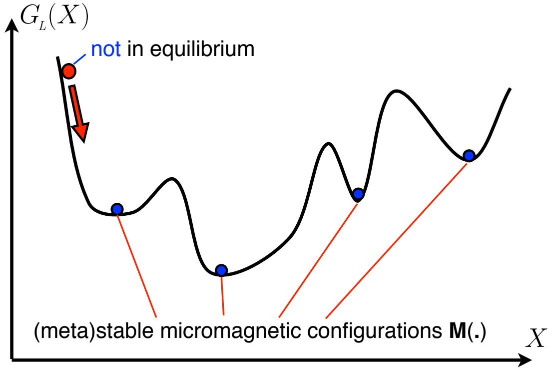
$$\mathbf{H}_{\text{eff}} = \frac{2A}{\mu_0 M_s} \nabla^2 \mathbf{m} - \frac{1}{\mu_0 M_s} \frac{\partial f_{AN}}{\partial \mathbf{m}} + \mathbf{H}_M + \mathbf{H}_{\text{a}}$$

exchange field; anisotropy field; magnetostatic field; external field

effective field

$$\delta\mathbf{m}(\mathbf{r}) = \mathbf{m}(\mathbf{r}) \times \delta\mathbf{v}(\mathbf{r})$$

$$\delta G_L = \mu_0 M_s \int_V (\mathbf{m} \times \mathbf{H}_{\text{eff}}) \cdot \delta\mathbf{v}\, dV + 2A \oint_S \left(\frac{\partial \mathbf{m}}{\partial n} \times \mathbf{m} \right) \cdot \delta\mathbf{v}\, dS$$



Brown's equations

minimization of micromagnetic energy (variational problem)

this is equivalent to zero surface anisotropy

zero magnetic torque everywhere inside the magnet

$$\mathbf{m} \times \mathbf{H}_{\text{eff}} = 0$$

$$\partial \mathbf{m} / \partial n = 0$$

zero normal derivative everywhere on the boundary of the magnet

$$\mathbf{m}_{AN} = (\mathbf{m} \cdot \mathbf{e}_{AN})\ \mathbf{e}_{AN}$$

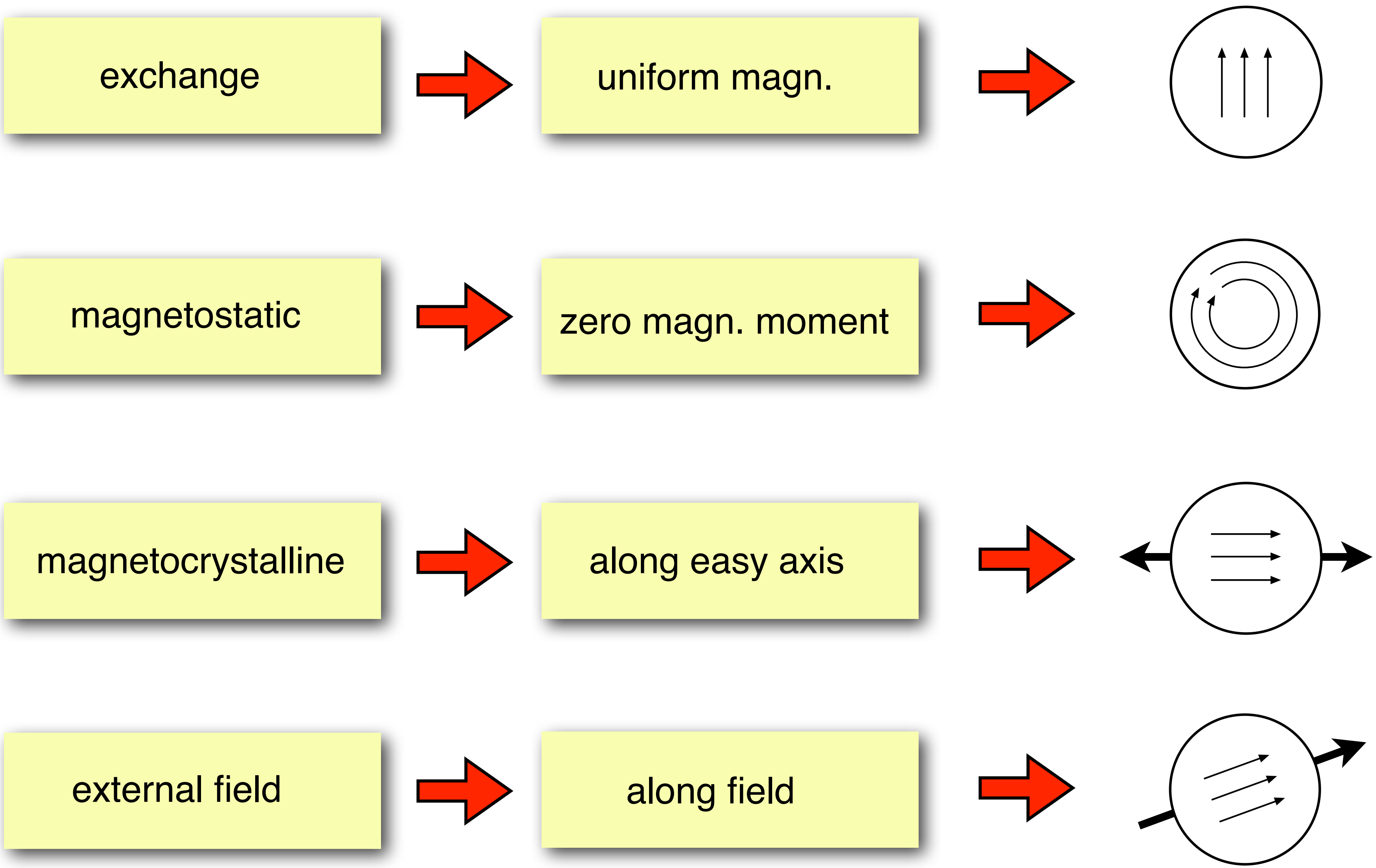
$$\mathbf{H}_{\text{eff}} = \frac{2A}{\mu_0 M_s} \nabla^2 \mathbf{m} + \frac{2K_1}{\mu_0 M_s} \mathbf{m}_{AN} + \mathbf{H}_M + \mathbf{H}_{\text{a}}$$

effective field

exchange field — anisotropy field — magnetostatic field — external field

- The effective field contains a Maxwellian part (applied and magnetostatic fields) and a non-Maxwellian part (exchange and anisotropy fields). They coexist and play identical roles.
- According to Brown's equations, at equilibrium there must be complete absence of internal magnetic torques in the magnet.
- Brown's equations are nonlinear because the effective field is itself a function of **m**.
- The component of the effective field along **m** plays no role as a consequence of the fact that the magnetization magnitude cannot change.

Competing energies



Characteristic parameters

- Micromagnetics is governed by a small number of fundamental parameters, which emerge once the micromagnetic energy is written in dimensionless form, by measuring energies in units of $\mu_0 M_s^2 V$ and fields in units of M_s :

$$g_{_L} = \frac{G_{_L}}{\mu_0 M_s^2 V} \quad , \quad \mathbf{h}_\mathrm{a} = \frac{\mathbf{H}_a}{M_s} \quad , \quad \mathbf{h}_{_M} = \frac{\mathbf{H}_{_M}}{M_s}$$

- The resulting dimensionless energy is (we consider for simplicity the case of uniaxial anisotropy):

$$g_{_L}\left(\mathbf{m}(.);\mathbf{h}_a,T\right) = \int_V \left(l_{_{EX}}^2 \frac{(\nabla \mathbf{m})^2}{2} - \kappa \frac{(\mathbf{m}\cdot\mathbf{e}_{_{AN}})^2}{2} - \frac{1}{2}\mathbf{h}_{_M}\cdot\mathbf{m} - \mathbf{h}_a\cdot\mathbf{m} \right) \frac{dV}{V}$$

exchange length ($l_{_{EX}}^2$)

hardness (or quality) factor (dimensionless) (κ)

$$l_{_{EX}} = \sqrt{\frac{2A}{\mu_0 M_s^2}}$$

The exchange length permits one to define what is large and what is small in micromagnetics.

$$\kappa = \frac{2K_1}{\mu_0 M_s^2}$$

The hardness parameter permits one to introduce the notion of magnetically soft and magnetically hard material.

Characteristic parameters

- The hardness parameter permits one to introduce the notion of soft versus hard material:

$$\kappa = \frac{2K_1}{\mu_0 M_s^2}$$

$\kappa \ll 1$ soft material

$\kappa \simeq 1$ hard material

For iron, where $K_1 \simeq 5 \cdot 10^4$ J/m³ and $\mu_0 M_s \simeq 2$ T , one finds $\kappa \simeq 0.03$.

- There are three characteristic lengths in micromagnetics corresponding to different combinations of the exchange length and the hardness parameter:

$$l_{EX} = \sqrt{\frac{2A}{\mu_0 M_s^2}}$$

$$l_W = \frac{l_{EX}}{\sqrt{\kappa}} = \sqrt{\frac{A}{K_1}}$$

The following order-of-magnitude estimate gives an idea of the typical values involved:

$A \simeq 10^{-11}$ J/m

$K_1 \simeq 10^4$ J/m³

$\mu_0 M_s \simeq 1$ T

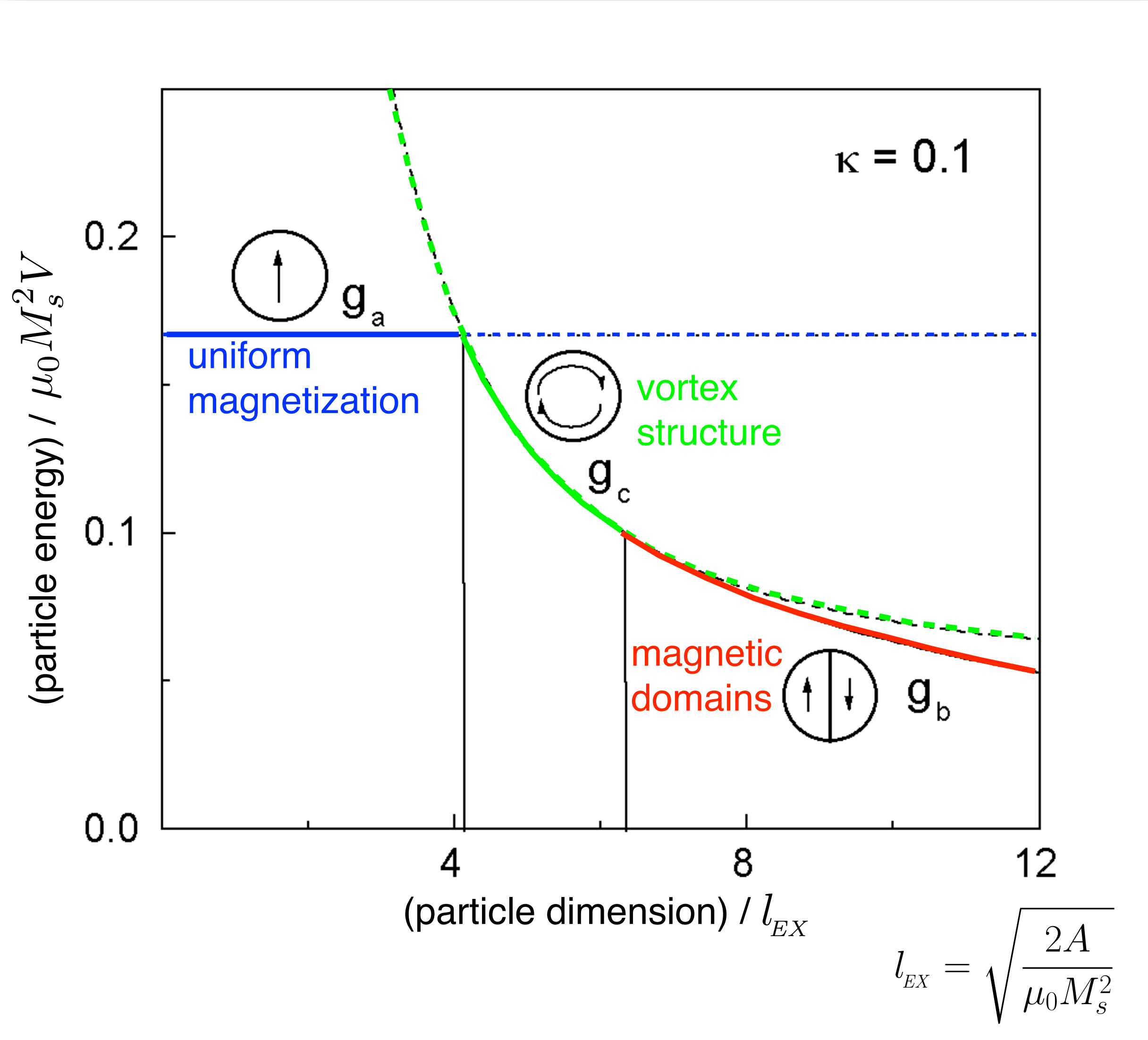
→

$l_{EX} \simeq 5$ nm

$l_W \simeq 60$ nm

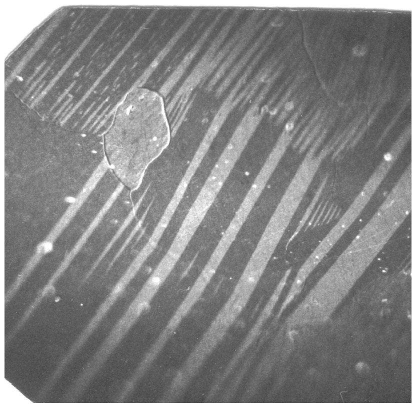
[there is also: $l_D = l_{EX}\sqrt{\kappa} = \dfrac{2\sqrt{AK_1}}{\mu_0 M_s^2}$, but this length is less important in present context]

Magnetic configurations in soft materials



magnetostatic energy is a volume effect (of the order of L^3) whereas domain wall energy is a surface effect (of the order of L^2)

large magnets develop magnetic domains



Moving to magnetization dynamics

- An isolated magnetic moment μ precesses around an external magnetic field $\mathbf{H}_a$ according to the equation:

$$\frac{d\boldsymbol{\mu}}{dt} = -\gamma\,\boldsymbol{\mu} \times \mathbf{H}_a$$

γ represents the absolute value of the gyromagnetic ratio. When the magnetic moment is due to the electron spin: $\gamma \simeq 2.2 \cdot 10^5$ m A^{-1} s^{-1}.

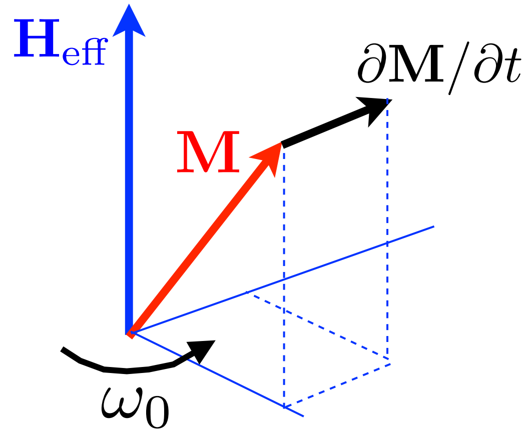
- In micromagnetics, this equation is generalized in several respects:
  - the magnetization $\mathbf{M}(\mathbf{r})$ takes the place of the individual magnetic moment;
  - interactions inside the medium are taken into account by the micromagnetic effective field $\mathbf{H}_{eff}$, which takes the place of the external magnetic field;

$$\frac{\partial \mathbf{M}}{\partial t} = -\gamma\,\mathbf{M} \times \mathbf{H}_{\text{eff}}$$

- Relaxation toward equilibrium is described by additional phenomenological damping terms, to be discussed shortly; the result is the so-called Landau-Lifshitz-Gilbert equation.

Magnetization precession

$$\frac{\partial \mathbf{M}}{\partial t} = -\gamma\, \mathbf{M} \times \mathbf{H}_{\mathrm{eff}}$$



$$\mathbf{m}_{AN} = (\mathbf{m} \cdot \mathbf{e}_{AN})\, \mathbf{e}_{AN}$$

$$\mathbf{H}_{\mathrm{eff}} = \frac{2A}{\mu_0 M_s} \nabla^2 \mathbf{m} + \frac{2K_1}{\mu_0 M_s} \mathbf{m}_{AN} + \mathbf{H}_{M} + \mathbf{H}_{\mathrm{a}}$$

- Magnetization precesses around the effective field, but the effective field is not constant, because it depends on magnetization
- The result can be more or less difficult to study, depending on the nature of the dependence of the effective field on magnetization
- The simplest situation is when the effective field reduces to the externally applied magnetic field only (i.e.: uniform magnetization, no surface anisotropy, no crystal anisotropy, spherical shape):

$$\mathbf{H}_{\mathrm{eff}} = \mathbf{H}_{\mathrm{a}} = H_a \mathbf{e}_z \;\Rightarrow\; \frac{d\mathbf{M}}{dt} = -\gamma H_a\, \mathbf{M} \times \mathbf{e}_z$$

counterclockwise precession at constant angular frequency ω_0 around the z axis

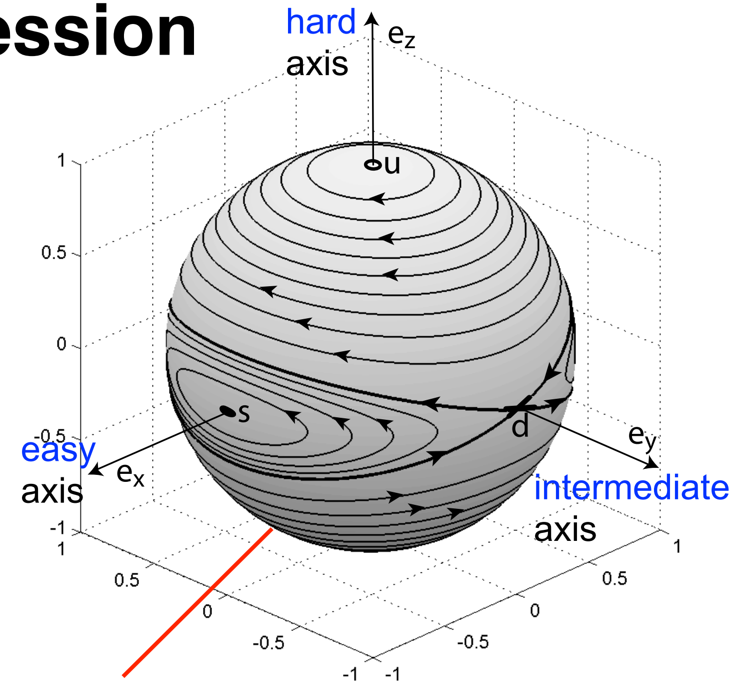
$$\omega_0 = \gamma H_a$$

$$\gamma \simeq 2.2 \cdot 10^5\, \mathrm{mA^{-1}s^{-1}} \;\Rightarrow\; \gamma/\mu_0 \simeq 176\, \mathrm{GHz/T}$$

$$\mu_0 H_a \simeq 1\, \mathrm{T} \;\Rightarrow\; f = \omega/2\pi \simeq 28\, \mathrm{GHz}$$

Magnetization precession

$$\frac{\partial \mathbf{M}}{\partial t} = -\gamma\, \mathbf{M} \times \mathbf{H}_{\text{eff}}$$



example of energy level curves under:
(i) uniform magnetization;
(ii) ellipsoidal anisotropy;
(iii) no external magnetic field.

- Magnetization precesses around the effective field, but the effective field is not constant, because it depends on magnetization; the result will reflect the nature of the dependence of the effective field on magnetization

- However, the following general laws control the qualitative properties of magnetization precession:

  - magnetization magnitude is preserved, because $2\mathbf{M} \cdot \partial \mathbf{M}/\partial t \equiv \partial |\mathbf{M}|^2/\partial t = 0$

  - energy is conserved if the external field is constant in time, because:

this expression is valid under constant external field $\mathbf{H}_a$

$$\delta G_L = -\mu_0 \int_V (\mathbf{H}_{\text{eff}} \cdot \delta \mathbf{M})\, dV \quad \Rightarrow \quad \frac{dG_L}{dt} = -\mu_0 \int_V \left(\mathbf{H}_{\text{eff}} \cdot \frac{\partial \mathbf{M}}{\partial t} \right) dV$$

$$\frac{\partial \mathbf{M}}{\partial t} = -\gamma\, \mathbf{M} \times \mathbf{H}_{\text{eff}} \quad \Rightarrow \quad \frac{dG_L}{dt} = 0$$

energy is conserved during magnetization precession

Characteristic length, time, and field scales

- Equation for magnetization precession:

$$\frac{\partial \mathbf{M}}{\partial t} = -\gamma\, \mathbf{M} \times \mathbf{H}_{\text{eff}}$$

$$\mathbf{m}_{AN} = (\mathbf{m} \cdot \mathbf{e}_{AN})\, \mathbf{e}_{AN}$$

$$\mathbf{H}_{\text{eff}} = \frac{2A}{\mu_0 M_s} \nabla^2 \mathbf{m} + \frac{2K_1}{\mu_0 M_s} \mathbf{m}_{AN} + \mathbf{H}_M + \mathbf{H}_{\text{a}}$$

- The above equation is nonlinear, of partial differential (due to exchange) as well as integral (due to magnetostatic interactions) character.

- In the problem, there exist a characteristic field scale, given by the saturation magnetization M_s (a typical value is $\mu_0 M_s \sim 1$ T, i.e., $M_s \sim 10^6$ A/m) and a characteristic time scale, given by $(\gamma\, M_s)^{-1}$ ($(\gamma M_s)^{-1} \sim 6$ ps when $\mu_0 M_s \sim 1$ T)

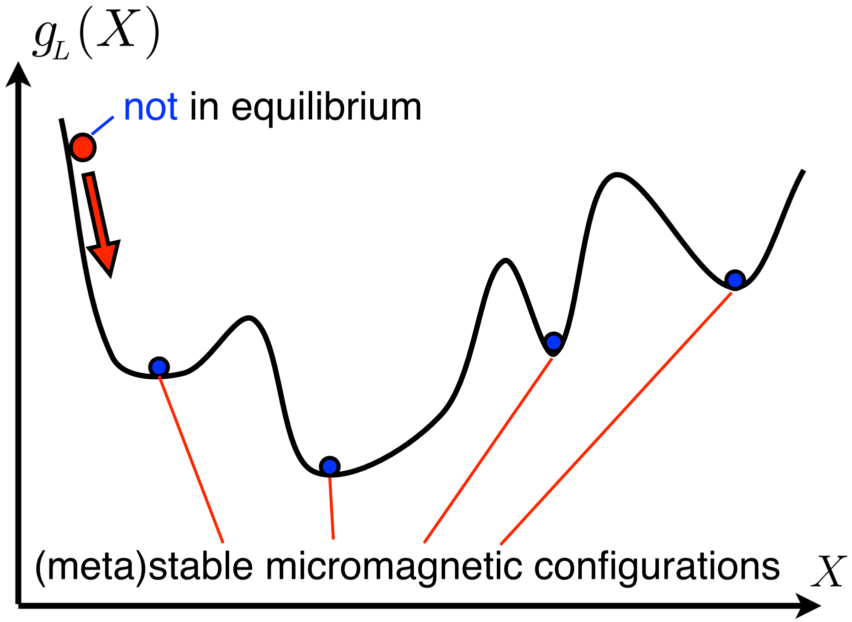
- By measuring time, magnetization, and fields in these units one obtains the dimensionless equation:

$$\frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m} \times \mathbf{h}_{\text{eff}}$$

$\mathbf{m} = \mathbf{M}/M_s$, $\mathbf{h}_{\text{eff}} = \mathbf{H}_{\text{eff}}/M_s$, time is measured in units of $(\gamma\, M_s)^{-1}$.

- There is also a characteristic energy scale, defined by the characteristic energy $\mu_0 M_s^2 V$, where *V* is the volume of the magnet. One can thus define a dimensionless energy $g_L = G_L / \mu_0 M_s^2 V$.

Relaxation toward equilibrium



g_L is a thermodynamic potential with the property that it is a decreasing function of time for any transformation taking place under constant external field $\mathbf{h}_a$ and temperature T

$$\frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m} \times \mathbf{h}_{\text{eff}} \quad \Rightarrow \quad \frac{dg_L}{dt} = 0$$

the system never relaxes toward equilibrium ! something is missing

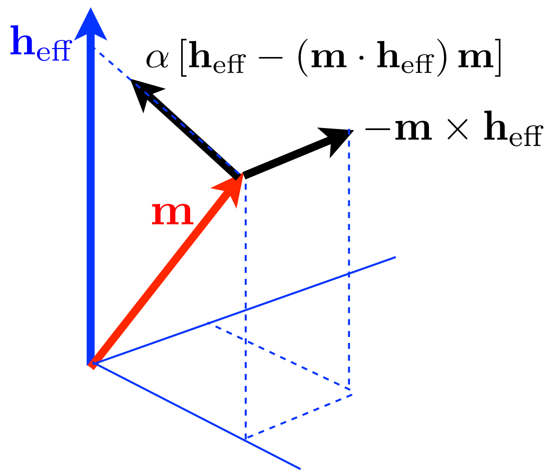
$$\frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m} \times \mathbf{h}_{\text{eff}} + (\text{dissipation}) + (\text{fluctuations})$$

constant energy — decreasing energy — random fluctuations

Landau-Lifshitz equation

- Energy relaxation mechanisms can be taken into account by suitable phenomenological terms. Relaxation must favor the progressive alignment of the magnetization to the effective field. In their 1935 paper, Landau and Lifshitz introduced a contribution to the magnetization rate of change proportional to the effective field component perpendicular to the magnetization:

$$\frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m} \times \mathbf{h}_{\text{eff}} + \alpha \left[\mathbf{h}_{\text{eff}} - (\mathbf{m} \cdot \mathbf{h}_{\text{eff}})\, \mathbf{m}\right]$$



- This equation can be written in the equivalent form:

$$\frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m} \times \mathbf{h}_{\text{eff}} - \alpha\, \mathbf{m} \times (\mathbf{m} \times \mathbf{h}_{\text{eff}})$$

this is the form in which the equation is most often written nowadays.

- The dimensionless parameter α measures the importance of damping effects. Usually $\alpha \ll 1$.

- Writing the equation in this form makes it evident that $\mathbf{m} \cdot \partial \mathbf{m}/\partial t = 0$ under all circumstances. This means that the micromagnetic condition of constant magnetization magnitude is preserved by the dynamics.

- The equation is consistent with Brown's equations, because $\partial \mathbf{m}/\partial t = 0$ when $\mathbf{m} \times \mathbf{h}_{\text{eff}} = 0$.

Gilbert form

- If one heuristically thinks of the effective field as the driving force and the magnetization rate as the velocity, one is led to consider the typical viscous relaxation law:

$$\mathbf{h}_{\text{eff}} - \alpha \frac{\partial \mathbf{m}}{\partial t} = 0$$

- This simple law is not completely satisfactory because it affects also the magnetization magnitude. Since we know that the magnetization modulus will stay constant irrespective of the forces acting on the system, we should restrict the validity of the relaxation law to the component that is perpendicular to the magnetization:

$$\mathbf{m} \times \left(\mathbf{h}_{\text{eff}} - \alpha \frac{\partial \mathbf{m}}{\partial t} \right) = 0$$

- In addition, we need to modify this law in order to make it consistent with the precessional law, $\partial \mathbf{m}/\partial t = -\mathbf{m} \times \mathbf{h}_{\text{eff}}$, that should be recovered in the limit of no relaxation. This suggest:

$$\frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m} \times \left(\mathbf{h}_{\text{eff}} - \alpha \frac{\partial \mathbf{m}}{\partial t} \right)$$

or equivalently:

$$\frac{\partial \mathbf{m}}{\partial t} - \alpha\, \mathbf{m} \times \frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m} \times \mathbf{h}_{\text{eff}}$$

This is the Gilbert form of the Landau-Lifshitz equation (Landau-Lifshitz-Gilbert (LLG) equation)

Equivalence of Landau-Lifshitz and Gilbert forms

- If one takes the vector product **m** x ... of both members of the equation:

$$\frac{\partial \mathbf{m}}{\partial t} - \alpha\, \mathbf{m} \times \frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m} \times \mathbf{h}_{\mathrm{eff}}$$

and one combines the resulting equation with the original equation, one obtains the following result:

$$\left(1 + \alpha^2\right) \frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m} \times \mathbf{h}_{\mathrm{eff}} - \alpha\, \mathbf{m} \times \left(\mathbf{m} \times \mathbf{h}_{\mathrm{eff}}\right)$$

which coincides with the Landau-Lifshitz equation apart from a renormalization of the time unit. In this sense the Landau-Lifshitz and Gilbert forms of the dynamic equation are mathematically equivalent.

- It should be noted that there are no strict reasons why the damping parameter α should be a simple constant. In general it may be expected to be a function of the dynamic state of the system.

Energy relations

- Rate of change of the system energy:

$$\frac{dg_L}{dt} = -\int_V \left(\mathbf{h}_{\text{eff}} \cdot \frac{\partial \mathbf{m}}{\partial t}\right) \frac{dV}{V} + \frac{\partial g_L}{\partial t}$$

this term is present if energy explicitly depends on time (e.g., if the external magnetic field is time-dependent)

- From the dynamic equation expressed in Gilbert form, one finds:

$$\mathbf{h}_{\text{eff}} \cdot \frac{\partial \mathbf{m}}{\partial t} = \alpha\, \mathbf{h}_{\text{eff}} \cdot \left(\mathbf{m} \times \frac{\partial \mathbf{m}}{\partial t}\right) = \alpha \frac{\partial \mathbf{m}}{\partial t} \cdot \left(\frac{\partial \mathbf{m}}{\partial t} - \alpha\, \mathbf{m} \times \frac{\partial \mathbf{m}}{\partial t}\right) = \alpha \left|\frac{\partial \mathbf{m}}{\partial t}\right|^2$$

- In the case when the energy explicitly depends on time only through the external magnetic field, one finally obtains:

$$\frac{dg_L}{dt} = -\int_V \alpha \left|\frac{\partial \mathbf{m}}{\partial t}\right|^2 \frac{dV}{V} - \langle \mathbf{m} \rangle \cdot \frac{d\mathbf{h}_{\text{a}}}{dt}$$

$$\langle \mathbf{m} \rangle = \frac{1}{V} \int_V \mathbf{m}\, dV$$

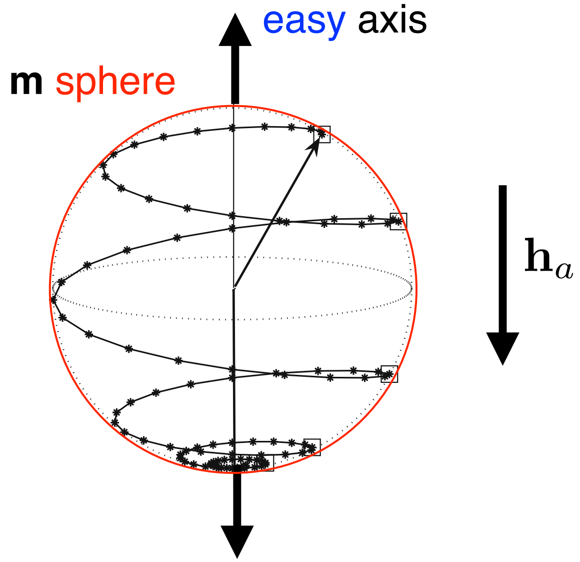
assuming that the field is uniform in space

- Under constant field the energy can only decrease. Consequently, the only admissible processes are those of relaxation toward micromagnetic configurations corresponding to local energy minima.

LLG dynamics in uniformly magnetized nanomagnets

- The state of a uniformly magnetized nanomagnet is described by a single vector **m**; magnetization dynamics take place on the surface of the unit sphere $|\mathbf{m}|^2 = 1$

- In the case of an ellipsoidal particle with principal axes along x,y,z, and crystal anisotropy characterized by the same symmetry as shape anisotropy, the system energy takes the simple quadratic form:

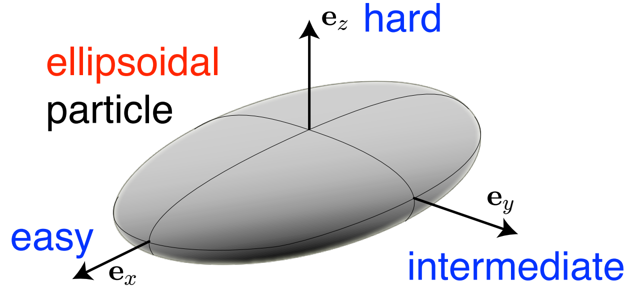


(D_x, D_y, D_z) describe shape + crystal anisotropy   external field

$$g_L(\mathbf{m}; \mathbf{h}_\mathrm{a}) = \frac{1}{2}\left(D_x m_x^2 + D_y m_y^2 + D_z m_z^2\right) - \mathbf{h}_\mathrm{a} \cdot \mathbf{m}$$

$$\mathbf{h}_\mathrm{eff} = -D_x m_x \mathbf{e}_x - D_y m_y \mathbf{e}_y - D_z m_z \mathbf{e}_z + \mathbf{h}_a$$

$$\mathbf{h}_\mathrm{eff} \equiv -\partial g_L / \partial \mathbf{m}$$



if the dynamics do not explicitly depend on time, e.g., the external field is constant, (autonomous dynamics):

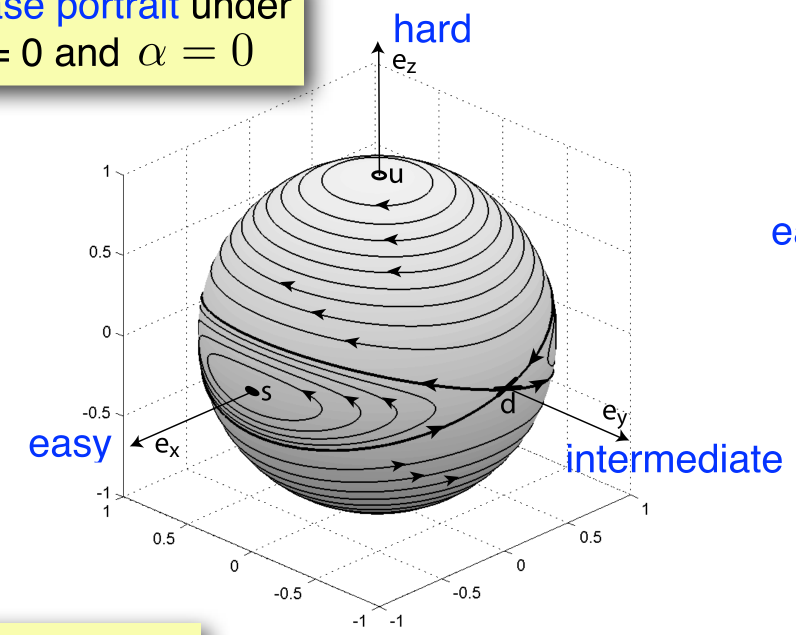
⬇

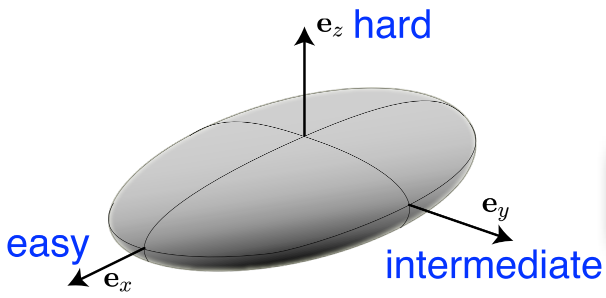
dynamics can be given a powerful geometrical representation through the associated phase portrait

If $D_x < D_y < D_z$, then x axis is the easy axis and z axis is the hard axis

Phase portraits of magnetization dynamics

phase portrait under $\mathbf{h}_a = 0$ and $\alpha = 0$



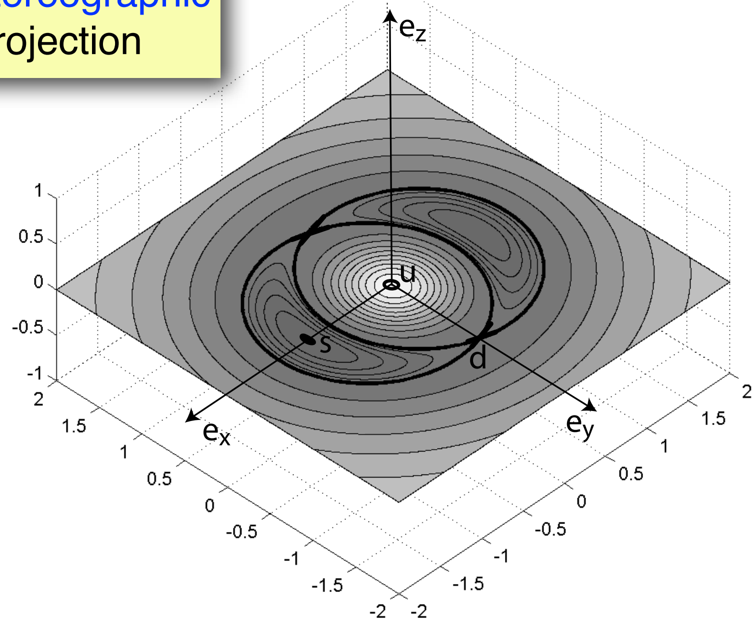


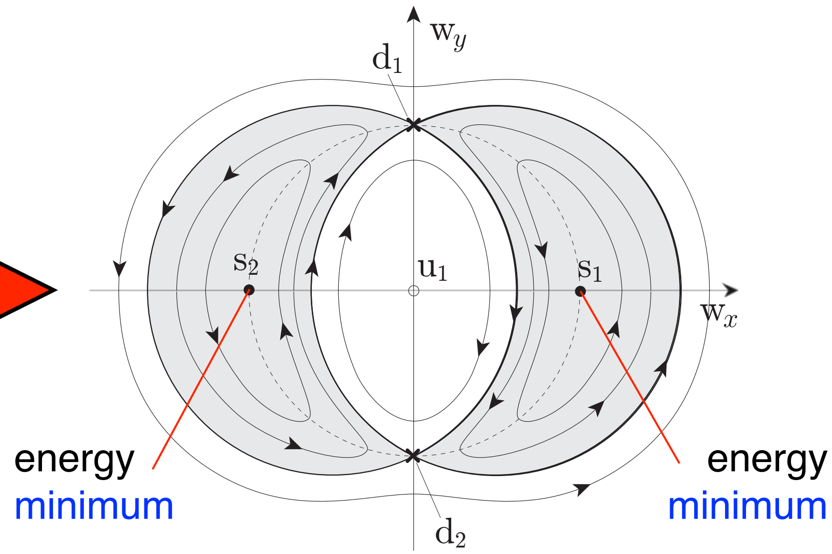
$$D_x < D_y < D_z$$

$$\frac{d\mathbf{m}}{dt} = -\mathbf{m} \times \mathbf{h}_{\text{eff}} \qquad \mathbf{h}_{\text{eff}} = -\frac{\partial g_L}{\partial \mathbf{m}}$$

$$g_L(\mathbf{m}) = \frac{1}{2}\left(D_x m_x^2 + D_y m_y^2 + D_z m_z^2\right)$$

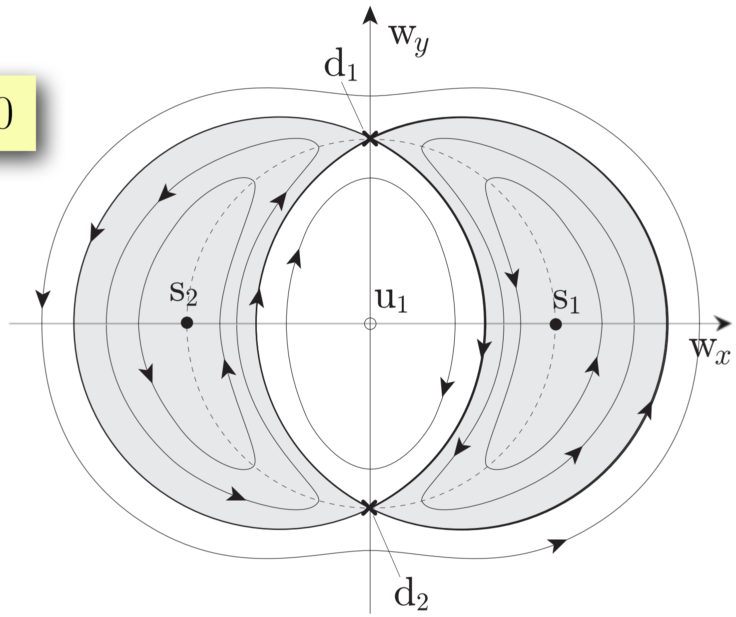
stereographic projection

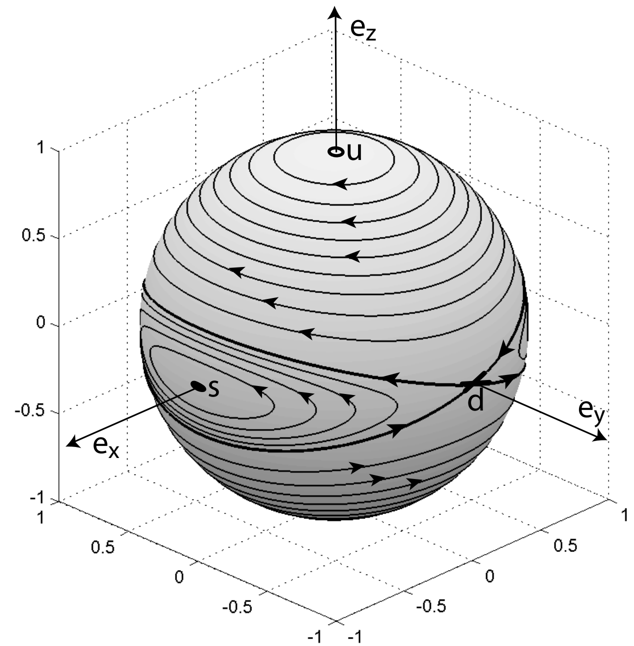




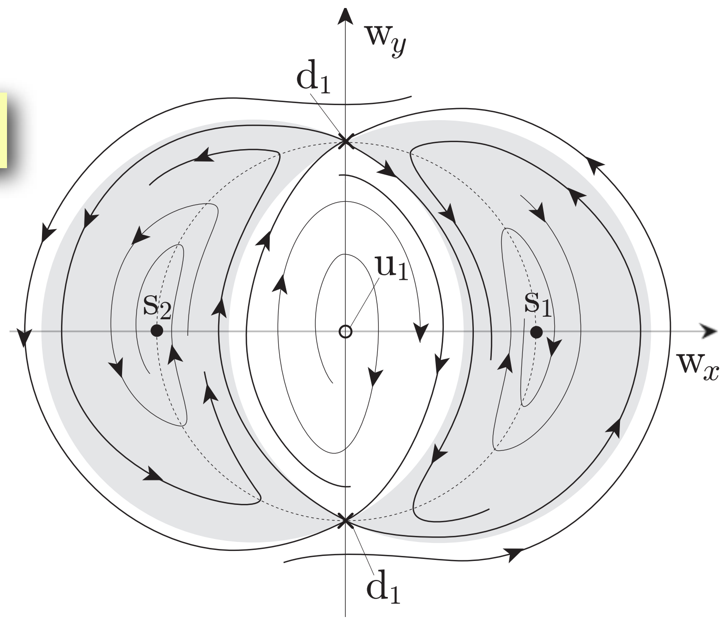
Effect of damping

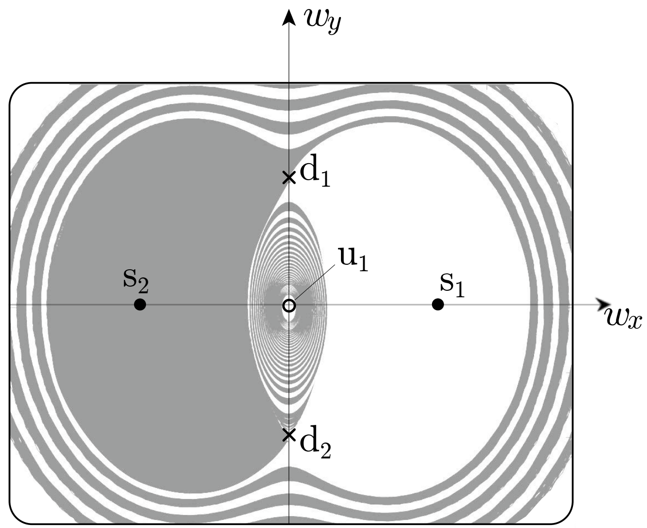
$\alpha = 0$



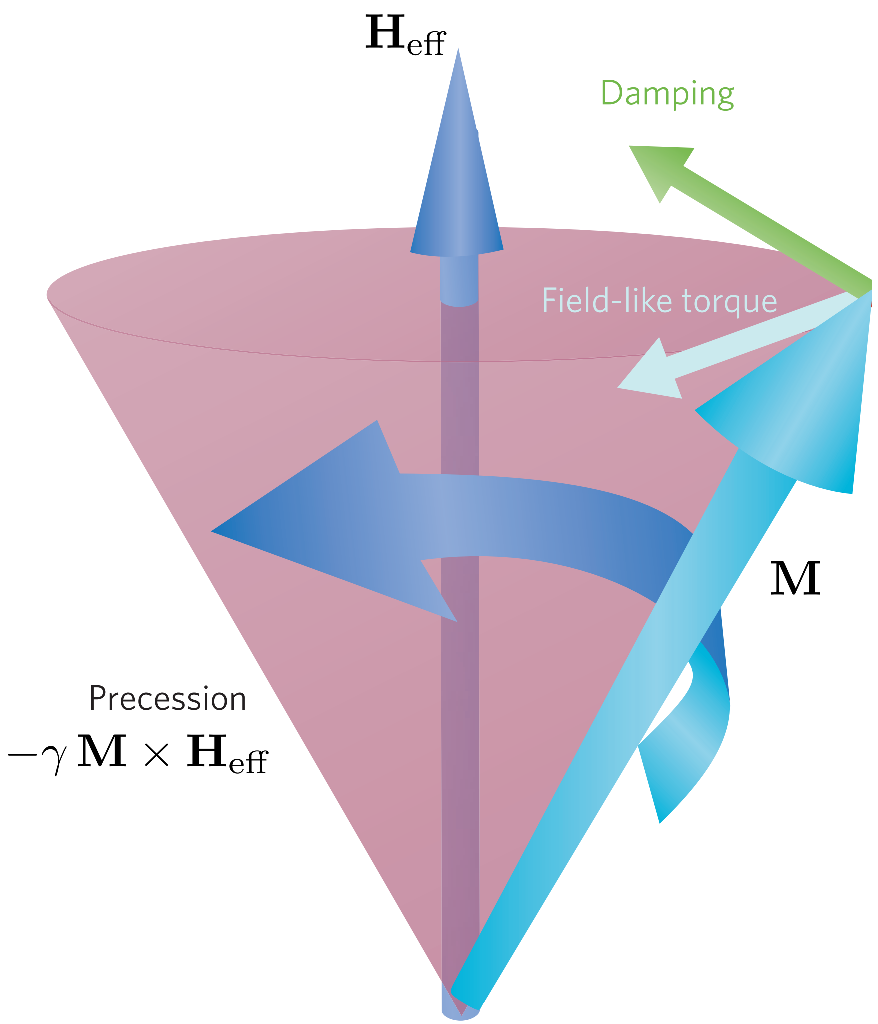


$\alpha \neq 0$





Landau-Lifshitz-Gilbert equation (summary)



$$\frac{\partial \mathbf{M}}{\partial t} = -\gamma\,\mathbf{M}\times\mathbf{H}_{\text{eff}} - \frac{\alpha\gamma}{M_s}\mathbf{M}\times(\mathbf{M}\times\mathbf{H}_{\text{eff}})$$

damping — precession

$$\frac{\partial \mathbf{M}}{\partial t} - \frac{\alpha}{M_s}\mathbf{M}\times\frac{\partial \mathbf{M}}{\partial t} = -\gamma\,\mathbf{M}\times\mathbf{H}_{\text{eff}}$$

the dimensionless parameter α measures the importance of damping effects: usually $\alpha \ll 1$

$$\frac{dG_L}{dt} = -\frac{\mu_0}{\gamma M_s}\int_V \alpha\left|\frac{\partial \mathbf{M}}{\partial t}\right|^2 dV - \mu_0\int_V \mathbf{M}\cdot\frac{d\mathbf{H}_a}{dt}\,dV$$

the energy invariably decreases in time when the applied magnetic field is kept constant in time

Landau-Lifshitz-Gilbert equation (summary)

- Landau-Lifshitz equation:

$$(1+\alpha^2)\,\frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m}\times\mathbf{h}_{\text{eff}} - \alpha\,\mathbf{m}\times(\mathbf{m}\times\mathbf{h}_{\text{eff}})$$

- Gilbert form:

$$\frac{\partial \mathbf{m}}{\partial t} - \alpha\,\mathbf{m}\times\frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m}\times\mathbf{h}_{\text{eff}}$$

$\mathbf{m} = \mathbf{M}/M_s\,,\ \mathbf{h}_{\text{eff}} = \mathbf{H}_{\text{eff}}/M_s$, time is measured in units of $(\gamma\, M_s)^{-1}$, α describes damping effects ($\alpha \ll 1$)

- Effective field:

$$\mathbf{h}_{\text{eff}} = \nabla^2\mathbf{m} + \mathbf{h}_{AN} + \mathbf{h}_{M} + \mathbf{h}_a$$

lengths are measured in units of the exchange length $l_{EX} = \sqrt{2A/\mu_0 M_s^2}$

- $\mathbf{m}\cdot\partial\mathbf{m}/\partial t = 0$

the dynamics preserves the magnetization magnitude condition $|\mathbf{m}(\mathbf{r},t)|^2 = 1$

- $\partial\mathbf{m}/\partial t = 0$ when $\mathbf{m}\times\mathbf{h}_{\text{eff}} = 0$

the dynamics preserves the micromagnetic equilibrium condition $\mathbf{m}\times\mathbf{h}_{\text{eff}} = 0$

- Rate of change of system energy:

$$\frac{dg_L}{dt} = -\int_V \alpha\left|\frac{\partial\mathbf{m}}{\partial t}\right|^2\frac{dV}{V} - \langle\mathbf{m}\rangle\cdot\frac{d\mathbf{h}_a}{dt}$$

under constant external field, the energy is always a decreasing function of time

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